Analytical solutions of one-dimensional advection–diffusion equation with variable coefficients in a finite domain

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Analytical solutions are obtained for one-dimensional advection–diffusion equation with variable coefficients in a longitudinal finite initially solute free domain, for two dispersion problems. In the first one, temporally dependent solute dispersion along uniform flow in homogeneous domain is studied. In the second problem the velocity is considered spatially dependent due to the inhomogeneity of the domain and the dispersion is considered proportional to the square of the velocity. The velocity is linearly interpolated to represent small increase in it along the finite domain. This analytical solution is compared with the numerical solution in case the dispersion is proportional to the same linearly interpolated velocity. The input condition is considered continuous of uniform and of increasing nature both. The analytical solutions are obtained by using Laplace transformation technique. In that process new independent space and time variables have been introduced. The effects of the dependency of dispersion with time and the inhomogeneity of the domain on the solute transport are studied separately with the help of graphs.

1. Introduction

Advection–diffusion equation describes the solute transport due to combined effect of diffusion and convection in a medium. It is a partial differential equation of parabolic type, derived on the principle of conservation of mass using Fick's law. Due to the growing surface and subsurface hydroenvironment degradation and the air pollution, the advection–diffusion equation has drawn significant attention of hydrologists, civil engineers and mathematical modelers. Its analytical/numerical solutions along with an initial condition and two boundary conditions help to understand the contaminant or pollutant concentration distribution behaviour through an open medium like air, rivers, lakes and porous medium like aquifer, on the basis of which remedial processes to reduce or eliminate the damages may be enforced. It has wide applications in other disciplines too, like soil physics, petroleum engineering, chemical engineering and biosciences.

In the initial works while obtaining the analytical solutions of dispersion problems in ideal conditions, the basic approach was to reduce the advection–diffusion equation into a diffusion equation by eliminating the convective term(s). It was done either by introducing moving co-ordinates (Ogata and Banks 1961; Harleman and Rumer 1963; Bear 1972; Guvanasen and Volker 1983; Aral and Liao 1996; Marshal *et al* 1996) or by introducing another dependent variable (Banks and Ali 1964; Ogata 1970; Lai and Jurinak 1971; Marino 1974 and Al-Niami and Rushton 1977). Then Laplace transformation technique has been used to get desired solutions. In addition to this method, Hankel transform method, Aris moment method, perturbation approach, method using Green's function, superposition method have also been used to get the analytical

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solutions of the advection–diffusion equations in one, two and three dimensions. But Laplace transformation technique has been commonly used because of being simpler than other methods and the analytical solutions using this method being more reliable in verifying the numerical solutions in terms of the accuracy and the stability.

Most of the works included the effects of adsorption, first order decay, zero order production on the concentration attenuation as the solute is transported down the stream. Such solutions have been compiled (van Genuchten and Alves 1982 and Lindstrom and Boersma 1989). Coming nearer to real problems layered porous medium was considered (Shamir and Harleman 1967), inhomogeneity of the porous medium was defined as a function of distance variable (Lin 1977), and non-linear adsorption has been considered (Liej *et al* 1993). Using the direct relationship between dispersion coefficient and velocity through porous medium (Ebach and White 1958), unsteady flow through porous medium has been considered to obtain the analytical solutions (Banks and Jerasate 1962; Hunt 1978 and Kumar 1983). Some one-dimensional analytical solutions have been given (Tracy 1995) by transforming the non-linear advection–diffusion equation into a linear one for specific forms of the moisture content *vs.* pressure head and relative hydraulic conductivity *vs.* pressure head curves which allow both two-dimensional and three-dimensional solutions to be derived. A method has been given to solve the transport equations for a kinetically adsorbing solute in a porous medium with spatially varying velocity field and dispersion coefficients (Van Kooten 1996). A methodology was presented (Manoranjan and Stauffer 1996) which enables to construct a closed form exact solution for transport with Langmuir sorption under nonequilibrium conditions without making any kind of restrictive *a priori* assumptions. A stochastic model for one-dimensional virus transport in homogeneous, saturated, semi-infinite porous media was developed (Chrysikopoulos and Sim 1996), the model accounts for first-order inactivation of liquid-phase, adsorbed viruses with different inactivation rate constants, and time dependent distribution coefficient.

Later it has been shown that some large subsurface formations exhibit variable dispersivity properties, either as a function of time or as a function of distance (Matheron and deMarsily 1980; Sposito *et al* 1986; Gelhar *et al* 1992). Analytical solutions were developed for describing the transport of dissolved substances in heterogeneous semi-infinite porous media with a distance dependent dispersion of exponential nature along the uniform flow (Yates 1990, 1992). The work of Yates was extended (Logan and Zlotnik 1995 and Logan 1996) by including the adsorption and decay effects and by studying their interaction with the inhomogeneity caused by scale-dependent dispersion along uniform flow for periodic input condition. One-dimensional analytical solutions were presented for the advection–diffusion equation for solute dispersion, being proportional to the square of velocity and velocity proportional to the position variable (Zoppou and Knight 1997). Time dependent dispersion along uniform flow has been considered (Aral and Liao 1996) to solve two-dimensional advection–diffusion equation. An analytical solution has been obtained for two dimensional steady state mass transports in a trapezoidal embankment in a spatially varying velocity field through its replacement with a hydrologically equivalent rectangular embankment (Tartakovsky and Federico 1997).

The temporal moment solution for one dimensional advective-dispersive solute transport with linear equilibrium sorption and first order degradation for time pulse sources has been applied to analyze soil column experimental data (Pang *et al* 2003). An analytical approach was developed for nonequilibrium transport of reactive solutes in the unsaturated zone during an infiltration–redistribution cycle (Severino and Indelman 2004). The solute is transported by advection and obeys linear kinetics. Analytical solutions were presented for solute transport in rivers including the effects of transient storage and first order decay (Smedt 2006). Pore flow velocity was assumed to be a nondivergence – free, unsteady and nonstationary random function of space and time for ground water contaminant transport in a heterogeneous media (Sirin 2006). A two-dimensional semi-analytical solution was presented to analyze stream–aquifer interactions in a coastal aquifer where groundwater level responds to tidal effects (Kim *et al* 2007).

In the present paper analytical solutions are obtained for two solute dispersion problems in a longitudinal finite domain, organized in sections 2 and 3, respectively. In the first problem time dependent solute dispersion of increasing or decreasing nature along a uniform flow through a homogeneous domain is studied. In the second problem the medium is considered inhomogeneous hence the velocity is considered dependent on position variable. The velocity is linearly interpolated in position variable which represents a small increase in the velocity from one end to the other end of the domain. This expression contains a parameter to represent a change in inhomogeneity from one medium to other medium. Dispersion is assumed proportional to square of velocity. In each problem the domain is initially solute free. The input condition is of uniform and varying nature, respectively.

Numerical solution has also been obtained for the case in which dispersion varies linearly with velocity and has been compared with the analytical solution obtained in the former case.

2. Temporally dependent dispersion along uniform flow

2.1 *Uniform continuous input*

Advection–diffusion equation in one dimension with variable coefficients is:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C}{\partial x} - u(x,t)C\right), \qquad (1)$$

where C represents the solute concentration at position x along the longitudinal direction at time t, D is the solute dispersion, if it is independent of position and time, is called dispersion coefficient, and u is the medium's flow velocity. To study the temporally dependent solute dispersion of a uniform input concentration of continuous nature in an initially solute free finite domain, we consider

$$D(x,t) = D_0 f(mt) \quad \text{and} \quad u(x,t) = u_0, \qquad (2)$$

where m is a coefficient whose dimension is inverse of that of the time variable. Thus $f(mt)$ is an expression in non-dimensional variable (mt). The expressions of $f(mt)$ are chosen such that $f(mt) = 1$ for $m = 0$ or $t = 0$. The former case represents the uniform solute dispersion and the latter case represents the initial dispersion. The coefficients D_0 and u_0 in equation (2) may be defined as initial dispersion coefficient and uniform flow velocity, respectively. Thus the partial differential equation (1) along with initial condition and boundary conditions may be written as:

$$\frac{\partial C}{\partial t} = D_0 f(mt)\frac{\partial^2 C}{\partial x^2} - u_0\frac{\partial C}{\partial x}, \qquad (3)$$

$$C(x,t) = 0, \quad 0 \le x \le L, \quad t = 0, \qquad (4)$$

$$C(x,t) = C_0, \quad x = 0, \quad t > 0, \qquad (5)$$

$$\frac{\partial C(x,t)}{\partial x} = 0, \quad x = L, \quad t \ge 0, \qquad (6)$$

where the input condition is assumed at the origin and a second type or flux type homogeneous condition is assumed at the other end $x = L$, of the domain. C_0 is a reference concentration.

To use the Laplace transform technique conveniently, it is necessary to bring the time dependent coefficient in differential equation (3) on the left hand side. For this purpose we introduce a new independent variable by a transformation

$$X = \int \frac{dx}{f(mt)} \quad \text{or} \quad \frac{dX}{dx} = \frac{1}{f(mt)}. \qquad (7)$$

As mt is a non-dimensional term so the dimension of X will be that x hence is referred to as a new space variable, a moving co-ordinate, though it is different from those considered in the references cited at the outset of the first section. The initial and boundary value problem in new space variable may be expressed as:

$$f(mt)\frac{\partial C}{\partial t} = D_0\frac{\partial^2 C}{\partial X^2} - u_0\frac{\partial C}{\partial X}, \qquad (8)$$

$$C(X,t) = 0, \quad 0 \le X \le X_0,$$

$$t = 0; \quad X_0 = \frac{L}{f(mt)}, \qquad (9)$$

$$C(X,t) = C_0, \quad X = 0, \quad t > 0, \qquad (10)$$

$$\frac{\partial C(X,t)}{\partial X} = 0, \quad X = X_0, \quad t \ge 0. \qquad (11)$$

To get rid of the time dependent coefficient following transformation (Crank 1975) is used:

$$T = \int_0^t \frac{dt}{f(mt)}. \qquad (12)$$

The dimension of T will be that of the variable t so it is referred to as a new time variable. Further it should also be ensured while choosing $f(mt)$ that $T = 0$ at $t = 0$ so that the nature of initial condition does not change in the new time domain. The initial and boundary value problem (equations 8–11) may be expressed in new time variable as:

$$\frac{\partial C}{\partial T} = D_0\frac{\partial^2 C}{\partial X^2} - u_0\frac{\partial C}{\partial X}, \qquad (13)$$

$$C(X,T) = 0, \quad 0 \le X \le X_0,$$

$$T = 0; \quad X_0 = \frac{L}{f(mt)}, \qquad (14)$$

$$C(X,T) = C_0, \quad X = 0, \quad T > 0, \tag{15}$$

$$\frac{\partial C(X,T)}{\partial X} = 0, \quad X = X_0, \quad T \geq 0. \tag{16}$$

Now the initial and boundary value problem (equations 13–16) in the (X,T) domain becomes similar to that of Cleary and Adrian (1973) in (x,t), quoted as problem A3 (van Genuchten and Alves 1982), hence or otherwise using Laplace transformation technique, the desired analytical solution may be written as follows:

$$C(X,T) = C_0 A(X,T), \tag{17}$$

where

$$\begin{aligned} A(X,T) &= \frac{1}{2}\,\mathrm{erfc}\left(\frac{X - u_0T}{2\sqrt{D_0T}}\right) \\ &+ \frac{1}{2}\exp\left(\frac{u_0X}{D_0}\right)\mathrm{erfc}\left(\frac{X + u_0T}{2\sqrt{D_0T}}\right) \\ &+ \frac{1}{2}\left[2 + \frac{u_0(2X_0 - X)}{D_0} + \frac{u_0^2T}{D_0}\right] \\ &\times \exp\left(\frac{u_0X_0}{D_0}\right)\mathrm{erfc}\left(\frac{(2X_0 - X) + u_0T}{2\sqrt{D_0T}}\right) \\ &- \left(\frac{u_0^2T}{\pi D_0}\right)^{1/2}\exp\left[\frac{u_0X_0}{D_0} - \frac{(2X_0 - X + u_0T)^2}{4D_0T}\right], \end{aligned}$$

$X = x/f(mt)$, $X_0 = L/f(mt)$ and T may be obtained from transformation (equation 12).

2.2 *Input condition of increasing nature*

The source of input concentration may increase with time due to variety of reasons. One way to represent such a condition is to consider a factor $C_0F(t)$ on right hand side of the input condition (equation 5) where $F(t)$ may be an increasing function. But this type of situation may also be described by a mixed type or third type condition written as follows:

$$-D(x,t)\frac{\partial C}{\partial x} + u(x,t)C = u_0C_0 \quad \text{at } x = 0,\ t > 0. \tag{18}$$

Using equations (2), (7) and (12) the above condition may be written in (X,T) domain as:

$$-D_0\frac{\partial C}{\partial X} + u_0C = u_0C_0 \quad \text{at } X = 0,\ T > 0. \tag{19}$$

Now the initial and boundary value problem composed of advection–diffusion equation (13), initial condition (14), input condition (19) and second boundary condition (16), in the (X,T) domain becomes similar to that of Bastian and Lapidus (1956) and Brenner (1962) in (x,t) domain, quoted as the problem A4 (van Genuchten and Alves 1982) hence or otherwise using Laplace transformation technique, the desired analytical solution may be written as follows:

$$C(X,T) = C_0 A(X,T) \tag{20}$$

where

$$\begin{aligned} A(X,T) &= \frac{1}{2}\,\mathrm{erfc}\left(\frac{X - u_0T}{2\sqrt{D_0T}}\right) + \left(\frac{u_0^2T}{\pi D_0}\right)^{1/2} \\ &\times \exp\left[-\frac{(X + u_0T)^2}{4D_0T}\right] - \frac{1}{2}\left(1 + \frac{u_0X}{D_0} + \frac{u_0^2T}{D_0}\right) \\ &\times \exp\left(\frac{u_0X}{D_0}\right)\mathrm{erfc}\left(\frac{X + u_0T}{2\sqrt{D_0T}}\right) \\ &+ \left(\frac{4u_0T}{\pi D_0}\right)^{1/2}\left[1 + \frac{u_0}{4D_0}(2X_0 - X + u_0T)\right] \\ &\times \exp\left[\frac{u_0X_0}{D_0} - \frac{(2X_0 - X + u_0T)}{4D_0T}\right] - \frac{u_0}{D_0} \\ &\times \left[(2X_0 - X) + \frac{3u_0T}{2} + \frac{u_0}{4D_0}(2X_0 - X + u_0T)^2\right] \\ &\times \exp\left(\frac{u_0X_0}{D_0}\right)\mathrm{erfc}\left(\frac{(2X_0 - X) + u_0T}{2\sqrt{D_0T}}\right), \end{aligned}$$

$X = x/f(mt)$, $X_0 = L/f(mt)$ and T may be obtained from transformation (12).

3. Spatially dependent dispersion along non-uniform flow

3.1 *Uniform input*

Though inhomogeneity of porous domain was defined by scale dependent dispersion and flow through the medium has been considered uniform (Yates 1992; Logan 1996) but the flow velocity may also depend upon position variable in case the domain is inhomogeneous. Zoppou and Knight (1997) have considered the velocity as $u = \beta x$, and the solute dispersion proportional to square of velocity, i.e., as $D = \alpha x^2$; in a semi-infinite domain $x_0 \leq x < \infty$. But these expressions do not reflect real variations due to inhomogeneity of the medium because as $x \to \infty$, dispersion and velocity also become too large. In fact the variation in velocity due to inhomogeneity should be small so that the velocity at each position satisfies the Darcy's law in case the medium is porous or satisfies the laminar condition of the flow in a non-porous medium, an essential condition for the velocity parameter, u in the advection–diffusion equation. This factor is taken care of in the present work and velocity is linearly interpolated in position variable such that it increases from a value u_0 at $x = 0$ to a value $(1 + b)u_0$ at $x = L$, where b may be a real constant. Thus

$$u(x,t) = u_0(1 + ax), \tag{21}$$

where $a = b/L$, is the parameter accounting for the inhomogeneity of the medium. It should be small so that the increase in velocity is of small order. Solute dispersion is assumed proportional to square of the velocity so we consider

$$D(x,t) = D_0(1 + ax)^2. \tag{22}$$

As ax is a non-dimensional term hence D_0 and u_0 are dispersion coefficient and velocity, respectively at the origin $(x = 0)$ of the medium. The domain is assumed initially solute free. An input concentration is assumed at the origin and a flux type homogeneous condition is assumed at the other end of the domain. Now to solve advection–diffusion equation (1) for the expressions in (21) and (22) along with the conditions (4–6) we introduce a new space variable, X by a transformation

$$X = -\int \frac{dx}{(1 + ax)^2} = \frac{1}{a(1 + ax)}, \tag{23}$$

in terms of which the advection–diffusion equation (1) assumes the form

$$\frac{\partial C}{\partial t} = a^2 D_0 X^2 \frac{\partial^2 C}{\partial X^2} + a u_0 X \frac{\partial C}{\partial X} - a u_0 C. \tag{24}$$

It is further reduced into a partial differential equation with constant coefficients by using a transformation

$$Z = -\log aX = \log(1 + ax). \tag{25}$$

Ultimately we get the present initial and boundary value dispersion problem reduced in following equations:

$$\frac{\partial C}{\partial t} = a^2 D_0 \frac{\partial^2 C}{\partial Z^2} - (a u_0 - a^2 D_0)\frac{\partial C}{\partial Z} - a u_0 C, \tag{26}$$

$$C(Z,t) = 0, \quad 0 \leq Z \leq Z_0, \quad t = 0; \quad Z_0 = \log(1 + aL), \tag{27}$$

$$C(Z,t) = C_0, \quad Z = 0, \quad t > 0, \tag{28}$$

$$\frac{\partial C}{\partial Z} = 0, \quad Z = Z_0, \quad t \geq 0. \tag{29}$$

It is similar to the problem of Selim and Mansell (1976) quoted as the problem C7 (van Genuchten and Alves 1982) hence or otherwise (using Laplace transformation technique) the desired analytical solution may be written as follows

$$C(Z,t) = C_0 A(Z,t)/B(Z), \tag{30}$$

where

$$A(Z,t) = \frac{1}{2}\exp\left\{\frac{(v - w)Z}{2D}\right\}\operatorname{erfc}\left(\frac{Z - wt}{2\sqrt{Dt}}\right) + \frac{1}{2}\exp\left\{\frac{(v + w)Z}{2D}\right\}\operatorname{erfc}\left(\frac{Z + wt}{2\sqrt{Dt}}\right) + \frac{(w - v)}{2(w + v)}\exp\left\{\frac{(v + w)Z - 2wZ_0}{2D}\right\}$$

$$\times \operatorname{erfc}\left(\frac{2Z_0 - Z - wt}{2\sqrt{Dt}}\right) + \frac{(w+v)}{2(w-v)}$$

$$\times \exp\left\{\frac{(v-w)Z + 2wZ_0}{2D}\right\}$$

$$\times \operatorname{erfc}\left(\frac{2Z_0 - Z + wt}{2\sqrt{Dt}}\right) - \frac{v^2}{2\mu D}$$

$$\times \exp\left(\frac{vZ_0}{D} - \mu t\right) \operatorname{erfc}\left(\frac{2Z_0 - Z + vt}{2\sqrt{Dt}}\right),$$

$$B(Z) = 1 + \frac{(w-v)}{2(w+v)} \exp\left(\frac{wZ_0}{D}\right),$$

$$w = v + \left(1 + \frac{4\mu D}{v^2}\right)^{\frac{1}{2}}$$

and

$$Z = \log(1 + ax); \quad Z_0 = \log(1 + aL);$$

$$D = a^2 D_0; \quad v = (au_0 - a^2 D_0); \quad \mu = au_0.$$

3.2 *Input condition of increasing nature*

The input condition of increasing nature introduced at the origin of the domain is of same type as condition (18) but is defined with slightly different coefficients as follows:

$$-D(x,t)\frac{\partial C}{\partial x} + \{u(x,t) - aD(x,t)\}\, C$$

$$= (u_0 - aD_0)C_0 \quad \text{at } x = 0, \quad t > 0. \tag{31}$$

This condition may be reduced in terms of new space variable Z defined by equation (25) as:

$$-a^2 D_0 \frac{\partial C}{\partial Z} + (au_0 - a^2 D_0)C = (au_0 - a^2 D_0)C_0,$$

$$Z = 0, \quad t > 0. \tag{32}$$

Now the initial and boundary value problem composed of advection–diffusion equation (26), initial condition (equation 27), input condition (equation 32) and second boundary condition (equation 29), in the (X, T) domain becomes similar to that problem C8 (van Genuchten and Alves 1982) hence or otherwise (using Laplace transformation technique) the desired analytical solution may be written as follows:

$$C(Z,t) = C_0 A(Z,t)/B(Z), \tag{33}$$

where

$$A(Z,t)$$

$$= \frac{v}{(v+w)} \exp\left\{\frac{(v-w)Z}{2D}\right\} \operatorname{erfc}\left(\frac{Z - wt}{2\sqrt{Dt}}\right)$$

$$+ \frac{v}{(v-w)} \exp\left\{\frac{(v+w)Z}{2D}\right\} \operatorname{erfc}\left(\frac{Z + wt}{2\sqrt{Dt}}\right)$$

$$+ \frac{v^2}{2\mu D} \exp\left\{\frac{vZ}{D} - \mu t\right\} \operatorname{erfc}\left(\frac{Z + vt}{2\sqrt{Dt}}\right)$$

$$+ \frac{v^2}{2\mu D}\left[\frac{v(2Z_0 - Z)}{D} + \frac{v^2 t}{D} + 3 + \frac{v^2}{\mu D}\right]$$

$$\times \exp\left\{\frac{vZ_0}{D} - \mu t\right\} \operatorname{erfc}\left(\frac{2Z_0 - Z + vt}{2\sqrt{Dt}}\right)$$

$$- \frac{v^3}{\mu D}\sqrt{\frac{t}{\pi D}} \exp\left[\frac{vZ_0}{D} - \mu t - \frac{(2Z_0 - Z + vt)^2}{4Dt}\right]$$

$$+ \frac{v(w-v)}{(w+v)^2} \exp\left\{\frac{(v+w)Z - 2wZ_0}{2D}\right\}$$

$$\times \operatorname{erfc}\left(\frac{2Z_0 - Z - wt}{2\sqrt{Dt}}\right)$$

$$- \frac{v(w+v)}{(w-v)^2} \exp\left\{\frac{(v-w)Z + 2wZ_0}{2D}\right\}$$

$$\times \operatorname{erfc}\left(\frac{2Z_0 - Z + wt}{2\sqrt{Dt}}\right)$$

$$B(Z) = 1 - \frac{(w-v)^2}{2(w+v)^2} \exp\left(-\frac{wZ_0}{D}\right),$$

$$w = v + \left(1 + \frac{4\mu D}{v^2}\right)^{\frac{1}{2}}$$

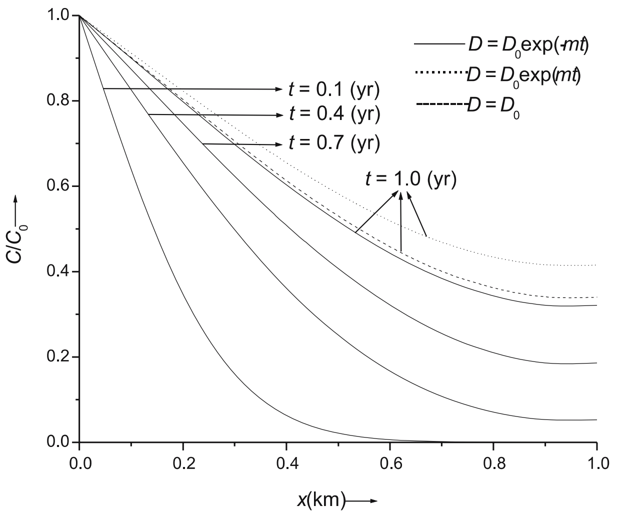

Figure 1. Temporal dependent solute dispersion along uniform flow of uniform input described by solution (equation 17). The four solid curves are drawn for $D = D_0 \exp(-mt)$. The two dotted curves are drawn for $D = D_0 \exp(mt)$ and $D = D_0$, respectively.

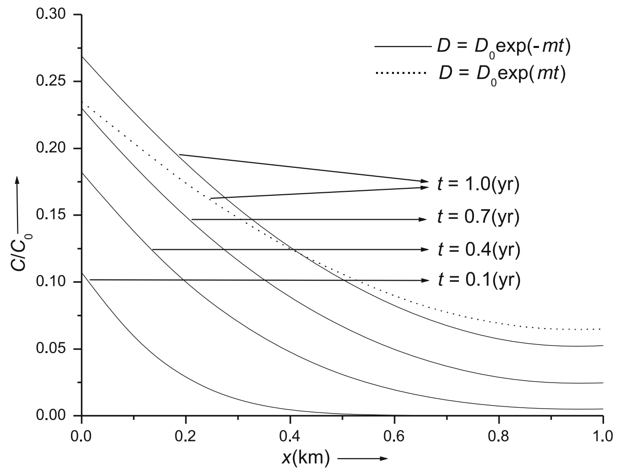

Figure 2. Temporal dependent solute dispersion along uniform flow of input of increasing nature described by solution (equation 20). The four solid curves are drawn for $D = D_0 \exp(-mt)$. The only dotted curve is drawn for $D = D_0 \exp(mt)$.

and

$$Z = \log(1 + ax); \quad Z_0 = \log(1 + aL);$$

$$D = a^2 D_0; \quad v = (au_0 - a^2 D_0); \quad \mu = au_0.$$

4. Results and discussions

Concentration values are evaluated from the four analytical solutions discussed in the sections 2 and 3, in a finite domain $0 \le x \le 1$ (i.e., $L = 1.0$ km is chosen) at times t (years) = 0.1, 0.4, 0.7 and 1.0, for input values $C_0 = 1.0$, $u_0 = 0.11$ (km/year), $D_0 = 0.21$ (km^2/year).

Figures 1 and 2 represent temporal dependent concentration dispersion pattern of uniform input and input of increasing nature, respectively along a uniform flow through a homogeneous medium, described by the analytical solutions, equations (17) and (20), respectively. In figure 1, the uniform input concentration value is 1.0 at all times. In figure 2, the concentration value at $x = 0$ increases with time. Thus the respective input boundary conditions are satisfied. In both the figures the solid curves represent the solutions for an expression $f(mt) = \exp(-mt)$ which is of decreasing nature. In both the figures the dotted curve represents the respective solutions at $t = 1.0$ (year), for another expression $f(mt) = \exp(mt)$, which is of increasing nature. It may be observed that in case of uniform input the concentration value at a particular position is higher for the latter expression of $f(mt)$ than that for the former expression of $f(mt)$. The difference increases with the distance along the domain. But in case of an input concentration of increasing nature its value is less for increasing nature of $f(mt)$ than that for decreasing nature of $f(mt)$. This trend is of diminishing nature up to $x = 0.4$, beyond which the trend reverses. For all the curves drawn in figures 1 and 2, a value $m(\text{year})^{-1} = 0.1$ is chosen. Both the analytical solutions of section 2 may be solved using other expressions of $f(mt)$ which satisfy the conditions stated at the outset of the section 2.

Figures 3 and 4 depict the concentration values evaluated from analytical solutions (equations 30 and 33) for spatially dependent dispersion of uniform input and input of increasing nature, respectively, along non-uniform flow, through an inhomogeneous domain. The solid curves in figure 3 represent the solution (equation 30) in which a value $a = 1.0\,(\text{km})^{-1}$ is taken. Using expressions (21–22) it may be evaluated that due to the inhomogeneity of the medium, the velocity u varies from a value of 0.11 (km/year) to a value of 0.22 (km/year) and dispersion D varies from a value of 0.21 (km/year) to a value of 0.42 (km/year), along the domain $0 \le x(\text{km}) \le 1$. This figure also shows the effect of inhomogeneity on the dispersion pattern. A dotted curve is drawn for the value $a = 0.1\,(\text{km})^{-1}$. It may be observed that the concentration values evaluated from the solution (equation 30) along a medium of lesser inhomogeneity (which introduces lesser variation in velocity and dispersion along the column), are slightly higher than those at the respective positions of a medium of higher inhomogeneity, near the origin but decrease at faster rate as the other

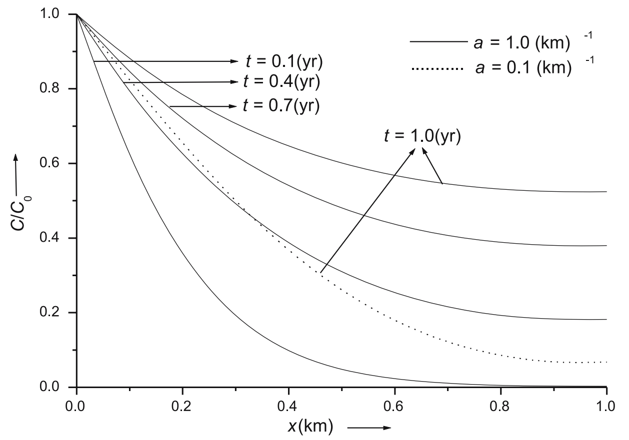

Figure 3. Spatially dependent solute dispersion along non-uniform flow of uniform input when $D \propto u^2$, described by solution (equation 30). The four solid curves are drawn for $a = 1.0\,(\text{km})^{-1}$. The only dotted curve is drawn for $a = 0.1\,(\text{km})^{-1}$.

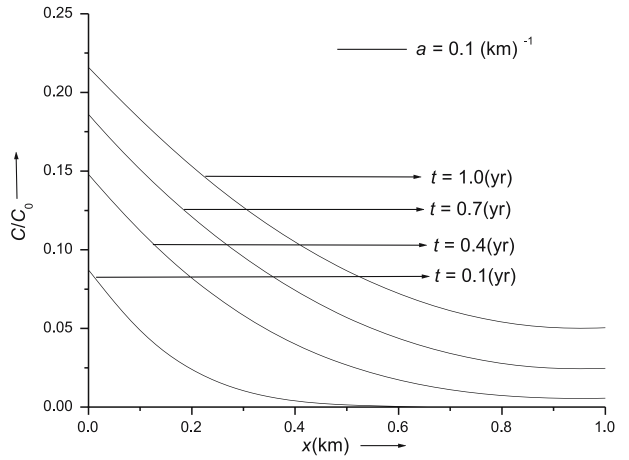

Figure 4. Spatially dependent solute dispersion along non-uniform flow of input of increasing nature when $D \propto u^2$ described by solution (equation 33). The four solid curves are drawn for $a = 1.0\,(\text{km})^{-1}$.

end of the medium is approached. This comparison is done at $t = 0.4$ (year). Such comparison is not shown in figure 4, in which all the four solid curves represent the concentration values in case of the input of increasing nature along a medium of inhomogeneity defined by a value $a = 0.1\,(\text{km})^{-1}$. This value is chosen to ensure that the factor $(u_0 - aD_0)$ in condition (32) remains positive for the values chosen for u_0 and D_0.

Numerical solutions are also been obtained using a two-level explicit finite difference scheme in case of uniform input and for the input of increasing nature. Step-sizes $\Delta x = 0.1$ and $\Delta t = 0.001$ along x-domain and t-domain, respectively, are chosen to ensure the stability criterion, $0 < \Delta t/(\Delta x)^2 < 0.5$, is satisfied. The numerical solutions have been obtained at $t = 1.0$ (year) using the same other input values, for temporally dependent problems for uniform and increasing inputs and spatially dependent dispersion problem for uniform input. These numerical solutions have been compared with the respective analytical solutions (equations 17, 20 and 30). A complete agreement between them has been found between the respective numerical and analytical solutions. So the same difference scheme has been used to obtain numerical solutions for spatially dependent dispersion of uniform input concentration along non-uniform velocity assuming dispersion being proportional to the velocity. In other words, in place of the expression in equation (22), $D(x,t) = D_0(1 + ax)$, has been considered. In this case analytical solution appears to be not possible. The numerical solutions are compared with the analytical solution (equation 30) at $t = 0.4$ (year) and $t = 1.0$ (years), respectively. The comparisons are explained with the help of table 1 and figure 5. It may be observed that the difference between the concentration values is almost negligible near the origin, those numerically obtained are slightly higher than the respective analytical values but beyond that region the trend reverses more evidently and increases with position. Thus the attenuation process along the column is slower in case the dispersion is directly proportional to velocity than that in case the dispersion is proportional to square of velocity, i.e., in the latter case, more solute concentration are driven away from its source than that in the former case. Such solutions are useful in predicting the danger level of pollutants concentration at a particular position/region away from its source. Due to human activities and other reasons, pollutants are reaching surface and subsurface hydro-environment at a particular poison and are transported down the stream degrading the water quality and depleting the health of flora and fauna. Usually the causes are of increasing nature. Similarly movement of particulate particles in the air, their effects on human health can also be understood by such studies.

Substituting $m = 0$, we get $f(mt) = 1$ and $T = t$, the initial and boundary value problem (equations 13–16) and its solution (equation 17) reduce to those for uniform dispersion along uniform velocity reported in the problem A3 (van Genuchten and Alves 1982), respectively. The concentration values are also evaluated from solution (equation 30) for $m = 0$ and are depicted in figure 1. It may be observed that these values are less than the values obtained for solute dispersion increasing with time and are higher than those obtained for the dispersion decreasing with time. The same substituting reduces the solution

Table 1. *Comparison of spatially dependent concentration dispersion along non-uniform flow of uniform input in cases (i) $D \propto u^2$ (analytical solution) and (ii) $D \propto u$ (numerical solution). In both the cases $a = 1.0$ (km)$^{-1}$.*

x (km)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$t = 0.4$ (yr)											
(i)	1.0	0.794	0.626	0.492	0.388	0.310	0.254	0.216	0.193	0.183	0.182
(ii)	1.0	0.810	0.642	0.498	0.379	0.285	0.213	0.163	0.131	0.115	0.114
$t = 1.0$ (yr)											
(i)	1.0	0.881	0.785	0.707	0.647	0.601	0.568	0.545	0.531	0.525	0.524
(ii)	1.0	0.887	0.786	0.697	0.622	0.559	0.510	0.474	0.450	0.438	0.438

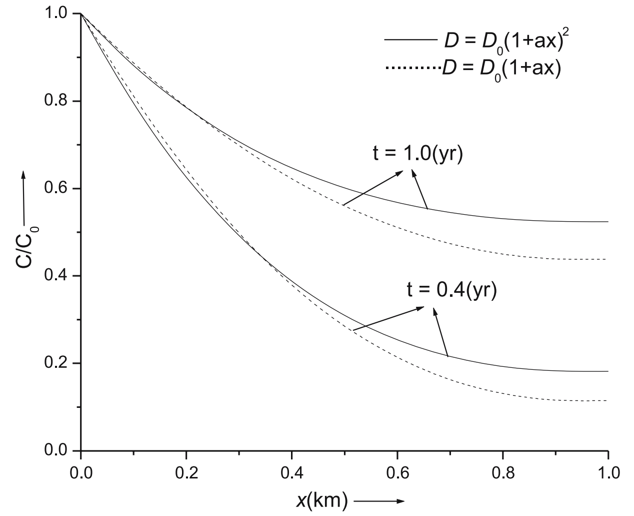

Figure 5. Comparison of spatially dependent concentration dispersion along non-uniform flow of uniform input in cases (i) $D \propto u^2$ (analytical solution) and (ii) $D \propto u$ (numerical solution). In both the cases, $a = 1.0\,(\text{km})^{-1}$.

(equation 20) to that of the problem A4 (van Genuchten and Alves 1982). But it may be noted that although for $a = 0$, from equations (21 and 22) we get $u = u_0$ and $D = D_0$, respectively, hence the advection–diffusion equation with spatially dependent coefficients reduces to that with constant coefficients but the solution of the latter along with the same initial and boundary conditions cannot be obtained by substituting $a = 0$ in the solution (equation 30 or 33) (as the case for the input may be). It may be verified and understood from the solutions of the two ordinary differential equations $(1 + ax)^2y'' - 3(1 + ax)y' + 2y = 0$ and $y'' - 3y' + 2y = 0$.

It may be observed that while getting the analytical solutions in section 2, the transformation equation (7) has played a key role. A similar transformation has been used in section 3 also. In fact in the advection–diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D_0 f_1(x,t)\frac{\partial C}{\partial x} - u_0 f_2(x,t)C\right),$$

a substitution like

$$\frac{\partial C}{\partial X} = f_1(x,t)\frac{\partial C}{\partial x} - f_2(x,t)C,$$

is a linear partial differential equation of first order hence it is equivalent to an auxilliary system of ordinary differential equations, one of which is

$$\frac{dX}{-1} = \frac{dx}{f_1(x,t)}.$$

Its solution is both the transformations (equations 7 and 23) for $f_1(x,t) = f(mt)$ and $f_1(x,t) = (1 + ax)^2$, respectively. Though minus sign is omitted in transformation (equation 7) it cannot be omitted in equation (23) otherwise the position $x = L$ in condition (12) will correspond to $Z_0 = \log(-1 - aL)$ in equation (27). In case the velocity is linearly interpolated as $u = u_0(1 - ax)$ to represent the decrease in it along the medium column $0 \le x \le L$, one has to proceed with positive sign in (equation 23).

5. Conclusions

Analytical solutions to one-dimensional advection–diffusion equation with variable coefficients along with two sets of boundary conditions (in one set input condition is uniform while in other it is of increasing nature while the second condition in each set is homogeneous of flux type) in an initially solute free finite domain have been obtained in two cases:

(i) temporal dependent dispersion along uniform flow through homogeneous medium and
(ii) spatially dependent dispersion along non-uniform flow through inhomogeneous medium in which solute dispersion is assumed proportional to the square of velocity.

The application of a new transformation which introduces another space variable, on the advection–diffusion equation makes it possible to use Laplace transformation technique in getting the analytical solutions. Numerical solutions have also been obtained using a two-level explicit finite difference scheme. The respective analytical and numerical solutions have also been compared and very good agreement between the two has been found. The analytical solution of the second problem in case of uniform input has been compared with the numerical solution of same problem but assuming dispersion varying with velocity. Such analytical solutions may serve as tools in validating numerical solutions in more realistic dispersion problems facilitating to assess the transport of pollutants solute concentration away from its source along a flow through soil medium, through aquifers and through oil reservoirs.

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