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ANALYTICAL SOLUTIONS OF THE KLEIN–FOCK–GORDON EQUATION WITH THE MANNING–ROSEN POTENTIAL PLUS A RING-SHAPED LIKE POTENTIAL

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In this work, on the condition that scalar potential is equal to vector potential, the bound state solutions of the Klein–Fock–Gordon equation of the Manning–Rosen plus ring-shaped like potential are obtained by Nikiforov–Uvarov method. The energy levels are worked out and the corresponding normalized eigenfunctions are obtained in terms of orthogonal polynomials for arbitrary l states. The conclusion also contain central Manning–Rosen, central and noncentral Hulthén potential.

Keywords: Nikiforov–Uvarov method; Manning–Rosen; ring-shaped potential.

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1. Introduction

Since the early years of quantum mechanics (QM) the study of exactly solvable problems for some special potentials of physical interest has attracted much attention in theoretical physics. Obtaining analytical solutions of the Klein–Fock–Gordon, Dirac and other wave equations is one of the interesting problems in high energy and nuclear physics. These wave equations are frequently used to describe the particle dynamics in relativistic QM. Already long time in literature, a great deal of effort has been spent to solve these relativistic wave equations for different potentials which also include mixing potentials.

The description of phenomena at high energies requires the investigation of relativistic wave equations, which are invariant under Lorentz transformation, to give correction for nonrelativistic QM.[1,2] If we consider the case where the interaction potential is not enough to create particle–antiparticle pairs, we can apply the KFG equation to the treatment of a zero-spin particle and apply the Dirac equation to that of a 1/2-spin particle. A particle is moving in a strong potential field, the relativistic effect must be considered. This effect gives the correction for nonrelativistic QM. Taking the relativistic effects into account, a particle including mixing potential should be described by the Klein–Fock–Gordon and Dirac equations.

In Refs. 3–37 analytical solutions of the Klein–Fock–Gordon and Dirac equations are widely studied.

Many methods were developed and have been used successfully in solving the Schrödinger, Dirac and Klein–Fock–Gordon (KFG) wave equations in the presence of some well-known potentials. In Refs. 18–36 some authors have assumed that the scalar potential is equal to the vector potential and using NU[38] method obtained bound states of the KFG and Dirac equation with some typical potential fields.

The noncentral potentials are needed to obtain better results than central potentials about the dynamical properties of the molecular structures and interactions. Some authors added ring-shaped potentials to certain potentials, for example Coulomb, Hulthén and Manning–Rosen potentials to obtain noncentral potentials.

Many works show the power and simplicity of NU method in solving central and noncentral potentials, for example Refs. 37, 39–41. This method is based on solving the second-order linear differential equation by reducing to a generalized equation of hypergeometric type which is a second-order homogeneous differential equation with polynomial coefficients of degree not exceeding the corresponding order of differentiation.

It would be interesting and important to study the relativistic bound states of the arbitrary l-wave KFG equation with Manning–Rosen potential plus a ring-shaped like potential, since it has been extensively used to describe the bound and continuum states of the interacting systems. The central Manning–Rosen[42,43] potential is defined by

$$V(r,\theta) = \frac{1}{kb^2}\left[\frac{\alpha(\alpha-1)\exp(-2r/b)}{(1-\exp(-r/b))^2} - \frac{A\exp(-r/b)}{(1-\exp(-r/b))}\right], \quad k = 2M/\hbar^2\,, \quad (1.1)$$

where A and α are dimensionless parameters, but the screening parameter b, determines the potential range, has dimension of length.

This potential is used as a mathematical model in the description of diatomic molecular vibrations and it constitutes a convenient model for other physical situations. It is known that for this potential the KFG equation can be solved exactly using suitable approximation scheme to deal with the centrifugal term.[44]

The potential which we used in this work

$$V(r,\theta)=\frac{1}{k}\left[\frac{\alpha(\alpha-1)\exp(-2r/b)}{b^2(1-\exp(-r/b))^2}-\frac{A\exp(-r/b)}{b^2(1-\exp(-r/b))}+\frac{\beta'}{r^2\sin^2\theta}+\frac{\beta\cos\theta}{r^2\sin^2\theta}\right], \tag{1.2}$$

is obtained by adding a ring-shaped like potential term.

Ring-shaped like potentials is usually used in quantum chemistry for describing the ring shaped organic molecules such as benzene and in nuclear physics for investigation the interaction between deformed pair of nucleus and spin–orbit coupling for the motion of the particle in the potential fields.

From the point of view of theoretical and experimental physics, Manning–Rosen plus a ring-shaped like potential is more informative relative to Manning–Rosen potential.

By taking into account these point the solution of the KFG equation for Manning–Rosen plus ring-shaped like potentials present a great interest in both theoretical and experimental studies.

Here we present the analytical solutions of the KFG equation with equal scalar and vector Manning–Rosen plus a ring-shaped potential.

The remainder of this paper is organized as follows. In Sec. 2, we provide KFG equation within Manning–Rosen plus a ring-shaped like potential. In Sec. 3, we present full details of bound state solution of the radial KFG equation by NU method. In Sec. 4, we present the solution of angle-dependent part of the KFG. Finally, we summarize our results and present our conclusions in Sec. 5.

2. The Klein–Fock–Gordon Equation with the Manning–Rosen Potential Plus a Ring-Shaped Like Potential

Since KFG equation contains two objects; the four-vector linear momentum operator and the scalar rest mass, one can introduce two different potentials in this equation. The first is a vector potential (V), introduced via minimal coupling and the second is a scalar potential (S) introduced via scalar coupling.[1] They allow us to introduce two types of potential coupling which are the four vector potential (V) and the space–time scalar potential (S).

The KFG equation with scalar potential $S(r,\theta)$ and vector potential $V(r,\theta)$ can be written in the following form in natural units ($\hbar=c=1$)

$$\left[-\nabla^2+(M+S(r,\theta))^2\right]\psi(r,\theta,\phi)=[E-V(r,\theta)]^2\psi(r,\theta,\phi), \tag{2.1}$$

where E is the relativistic energy of the system and M denotes the rest mass of a scalar particle.

Here, we consider the case when the scalar potential and vector potential are equal to the Manning–Rosen plus a ring-shaped potential as done in Ref. 45. By taking the wave function of the form

$$\psi(r,\theta,\phi)=\frac{\chi(r)}{r}\Theta(\theta)e^{im\phi}, \quad m=0,\pm1,\pm2,\pm3,\ldots \tag{2.2}$$

and substituting this into Eq. (2.1) leads to the following second-order differential equations

$$\chi''(r)+\left[(E^2-M^2)-\frac{M+E}{Mb^2}\left(\frac{\alpha(\alpha-1)e^{-2r/b}}{(1-e^{-r/b})^2}-\frac{Ae^{-r/b}}{1-e^{-r/b}}\right)-\frac{\lambda}{r^2}\right]\chi(r)=0\,, \quad (2.3)$$

$$\Theta''(\theta)+\cot\theta\Theta'(\theta)+\left[\frac{-1}{\sin^2\theta}\left(\frac{M+E}{M}(\beta'+\beta\cos\theta)+m^2\right)+\lambda\right]\Theta(\theta)=0\,. \quad (2.4)$$

3. Bound State Solution of the Radial Klein–Fock–Gordon Equation

When $l \neq 0$, the differential equation in Eq. (2.3) cannot be solved analytically due to the centrifugal term. Therefore, we must use a proper approximation for the centrifugal term in which similar approach was also employed previously.[41,46,47] In this work, we attempt to use the following improved approximation scheme to deal with the centrifugal term

$$\frac{1}{r^2}\approx\frac{1}{b^2}\left[C_0+\frac{e^{-r/b}}{(1-e^{-r/b})^2}\right], \quad (3.1)$$

which reduces to convectional approximation scheme suggested by Greene and Aldrich when $C_0 = 0$.[46] For bound states $|E| < M$, inserting this new centrifugal term into Eq. (2.3) allows us to obtain

$$\chi''(r)+\left[E^2-M^2-\frac{M+E}{Mb^2}\left(\frac{\alpha(\alpha-1)e^{-2r/b}}{(1-e^{-r/b})^2}-\frac{Ae^{-r/b}}{1-e^{r/b}}\right)\right.$$
$$\left.-\frac{\lambda}{b^2}\left[C_o+\frac{e^{-r/b}}{(1-e^{-r/b})^2}\right]\right]\chi(r)=0\,. \quad (3.2)$$

Equation (3.2) can be further written in the form

$$\chi''(s)+\frac{\tilde{\tau}}{\sigma}\chi'(s)+\frac{\tilde{\sigma}}{\sigma^2}\chi(s)=0\,, \quad (3.3)$$

which is known equation of the generalized hypergeometric-type by using the transformation $s = e^{-r/b}$. Hence we obtain

$$\chi''(s)+\chi'(s)\frac{1-s}{s(1-s)}+\left[\frac{1}{s(1-s)}\right]^2$$
$$\times\left[-\epsilon^2(1-s)^2+A\eta s(1-s)-\alpha\eta(\alpha-1)s^2\right.$$
$$\left.-(1-s)^2\lambda\left(C_0+\frac{s}{(1-s)^2}\right)\right]\chi(s)=0\,, \quad (3.4)$$

where we use the following notation for bound states

$$\epsilon=b\sqrt{M^2-E^2}\,,\quad \eta=\frac{M+E}{M}\,. \quad (3.5)$$

Now, we can successfully apply NU method of definition for eigenvalues of energy. By comparing Eqs. (3.4) with (3.3) we can define the following

$$\begin{aligned} \tilde{\tau}(s) &= 1 - s\,, \quad \sigma(s) = s(1-s)\,, \\ \tilde{\sigma}(s) &= s^2[-\epsilon^2 - A\eta - \alpha\eta(\alpha-1) - \lambda C_0] \\ &\quad + s[2\epsilon^2 + A\eta + 2\lambda C_0 - \lambda] + [-\epsilon^2 - \lambda C_0]\,. \end{aligned} \tag{3.6}$$

If we take the following factorization

$$\chi(s) = \phi(s) y(s)\,, \tag{3.7}$$

for the appropriate function $\phi(s)$ Eq. (3.3) takes the form of the well-known hypergeometric-type equation,

$$\sigma(s) y''(s) + \tau(s) y'(s) + \bar{\lambda} y(s) = 0\,. \tag{3.8}$$

The appropriate $\phi(s)$ function must satisfy the following condition

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}\,, \tag{3.9}$$

where $\pi(s)$, the polynomial of degree at most one, is defined as

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}\,. \tag{3.10}$$

Finally the equation, where $y(s)$ is one of its solutions, takes the form known as hypergeometric-type if the polynomial $\bar{\sigma}(s) = \tilde{\sigma}(s) + \pi^2(s) + \pi(s)[\tilde{\tau}(s) - \sigma'(s)] + \pi'(s)\sigma(s)$, is divisible by $\sigma(s)$, i.e. $\bar{\sigma} = \bar{\lambda}\sigma(s)$.

The constant $\bar{\lambda}$ and polynomial $\tau(s)$ in Eq. (3.8) is defined as

$$\bar{\lambda} = k + \pi' \tag{3.11}$$

and

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s)\,, \tag{3.12}$$

respectively. For our problem, the $\pi(s)$ function is written as

$$\pi(s) = \frac{-s}{2} \pm \sqrt{s^2[a-k] - s[b-k] + c}\,, \tag{3.13}$$

where the values of the parameters are

$$\begin{aligned} a &= \frac{1}{4} + \epsilon^2 + A\eta + \alpha\eta(\alpha-1) + \lambda C_0\,, \\ b &= 2\epsilon^2 + A\eta + 2\lambda C_0 - \lambda\,, \\ c &= \epsilon^2 + \lambda C_0\,. \end{aligned}$$

The constant parameter k can be found complying with the condition that the discriminant of the expression under the square root is equal to zero. Hence we obtain

$$k_{1,2} = (b - 2c) \pm 2\sqrt{c^2 + c(a - b)}\,. \tag{3.14}$$

When the individual values of k given in Eq. (3.14) are substituted into Eq. (3.13), the four possible forms of $\pi(s)$ are written as follows

$$\pi(s) = \frac{-s}{2} \pm \begin{cases} \left(\sqrt{c} - \sqrt{c + a - b}\right)s - \sqrt{c} & \text{for } k = (b - 2c) + 2\sqrt{c^2 + c(a - b)}\,, \\ \left(\sqrt{c} + \sqrt{c + a - b}\right)s - \sqrt{c} & \text{for } k = (b - 2c) - 2\sqrt{c^2 + c(a - b)}\,. \end{cases} \tag{3.15}$$

According to NU method, from the four possible forms of the polynomial $\pi(s)$, we select the one for which the function $\tau(s)$ has the negative derivative. Other forms are not suitable physically. Therefore, the appropriate functions $\pi(s)$ and $\tau(s)$ are

$$\pi(s) = \sqrt{c} - s\left[\frac{1}{2} + \sqrt{c} + \sqrt{c + a - b}\right], \tag{3.16}$$

$$\tau(s) = 1 + 2\sqrt{c} - 2s\left[1 + \sqrt{c + a - b}\right], \tag{3.17}$$

for

$$k = (b - 2c) - 2\sqrt{c^2 + c(a - b)}\,. \tag{3.18}$$

Also by Eq. (3.11) we can define the constant $\bar{\lambda}$ as

$$\bar{\lambda} = b - 2c - 2\sqrt{c^2 + c(a - b)} - \left[\frac{1}{2} + \sqrt{c} + \sqrt{c + a - b}\right]. \tag{3.19}$$

Given a nonnegative integer n, the hypergeometric-type equation has a unique polynomials solution of degree n if and only if

$$\bar{\lambda} = \bar{\lambda}_n = -n\tau' - \frac{n(n-1)}{2}\sigma'' \quad (n = 0, 1, 2, \dots)\,, \tag{3.20}$$

and $\bar{\lambda}_m \neq \bar{\lambda}_n$ for $m = 0, 1, 2, \dots, n - 1$,[48] then it follows that,

$$\begin{aligned} \bar{\lambda}_{n_r} &= b - 2c - 2\sqrt{c^2 + c(a - b)} - \left[\frac{1}{2} + \sqrt{c} + \sqrt{c + a - b}\right] \\ &= 2n_r\left[1 + \left(\sqrt{c} + \sqrt{c + a - b}\right)\right] + n_r(n_r - 1)\,. \end{aligned} \tag{3.21}$$

We can solve Eq. (3.21) explicitly for c by using the relation $c = \epsilon^2 + \lambda C_0$ which brings

$$\epsilon^2 = \left[\frac{\lambda + 1/2 + \Lambda(1 + 2n_r) + n_r(n_r + 1) - A\eta}{2\Lambda + 1 + 2n_r}\right]^2 - \lambda C_0\,, \tag{3.22}$$

where $\Lambda = \sqrt{1/4 + \eta\alpha(\alpha - 1) + \lambda}$. After inserting ϵ^2 into Eq. (3.5) with $\lambda = l(l+1)$ for energy levels we find

$$M^2 - E^2_{n_r,l} = \frac{1}{b^2}\left[\left[n_r + \frac{1}{2} + \frac{(l-n_r)(l+n_r+1) - A\eta}{2\Lambda + 1 + 2n_r}\right]^2 - l(l+1)C_0\right]. \tag{3.23}$$

The energy levels $E_{n_r,l}$ are determined by the energy equation (3.23), which is rather complicated transcendental equation.

Now, using NU method we can obtain the radial eigenfunctions. After substituting $\pi(s)$ and $\sigma(s)$ into Eq. (3.9) and solving first-order differential equation, it is easy to obtain

$$\phi(s) = s^{\sqrt{c}}(1-s)^K\,, \tag{3.24}$$

where $K = 1/2 + \Lambda$.

Furthermore, the other part of the wave function $y_n(s)$ is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \tag{3.25}$$

where B_n is a normalizing constant and $\rho(s)$ is the weight function which is the solution of the Pearson differential equation. The Pearson differential equation and $\rho(s)$ in our case have the form,

$$(\sigma\rho)' = \tau\rho\,, \tag{3.26}$$

$$\rho(s) = (1-s)^{2K-1}s^{2\sqrt{c}}\,, \tag{3.27}$$

respectively.

Substitute Eq. (3.27) into Eq. (3.25) then we get

$$y_{n_r}(s) = B_{n_r}(1-s)^{1-2K}s^{2\sqrt{c}}\frac{d^{n_r}}{ds^{n_r}}\left[s^{2\sqrt{c}+n_r}(1-s)^{2K-1+n_r}\right]. \tag{3.28}$$

Then by using the following definition of the Jacobi polynomials[49]

$$P_n^{(a,b)}(s) = \frac{(-1)^n}{n!2^n(1-s)^a(1+s)^b}\frac{d^n}{ds^n}\left[(1-s)^{a+n}(1+s)^{b+n}\right], \tag{3.29}$$

we can write

$$P_n^{(a,b)}(1-2s) = \frac{C_n}{s^a(1-s)^b}\frac{d^n}{ds^n}\left[s^{a+n}(1-s)^{b+n}\right] \tag{3.30}$$

and

$$\frac{d^n}{ds^n}\left[s^{a+n}(1-s)^{b+n}\right] = C_n s^a(1-s)^b P_n^{(a,b)}(1-2s)\,. \tag{3.31}$$

If we use the last equality in Eq. (3.28), we can write

$$y_{n_r}(s) = C_{n_r}P_{n_r}^{(2\sqrt{c},2K-1)}(1-2s)\,. \tag{3.32}$$

Substituting $\phi(s)$ and $y_{n_r}(s)$ into Eq. (3.7), we obtain

$$\chi_{n_r}(s) = C_{n_r} s^{\sqrt{c}} (1-s)^K P_{n_r}^{(2\sqrt{c},2K-1)}(1-2s)\,. \tag{3.33}$$

Using the following definition of the Jacobi polynomials[49]

$$P_n^{(a,b)}(s) = \frac{\Gamma(n+a+1)}{n!\Gamma(a+1)} \underset{21}{F}\left(-n, a+b+n+1, 1+a; \frac{1-s}{2}\right), \tag{3.34}$$

we are able to write Eq. (3.33) in terms of hypergeometric polynomials as

$$\chi_{n_r}(s) = C_{n_r} s^{\sqrt{c}} (1-s)^K \frac{\Gamma(n_r+2\sqrt{c}+1)}{n_r!\Gamma(2\sqrt{c}+1)} \underset{21}{F}\left(-n_r, 2\sqrt{c}+2K+n_r, 1+2\sqrt{c}; s\right). \tag{3.35}$$

The normalization constant C_{n_r} can be found from normalization condition

$$\int_0^\infty |R(r)|^2 r^2\, dr = \int_0^\infty |\chi(r)|^2 dr = b\int_0^1 \frac{1}{s}|\chi(s)|^2 ds = 1\,, \tag{3.36}$$

by using the following integral formula[50]

$$\int_0^1 (1-z)^{2(\delta+1)} z^{2\lambda-1}\left\{\underset{21}{F}(-n_r, 2(\delta+\lambda+1)+n_r, 2\lambda+1; z)\right\}^2 dz$$
$$= \frac{(n_r+\delta+1)n_r!\Gamma(n_r+2\delta+2)\Gamma(2\lambda)\Gamma(2\lambda+1)}{(n_r+\delta+\lambda+1)\Gamma(n_r+2\lambda+1)\Gamma(2(\delta+\lambda+1)+n_r)} \tag{3.37}$$

for $\delta > \frac{-3}{2}$ and $\lambda > 0$. After simple calculations, we obtain normalization constant as

$$C_{n_r} = \sqrt{\frac{n_r!2\sqrt{c}(n_r+K+\sqrt{c})\Gamma(2(K+\sqrt{c})+n_r)}{b(n_r+K)\Gamma(n_r+2\sqrt{c}+1)\Gamma(n_r+2K)}}\,. \tag{3.38}$$

4. Solution of Azimuthal Angle-Dependent Part of the Klein–Fock–Gordon Equation

We may also derive the eigenvalues and eigenvectors of the azimuthal angle dependent part of the KFG equation in Eq. (2.4) by using NU method. Introducing a new variable $x = \cos\theta$, Eq. (2.4) is brought to the form

$$\Theta''(x) - \frac{2x}{1-x^2}\Theta'(x) + \frac{1}{(1-x^2)^2}\left[\lambda(1-x^2) - m^2 - \eta(\beta' + \beta x)\right]\Theta(x) = 0\,. \tag{4.1}$$

After the comparison of Eq. (4.1) with Eq. (3.3) we have

$$\tilde{\tau}(x) = -2x\,, \quad \sigma(x) = 1-x^2\,, \quad \tilde{\sigma}(x) = -\lambda x^2 - \eta\beta x + (\lambda - m^2 - \eta\beta')\,. \tag{4.2}$$

In the NU method the new function $\pi(x)$ is calculated for angle-dependent part as

$$\pi(x) = \pm\sqrt{x^2(\lambda-k) + \eta\beta x - (\lambda - \eta\beta' - m^2 - k)}\,. \tag{4.3}$$

The constant parameter k can be determined as

$$k_{1,2} = \frac{2\lambda - m^2 - \eta\beta'}{2} \pm \frac{u}{2}\,, \tag{4.4}$$

where $u = \sqrt{(m^2 + \eta\beta')^2 - \eta^2\beta^2}$. The appropriate function $\pi(x)$ and parameter k are

$$\pi(x) = -\left[x\sqrt{\frac{m^2 + \eta\beta' + u}{2}} + \sqrt{\frac{m^2 + \eta\beta' - u}{2}}\right], \tag{4.5}$$

$$k = \frac{2\lambda - m^2 - \eta\beta'}{2} - \frac{u}{2}\,. \tag{4.6}$$

The following track in this selection is to achieve the condition $\tau' < 0$. Therefore $\tau(x)$ becomes

$$\tau(x) = -2x\left[1 + \sqrt{\frac{m^2 + \eta\beta' + u}{2}}\right] - 2\sqrt{\frac{m^2 + \eta\beta' - u}{2}}\,. \tag{4.7}$$

We can also write the values $\bar{\lambda} = k + \pi'(s)$ as

$$\bar{\lambda} = \frac{2\lambda - \eta\beta' - m^2}{2} - \frac{u}{2} - \sqrt{\frac{m^2 + \eta\beta' + u}{2}}\,, \tag{4.8}$$

also using Eq. (3.20), then from Eq. (4.8) we can obtain

$$\begin{aligned} \bar{\lambda}_N &= \frac{2\lambda - \eta\beta' - m^2}{2} - \frac{u}{2} - \sqrt{\frac{m^2 + \eta\beta' + u}{2}} \\ &= 2N\left[1 + \sqrt{\frac{m^2 + \eta\beta' + u}{2}}\right] + N(N-1)\,. \end{aligned} \tag{4.9}$$

In order to obtain unknown λ we can solve Eq. (4.9) explicitly for $\lambda = l(l+1)$

$$\lambda - \zeta^2 - \zeta = 2N(1+\zeta) + N(N-1)\,, \tag{4.10}$$

where $\zeta = \sqrt{\frac{m^2+\eta\beta'+u}{2}}$, and

$$\lambda = \zeta^2 + \zeta + 2N\zeta + N(N+1) = (N+\zeta)(N+\zeta+1) = l(l+1)\,, \tag{4.11}$$

then

$$l = N + \zeta\,. \tag{4.12}$$

Substitution of this result in Eq. (3.23) yields the desired energy spectrum, in terms of n_r and N quantum numbers. Similarly, the wave function of azimuthal angle dependent part of KFG equation can be formally derived by a process to the derivation of radial part of KFG equation. Thus using Eq. (3.9), we obtain

$$\phi(x) = (1-x)^{(B+C)/2}\,, \tag{4.13}$$

where $B = \sqrt{\frac{m^2+\eta\beta'+u}{2}}$, $C = \sqrt{\frac{m^2+\eta\beta'-u}{2}}$.

On the other hand, to find a solution for $y_N(s)$ we should first obtain the weight function $\rho(s)$. From Pearson equation, we find weight function as

$$\rho(x) = (1-x)^{B+C}(1+x)^{B-C}\,. \tag{4.14}$$

Substituting $\rho(s)$ into Eq. (3.25) allows us to obtain the polynomial $y_N(s)$ as follows

$$y_N(x) = B_N(1-x)^{-(B+C)}(1+x)^{C-B}\frac{d^N}{dx^N}\left[(1-x)^{B+C+N}(1+x)^{B-C+N}\right]. \tag{4.15}$$

From the definition of Jacobi polynomials, we can write

$$\frac{d^N}{dx^N}\left[(1-x)^{B+C+N}(1+x)^{B-C+N}\right] = (-1)^N 2^N (1-x)^{B+C}(1+x)^{B-C}P_N^{(B+C,B-C)}(x)\,. \tag{4.16}$$

Substitution of Eq. (4.16) into Eq. (4.15) and after long but straightforward calculations we obtain the following result,

$$\Theta_N(x) = C_N(1-x)^{(B+C)/2}(1+x)^{(B-C)/2}P_N^{(B+C,B-C)}(x)\,, \tag{4.17}$$

where C_N is the normalization constant. Using orthogonality relation of the Jacobi polynomials[49] the normalization constant can be found as

$$C_N = \sqrt{\frac{(2N+2B+1)\Gamma(N+1)\Gamma(N+2B+1)}{2^{2B+1}\Gamma(N+B+C+1)\Gamma(N+B-C+1)}}\,. \tag{4.18}$$

5. Conclusion

In this work we have applied NU method to the calculation of the nonzero angular momentum solutions for the KFG equation of the Manning–Rosen plus a ring-shaped like potential. For any state energy eigenvalues can be obtained from Eq. (3.23), which is rather complicated transcendental equation. In case $\beta = \beta' = 0$, one can obtain central potential solutions and in case $\alpha = 1$ or $\alpha = 0$ gives solutions of Hulthén potential. We also obtain normalized eigenfunctions in terms of orthogonal Jacobi polynomials.

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