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# Analytical Solutions to Temperature Field in Various Relative-Scale Media Subjected to a Reciprocating Motion Point Heat Source

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**Abstract:** To reveal the temperature rise evolution mechanism of isotropic media subjected to reciprocating motion constant-strength point heat source, various forms of analytical solutions are derived on the basis of differentiated relative scales, and non-dimensionalized parameters are designed to characterize the thermal distribution regularities by utilizing numerical calculations. Temperature rise curves of media subjected to a reciprocating motion point heat source allow similar quasi-steady-state characteristics to appear, which finally achieve a stable state, so that the values of surplus temperature oscillate around the constant time-average quantity. The time to reach quasi-steady state, the time-averaged quantity and the fluctuation amplitude of surplus temperature are comprehensively impacted by the dimensionless distance parameter $\gamma$, the convective heat transfer parameter $\omega$ and the velocity and travel parameter $\beta$. This work discusses influence rules of temperature evolution in various relative-scale media and further enriches the moving heat source theory.

**Keywords:** reciprocating motion point heat source; analytical solutions; dimensionless parameters; temperature evolution influence rules

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## 1. Introduction

Studies on heat transfer in media subjected to moving heat sources have been motivated by a broad range of applications. The research directions mainly focus on the heat transfer in various relative-scale media subjected to heat source with different shapes and moving paths. Heat transfer characteristics may appear differentiated among media with the irregular shapes and discrete heating area [1]. Generally, the studies can be classified by relative scales including the infinite [2], the semi-infinite [3] and the finite [4] media. Zhao et al. [5] presented a mapped transient infinite element in the global coordinate system, of which heat transfer functions were derived to solve thermal problems in the infinite media. Sheng et al. [6] revealed the temperature evolution mechanism of infinite/finite-length cylindrical solids and helped successfully predict the temperature rise in the ball screw for engineering applications. Winczek et al. [7] proposed an analytical semi-infinite body model for temperature field description of multi-pass arc weld surfacing. Discussed in a rectangular coordinate system, when the sizes are negligible compared with these of the heat transfer media in all three directions, the heat source can be regarded as a point heat source [8]. Division of the line and the surface heat source [9] obeys the similar rule. In addition, studies defined the spherical [10], the circular [11] and the cylindrical [12] heat sources in their respective coordinate systems. Wang et al. [13] established the common point and linear heat source model to simulate the temperature fields of electron beam welding and predict the weld shapes. Azar et al. [14] utilized analytical and numerical approaches to model a gas metal arc welding process with the discretely distributed point heat source. According to the various heat generation laws, the temperature solutions to a moving point heat source can also be divided into periodic-type [15], gamma-type [16] variable power and so on. The temperature field distribution in media subjected to straight [17], alterable-direction [18] and specific included angle motion heat sources have also been investigated; however, few studies considering the reciprocating characteristics of heat source can be seen in our visible references.

For the specific media subjected to an excitation heat source with constant unidirectional velocity, the transient heat conduction equation is commonly utilized to derive the corresponding temperature-response function, of which the simplified solution has been demonstrated effective by transforming into a moving coordinate whose origin moving at the coincident velocity with the heat source [16]. Green's function method is one of the most significant tools for applying field theory in different branches of physic systems [19]. In the light of heat sources with the reciprocating characteristics, Green's functions are no doubt able to construct a suitable mathematical description for the temperature field distribution.

Green's functions can be utilized to deal with all kinds of unsteady-state heat conduction equations with heat source, non-homogeneous boundary and specific initial conditions. Green's function method is also known as pulse decomposition or point source superposition method, which decomposes the physical quantities into pulses according to the time and space instead of the spectrum. The decomposed active factors including sources, boundary and initial disturbances are uniformly regarded as point sources. Sources generate fields and the heat conduction equations formed as partial differential are used to describe the distribution and variation of temperature fields.

The early investigations mainly focused on heat generation laws, shapes of heat sources, boundary conditions, etc., barely touching on the reciprocating motion of heat sources. In this work, periodically reciprocating characteristics of point heat source has been taken into prior consideration. Based on Green's function method, analytical solutions in the form of generalized incomplete Gamma functions to temperature field of various relative-scale media are derived. Non-dimensionalized parameters are put forward to characterize the temperature field distribution. Influences of the variables $\xi$, $\gamma$, $\kappa$, $\omega$ and the reduced parameter $\beta$ on the temperature field are discussed.

## 2. Problem Description and Mathematical Modelling

### 2.1. Path Simplification

Inevitable acceleration and deceleration exists in the actual reciprocating motion process of a heat source. The accompanying quadratic function of time presents great difficulties in theoretical derivation [20]. The Laplace transform is one of the most effective means to eliminate the partial derivatives of time variables in differential equations [21]. Nevertheless, neglecting the stages of variable speed motion is another more simple and convenient method, and displacement curves of the practical and the simplified motion are exhibited in Figure 1. Relationship between practical and simplified displacements can be deducted as

$$\frac{S_1 - S_2}{S_2} = \frac{1}{1 + \tau_u / \tau_v} \tag{1}$$

The two displacement curves trends to be close enough if the motion time $\tau_v$ of variable speed is short enough compared with that $\tau_u$ of uniform speed, so that the motion of point heat source is assumed as periodically reciprocating motion at the uniform velocity of $U$ from the outset $O$. The period of motion is $T$. Then, the displacement function $S(\tau)$ can be expressed as

$$S(\tau) = \begin{cases} U(\tau - KT), & KT \leq \tau \leq (K + 1/2)T \\ U[(K+1)T - \tau], & (K + 1/2)T < \tau < (K+1)T \end{cases} \tag{2}$$

where $K$ is the integer of the ratio of time to period, $K = [\tau/T]$. Representation method of time-displacement in the following article obeys the calculation rule of $S(\tau)$.

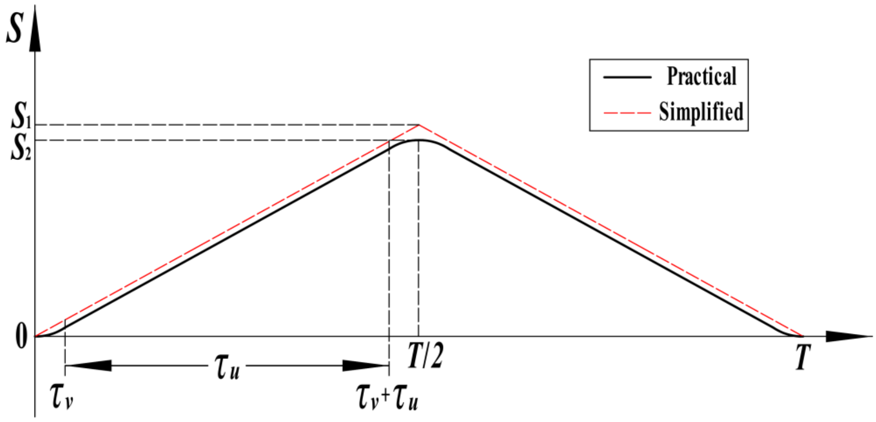


**Figure 1.** Displacement curves of practical and simplified motion.

*2.2. Mathematical Modelling*

The reciprocating motion means that the heat source starts to make straight movement from some point in the medium and return to the outset after a full motion cycle. Then, the heat source repeats this kind of reciprocating process. As the heat transfer medium is infinite, we assume the outset of heat source as origin point and the move path direction as $x$ axis. Then, we construct the corresponding rectangular coordinate system to study the heat transfer problem. The schematic diagram of the physical model is exhibited as Figure 2.

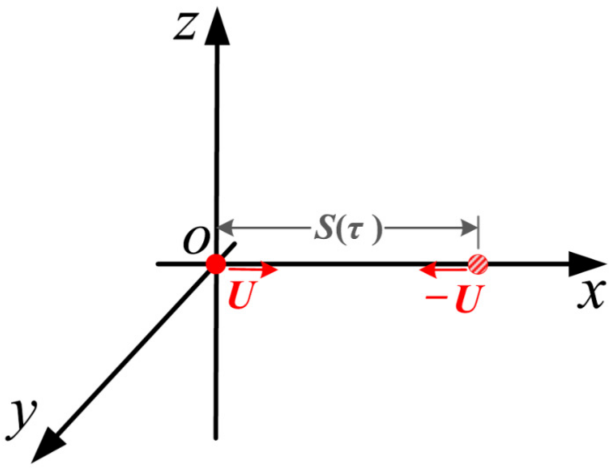


**Figure 2.** Schematic diagram of moving heat source in an infinite medium.

Heat conduction happens in a homogeneous and isotropic medium of the infinite extension in all directions, of which the initial temperature is $t_\infty$. The point heat source starts periodically reciprocating motion along the $x$ axis in a rectangular coordinate region ($x$, $y$, $z$). Introducing the surplus temperature $\theta = t(x, y, z, t) - t_\infty$ and heating rate operator $g_\text{p}$ of moving heat source, the general mathematical model of three-dimensional heat transfer problem subjected to a point heat source can be expressed as

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} + \frac{1}{k} g_\text{p}(t)\delta(x - x') = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}, \quad -\infty < x, y, z < +\infty \tag{3}$$

with initial condition

$$\theta = I(x, y, z, t), \quad t = 0, \quad -\infty < x, y, z < +\infty \tag{4}$$

where $\delta$ function is the density representation of point source as a centrally distributed physical quantity, $\delta(x - x')$ represents the space distribution of point heat resource in the $x$

axis [22], m, $k$ is the thermal conductivity, and W/m·K, $k = \alpha\rho c$. $\alpha$ is the thermal diffusivity, $m^2/s$.

Due to the periodical characteristics of point heat source, departure and back travels consume the same time, so that the integral limit 0~$t$ can be split by the interval $T/2$. Solutions to temperature field can be calculated as

$$\theta = \left(\int_0^{\frac{T}{2}} + \int_{\frac{T}{2}}^{T} + \int_{T}^{\frac{3T}{2}} + \cdots \int_{(K-1)T}^{(K-\frac{1}{2})T} + \int_{(K-\frac{1}{2})T}^{KT} + \int_{KT}^{t}\right) G g_{\mathrm{p}}(\tau)\mathrm{d}\tau \tag{5}$$

Tallying up the reciprocating regular pattern, the integral form of surplus temperature can be generally expressed as

$$\theta = \left(\sum_{N=0}^{K-1}\int_{NT}^{(N+\frac{1}{2})T} + \sum_{N=0}^{K-1}\int_{(N+\frac{1}{2})T}^{(N+1)T} + \int_{KT}^{t}\right) G g_p(\tau)\mathrm{d}\tau \tag{6}$$

where $K$ = 1, 2, ⋯ and $NT$~$(N + 1/2)T$ represents the time period of departure while $(N + 1/2)T$~$(N + 1)T$ represents that of back tracking. $KT$~$t$ represents the rest time after deducting all the complete cycles. $G$ is Green's function of the corresponding heat transfer problem.

### 2.3. Integral Form of Surplus Temperature

Substituting $(t - \tau)$ for $t$, separated variable method has been utilized to work out the one-dimensional Green's function in $x$ direction formed as [23]

$$G(x, t|x', \tau) = \frac{\alpha}{k}\frac{1}{\sqrt{4\pi\alpha(t-\tau)}}\exp\left[-\frac{(x-x')^2}{4\alpha(t-\tau)}\right] \tag{7}$$

Green's functions in the other two directions have the similar forms so that corresponding Green's function of the multidimensional heat transfer problem in an infinite medium can be expressed as the product form of three one-dimensional problem solutions. It is worth noting that the coefficient $(\alpha/k)$ can only appear once, no matter whether the heat transfer problem is one or multidimensional [22]. So, the Green's function of above 3D heat transfer problem can be expressed as

$$G(x, y, z, t|x', y, z, \tau) = \frac{\alpha}{k}\frac{1}{[4\pi\alpha(t-\tau)]^{3/2}}\exp[-\frac{[x-S(\tau)]^2 + y^2 + z^2}{4\alpha(t-\tau)}] \tag{8}$$

In our study, the heating power is a constant unchangeable with time so that operator $g_{\mathrm{p}}(t) = Q$. For a reciprocating motion heat source, $x' = S(\tau)$ is introduced in calculation. Then, the integral form of surplus temperature can be calculated as

$$\theta = \frac{Q}{(4\pi\alpha)^{3/2}}\frac{\alpha}{k}\left(\sum_{N=0}^{K-1}\int_{NT}^{(N+\frac{1}{2})T} + \sum_{N=0}^{K-1}\int_{(N+\frac{1}{2})T}^{(N+1)T} + \int_{KT}^{t}\right)\frac{1}{(t-\tau)^{3/2}}\exp[-\frac{[x-S(\tau)]^2 + y^2 + z^2}{4\alpha(t-\tau)}]\mathrm{d}\tau \tag{9}$$

where $Q$ is the rate of heat transfer from a point source [24], W. Locating point heat source at $x = x'$, the temperature distribution produced by point heat source can be represented as $T(x) = (Q/\rho c) \times \delta(x - x')$. This formula constructs a bridge between temperature and heat conversion.

The formula can be solved in three sections, introducing

$$\theta_1^N = \frac{Q/(\rho c)}{(4\pi\alpha)^{3/2}}\int_{NT}^{(N+\frac{1}{2})T}\frac{1}{(t-\tau)^{3/2}}\exp\left[-\frac{[x-S(\tau)]^2 + y^2 + z^2}{4\alpha(t-\tau)}\right]\mathrm{d}\tau \tag{10}$$

Bringing in the displacement calculation formula of the first half cycle, the position variable $(x - x')$ over time can be noted: $x - U(\tau - NT) = x + UNT - Ut + U(t - \tau)$.

Substituting variables as $X_N = x + UNT - Ut$; $R_N{}^2 = X_N{}^2 + y^2 + z^2$ and $\varphi = R_N{}^2/[4\alpha(t-\tau)]$, then Equation (10) can be rewritten in the following form

$$\begin{aligned}\theta_1^N = \; & \frac{Q}{4k\pi^{3/2}R_N}\exp\!\left(-\frac{X_N U}{2\alpha}\right) \\ & \times \int_{R_N{}^2/[4\alpha(t-NT)]}^{R_N{}^2/\{4\alpha[t-(N+1/2)T]\}} \phi^{-\frac{1}{2}}\exp\!\left(-\phi - \frac{U^2R_N{}^2}{16\alpha^2}\phi^{-1}\right)\mathrm{d}\phi\end{aligned} \quad (11)$$

The expression can be denoted by the generalized incomplete Gamma functions [12] in the form as

$$\begin{aligned}\theta_1^N = \; & \frac{Q}{4k\pi^{3/2}R_N}\exp\!\left(-\frac{X_N U}{2\alpha}\right) \\ & \times\left[\Gamma\!\left(\frac{1}{2},\frac{R_N{}^2}{4\alpha(t-NT)};\frac{U^2R_N{}^2}{16\alpha^2}\right) - \Gamma\!\left(\frac{1}{2},\frac{R_N{}^2}{4\alpha[t-(N+1/2)T]};\frac{U^2R_N{}^2}{16\alpha^2}\right)\right]\end{aligned} \quad (12)$$

The generalized incomplete Gamma function is proved useful in the analytical study of heat conduction problems in an infinite medium [16], of which the general form can be expressed as

$$\Gamma(\alpha, t; b) = \int_t^{\infty} x^{\alpha-1}\exp(-x - bx^{-1})\mathrm{d}x \quad (13)$$

Similarly, substituting variables as $Y_N = x + Ut - U(N+1)T$; $S_N{}^2 = Y_N{}^2 + y^2 + z^2$ and $\eta = R_N{}^2/[4\alpha(t-\tau)]$. The second half cycle part can be expressed as

$$\begin{aligned}\theta_2^N = \; & \frac{Q}{4k\pi^{3/2}S_N}\exp\!\left(-\frac{Y_N U}{2\alpha}\right) \\ & \times\left[\Gamma\!\left(\frac{1}{2},\frac{S_N{}^2}{4\alpha[t-(N+1/2)T]};\frac{U^2S_N{}^2}{16\alpha^2}\right) - \Gamma\!\left(\frac{1}{2},\frac{S_N{}^2}{4\alpha[t-(N+1)T]};\frac{U^2S_N{}^2}{16\alpha^2}\right)\right]\end{aligned} \quad (14)$$

In particular, the role of rest time should be discussed separately according to whether the half cycle is full or not.

If $t - KT \le T/2$,

$$\theta_r = \frac{Q}{4k\pi^{3/2}R_K}\exp\!\left(-\frac{X_K U}{2\alpha}\right) \times \Gamma\!\left(\frac{1}{2},\frac{R_K{}^2}{4\alpha(t-KT)};\frac{U^2R_K{}^2}{16\alpha^2}\right) \quad (15)$$

If $t - KT > T/2$,

$$\theta_r = \theta_1^K + \frac{Q}{4k\pi^{3/2}S_K}\exp\!\left(-\frac{Y_K U}{2\alpha}\right) \times \Gamma\!\left(\frac{1}{2},\frac{S_K{}^2}{4\alpha[t-(K+1/2)T]};\frac{U^2S_K{}^2}{16\alpha^2}\right) \quad (16)$$

where $\theta_1^K$, $X_K$, $Y_K$ are obtained from $\theta_1^N$, $X_N$, $Y_N$ by changing $N$ to $K$. Then, the complete integral form of temperature field can be expressed as

$$\theta = \sum_{N=0}^{K-1}\left(\theta_1^N + \theta_2^N\right) + \theta_{\mathrm{r}} \quad (17)$$

stipulating when $K = 0$, $\theta = \theta_{\mathrm{r}}$.

## 3. Non-Dimensionalization and Parameter Discussions

### *3.1. Dimensionless Parameters Design and Non-Dimensionalization*

The essence of dimension is expansion of the quantity coefficient according to basic quantity class. The different selection of basic quantity class, or even of families the unit system belongs to, may cause the expression form of dimensional equations to be changed. Non-dimensionalization means all the dimensions are constructed into one through appropriate variable substitutions. Thus, some or all units involved in the physical quantity equations are removed to ensure the form of the equations remains independent from unit system families. For convenient analysis, dimensionless parameters are designed by introducing the variables group listed as

$\Theta = \frac{8\alpha k\theta\ \pi^{3/2}}{QU}$: dimensionless surplus temperature;
$\xi = \frac{Ux}{2\alpha}$; $\mu = \frac{Uy}{2\alpha}$; $\varsigma = \frac{Uz}{2\alpha}$: dimensionless coordinate;
$\gamma^2 = \mu^2 + \varsigma^2$: dimensionless distance away from the $x$ axis;
$\kappa = \frac{t}{T}$: dimensionless time;
$\beta = \frac{U^2T}{2\alpha}$: dimensionless velocity and travel parameter.

Then, the non-dimensionalized forms of each parts can be expressed as

$$\Theta_1^N = \frac{e^{-[\xi+\beta(N-\kappa)]}}{\sqrt{[\xi+\beta(N-\kappa)]^2+\gamma^2}} \times \begin{bmatrix} \Gamma\left(\frac{1}{2}, \frac{[\xi+\beta(N-\kappa)]^2+\gamma^2}{2\beta(\kappa-N)}; \frac{[\xi+\beta(N-\kappa)]^2+\gamma^2}{4}\right) \\ -\Gamma\left(\frac{1}{2}, \frac{[\xi+\beta(N-\kappa)]^2+\gamma^2}{2\beta(\kappa-N-1/2)}; \frac{[\xi+\beta(N-\kappa)]^2+\gamma^2}{4}\right) \end{bmatrix} \tag{18}$$

$$\begin{aligned} \Theta_2^N = & \frac{e^{\xi+\beta(\kappa-N-1)}}{\sqrt{[\xi+\beta(\kappa-N-1)]^2+\gamma^2}} \\ & \times \begin{bmatrix} \Gamma\left(\frac{1}{2}, \frac{[\xi+\beta(\kappa-N-1)]^2+\gamma^2}{2\beta(\kappa-N-1/2)}; \frac{[\xi+\beta(\kappa-N-1)]^2+\gamma^2}{4}\right) \\ -\Gamma\left(\frac{1}{2}, \frac{[\xi+\beta(\kappa-N-1)]^2+\gamma^2}{2\beta(\kappa-N-1)}; \frac{[\xi+\beta(\kappa-N-1)]^2+\gamma^2}{4}\right) \end{bmatrix} \end{aligned} \tag{19}$$

If $\kappa - K \leq 1/2$,

$$\Theta_r = \frac{e^{-[\xi+\beta(K-\kappa)]}}{\sqrt{[\xi+\beta(K-\kappa)]^2+\gamma^2}} \left[\Gamma\left(\frac{1}{2}, \frac{[\xi+\beta(K-\kappa)]^2+\gamma^2}{2\beta(\kappa-K)}; \frac{[\xi+\beta(K-\kappa)]^2+\gamma^2}{4}\right)\right] \tag{20}$$

If $\kappa - K > 1/2$,

$$\begin{aligned} \Theta_r = & \Theta_1^K + \frac{e^{\xi+\beta(\kappa-K-1)}}{\sqrt{[\xi+\beta(\kappa-K-1)]^2+\gamma^2}} \\ & \times\left[\Gamma\left(\frac{1}{2}, \frac{[\xi+\beta(\kappa-K-1)]^2+\gamma^2}{2\beta(\kappa-K-1/2)}; \frac{[\xi+\beta(\kappa-K-1)]^2+\gamma^2}{4}\right)\right] \end{aligned} \tag{21}$$

specifying when $K = 0$, $\Theta = \Theta$r.

### *3.2. Discussions on Dimensionless Parameters*

Non-dimensionalization helps appropriately reduce the number of variables in equations, reveal the mathematical essence of physical formulas, and facilitate the relative value comparisons of various items, so as to simplify the calculation and facilitate the analysis. This is guided by the generalized incomplete Gamma formula [13]

$$\Gamma\left(\frac{1}{2}, x; b\right) = \frac{\sqrt{\pi}}{2}\left[e^{-2\sqrt{b}}\mathrm{erfc}(\sqrt{x} - \sqrt{\frac{b}{x}}) + e^{2\sqrt{b}}\mathrm{erfc}(\sqrt{x} + \sqrt{\frac{b}{x}})\right] \tag{22}$$

where erfc is complementary error function. Numerical results with various values of dimensionless parameters are graphically discussed in this section.

To reveal the temperature evolution regularities over time, Figure 3a shows that the curves experience a rapid ascending process at the initial period of heat source effect and then gradually flatten out. The dimensionless surplus temperature finally rises to a stable fluctuation state over time. The curves perform the characteristics of almost unchangeable time-averaged quantity but periodically fluctuated instantaneous quantity. The surplus temperature fluctuation mainly results from periodic motion of the point heat source. Dimensionless parameter $\gamma$ represents the distance away from $x$ axis. Meanwhile, the smaller the $\gamma$ values, the higher the time-averaged quantity and the more violent fluctuation the dimensionless surplus temperature develops, the less time used to reach a stable state. As moving along the $x$ axis, the point heat source impacts much more on the regions nearby (smaller $\gamma$ values) rather than those far away. The fluctuation amplitude

A is defined as the maximum offset quantity compared to the time-average value during oscillation process. Figure 3b clarifies more clearly the relationship between the fluctuation amplitude and the distance parameter. With fixed $\beta$ values, the smaller the distance away from the $x$ axis, the larger fluctuation amplitude. The fluctuation amplitudes A over dimensionless velocity and travel parameter $\beta$ with various $\gamma$ values show a similar rule, which performs a rapid increase at first and then reaches a stable state when parameter $\beta$ starts to increase from zero. However, $\beta$ only affects the fluctuation amplitude within a certain range, in which the larger $\beta$ values aggravate the oscillation of surplus temperature With smaller $\gamma$ values, fluctuation amplitudes reach a stable state more rapidly and remain unchanged as parameter $\beta$ increases. Moreover, the values of A are larger at a stable state, which indicates that the distance parameter is still the main influence factor of fluctuation amplitude compared with parameter $\beta$.

Figure 3c reveals the influence of reduced parameter $\beta$, the smaller $\beta$ values, the greater time consumption for surplus temperature to reach a stable state and the lower final time-average value it ascends to. For a reciprocating moving heat source with a fixed travel, the lower velocity means larger time intervals between twice heating effects, and untimely heat compensation for that transferred to low-temperature regions causes the difficulty in temperature rise. Additionally, the value of $\beta$ is inversely proportional to that of the thermal diffusivity. For regions in the infinite medium with inapparent temperature fluctuation, a lower $\beta$ value means a faster heat transfer of media, so that the surplus temperature increases more slowly due to the increased heat dissipation. Moreover, more time is needed to reach a steady state, and its steady-state value is higher.

Figure 3d displays the evolution process of dimensionless surplus temperature. For suitable $\beta$ and $\gamma$ values, the received heat is more than the dissipated. Because of the reciprocating motion, for each value the point heat source passes by, it will cause the accumulation of residual heat, so that the surplus temperature continuously increases over time. The $\xi$-$\Theta$ curves have a peak value and symmetry about some point on the horizontal coordinate $\xi$. As the point heat source periodically moves between (0, 0, 0) and $(UT/2, 0, 0)$ on the $x$ axis, the most affected point is located at $(UT/2)$, where the influence of the heat source continues for the longest time, and surplus temperature values are at the maximum. The point location at the converted horizontal coordinate is $\xi = \beta/4$. What needs to be particularly indicated is that the heating conditions are not symmetrical about $\xi = \beta/4$ at the early stage of the heat source motion, which is also the case for the temperature distribution. Only when the reciprocating time is enough, $\xi = \beta/4$ is flanked by almost the same heating conditions, and the $\Theta$ curve is considered symmetrical.

Figure 3e further confirms the symmetry as peak points of the fluctuation amplitude $a$ and $b$, $d$ and $e$ are symmetrical about $\xi = \beta/4$, respectively. The most violent oscillation appears in the adjacent regions outside the abscissa interval $[0, \beta/2]$ and then decreases quickly to almost zero. Due to shorter heating intervals, the fluctuation amplitude curves of surplus temperature reach a local trough of the wave at the symmetrical points, which indicates that, on the contrary, oscillation at the position of the heat source motion center is not violent. Fluctuation curves in the regions too far away from the symmetric center show more smooth features for the opposite reason. The regions of maximum surplus temperature and most violent fluctuation amplitude are abhorrent.

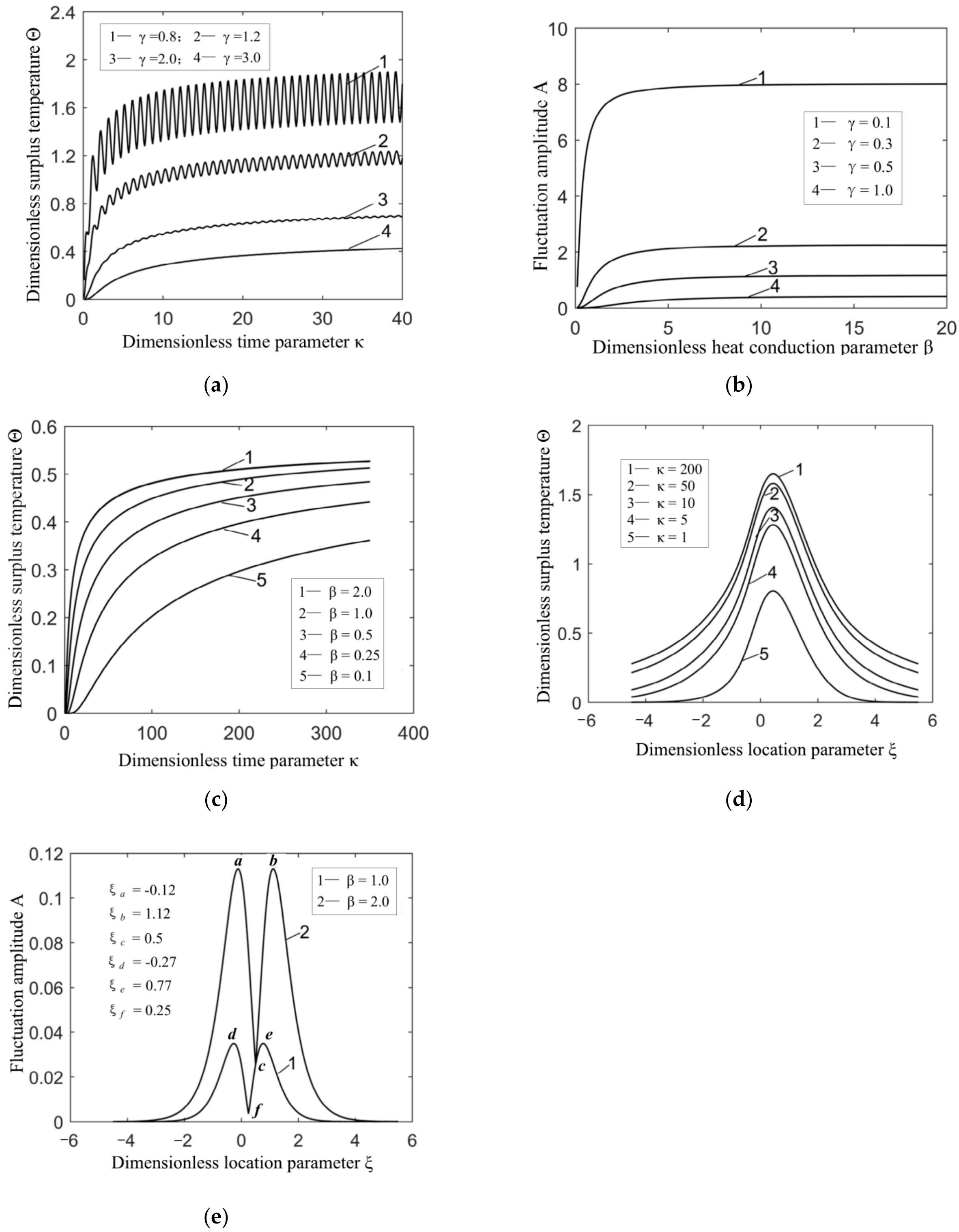


**Figure 3.** Graphical exhibition of numerical results with various dimensionless parameter values: (**a**) Dimensionless surplus temperature profiles over time parameter for various values of γ with fixed ξ = 0 and β = 2; (**b**) Fluctuation amplitude profiles over heat conduction parameter for various values of γ with fixed ξ = 0; (**c**) Dimensionless surplus temperature profiles over time parameter for various values of β with fixed ξ = 0 and γ = 2; (**d**) Dimensionless surplus temperature profiles over location parameter for various values of κ with fixed β = 2 and γ = 1; (**e**) Fluctuation amplitude profiles over location parameter for various values of β with fixed γ = 1.

## 4. Analytical Solutions of Semi-Infinite Media

Assuming the boundary surface *xoy* of a specific semi-infinite medium, when $t > 0$, the medium conducts convective heat transfer with ambiance on the boundary $z = 0$. The initial temperature is set $t_i$, while the convective heat transfer coefficient is $h$. Rate of heat transfer from a point heat source valued $Q$ starts from the original point $o$ to perform a periodically reciprocating motion along the positive direction of the $x$ axis with the velocity $U$ and the period $T$. A graphical description of the heat transfer problem is shown in Figure 4.

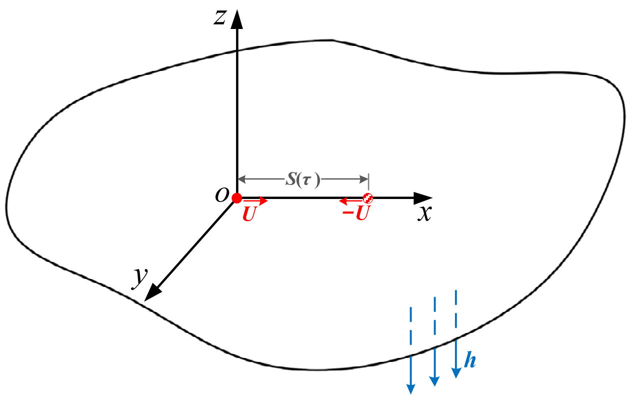


**Figure 4.** Semi-infinite medium subjected to reciprocating a point heat source.

The mathematical model of the heat conduction problem in semi-infinite media can be expressed as

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} + \frac{1}{k} g_{\mathrm{p}}(t)\delta(x - x') = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}, \quad -\infty < x, y < +\infty,\ 0 \le z < +\infty \tag{23}$$

The boundary condition is

$$-k\frac{\partial\theta}{\partial z} + h\theta = 0, \quad z = 0, \quad t > 0 \tag{24}$$

The initial condition is

$$\theta = I(z, t), \quad t = 0, \quad 0 \le z < +\infty \tag{25}$$

### *4.1. Solutions of Green's Function*

The solution process of the Green's function is similar with that of the heat transfer problem in an infinite medium. In fact, the three-dimensional unsteady-state heat transfer process, in a semi-infinite medium with the third kind of boundary condition, can been seen as the superposition of three one-dimensional problems. The Green's function finally appears in the form of the product of three single functions.

$$G(x, t|x', \tau) = \frac{\alpha}{k}\frac{1}{\sqrt{4\pi\alpha(t-\tau)}}\exp\left[-\frac{(x-x')^2}{4\alpha(t-\tau)}\right] \tag{26}$$

$$G(y, t|y', \tau) = \frac{1}{\sqrt{4\pi\alpha(t-\tau)}}\exp\left[-\frac{(y-y')^2}{4\alpha(t-\tau)}\right] \tag{27}$$

$$G(z, t|z', \tau) = \left[\begin{array}{c} \frac{1}{\sqrt{4\pi\alpha(t-\tau)}}\left\{\exp\left[-\frac{(z-z')^2}{4\alpha(t-\tau)}\right] + \exp\left[-\frac{(z+z')^2}{4\alpha(t-\tau)}\right]\right\} \\ -He^{H(z+z')+H^2\alpha(t-\tau)}\mathrm{erfc}\left(\frac{z+z'}{\sqrt{4\alpha(t-\tau)}} + H\sqrt{\alpha(t-\tau)}\right) \end{array}\right] \tag{28}$$

where $H = h/k$. So, for the r = $(x, y, z)$ in semi-infinite media, Green's function can be calculated as

$$G(\mathbf{r}, t|\mathbf{r}', \tau) = G(x, t|x', \tau) \cdot G(y, t|y', \tau) \cdot G(z, t|z', \tau) \tag{29}$$

*4.2. Integral Form of Surplus Temperature*

For a moving inner point heat source, the integral form of surplus temperature $\theta = t(\mathrm{r},\tau) - t_\infty$ should be calculated as [22,23]

$$\theta(\mathbf{r}, t) = \frac{k}{\alpha}\int_R G|_{\tau=0}F(\mathbf{r}')\,\mathrm{d}v' + \int_0^\tau G(\mathbf{r}, t|\mathbf{r}', \tau)g_\mathrm{p}(\mathbf{r}', \tau)\,\mathrm{d}\,\tau \tag{30}$$

where $F(\mathrm{r}')$ is the initial surplus temperature distribution and $F(\mathrm{r}') = t_\mathrm{i} - t_\infty$. $R$ is the whole integral domain. In particular, the ambient surplus temperature can be expressed as [22]

$$\begin{aligned}\theta_\mathrm{amb} &= \theta_0 \iiint\limits_R G(x, y, z, t|x', y', z', 0)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' \\ &= \theta_0\left[\mathrm{erf}\ \left(\tfrac{z}{4\alpha t}\right) + e^{Hz+H^2\alpha t}\mathrm{erfc}\ \left(\tfrac{z}{4\alpha t} + H\sqrt{\alpha t}\right)\right]\end{aligned} \tag{31}$$

where erf is the error function.

The calculation of the second half obeys the rule of Equation (4), and the resulting expressions are given directly to avoid repeats and save article space. So,

$$\theta_{12}^N = \frac{QHe^{Hz-\frac{X_N U}{2\alpha}}}{2k\pi}\left[\begin{array}{c}\chi\left(0, \frac{z}{\sqrt{4\alpha(t-NT)}}; \frac{X_N{}^2+y^2}{x^2}, \frac{z^2(U^2-4\alpha^2H^2)}{16\alpha^2}, \frac{Hz}{2}\right) \\ -\chi\left(0, \frac{z}{\sqrt{4\alpha[t-(N+1/2)T]}}; \frac{X_N{}^2+y^2}{x^2}, \frac{z^2(U^2-4\alpha^2H^2)}{16\alpha^2}, \frac{Hz}{2}\right)\end{array}\right] \tag{32}$$

where $\chi(\alpha, t; a, b, c)$ is the custom expression which cannot be defined by the ready-made functions. The form of $\chi$ can be expressed as

$$\chi(\alpha, t; a, b, c) = \int_t^\infty x^{\alpha-1}\exp\left(-ax^2 - bx^{-2}\right)\mathrm{erfc}(x + cx^{-1})\mathrm{d}x \tag{33}$$

Similarly,

$$\theta_{22}^N = \frac{QHe^{Hz-\frac{Y_N U}{2\alpha}}}{2k\pi}\left[\begin{array}{c}\chi\left(0, \frac{z}{\sqrt{4\alpha[t-(N+1/2)T]}}; \frac{Y_N{}^2+y^2}{x^2}, \frac{z^2(U^2-4\alpha^2H^2)}{16\alpha^2}, \frac{Hz}{2}\right) \\ -\chi\left(0, \frac{z}{\sqrt{4\alpha[t-(N+1)T]}}; \frac{Y_N{}^2+y^2}{x^2}, \frac{z^2(U^2-4\alpha^2H^2)}{16\alpha^2}, \frac{Hz}{2}\right)\end{array}\right] \tag{34}$$

If $\mathrm{t} - KT \le T/2$

$$\theta_{\mathrm{r}2} = \frac{QHe^{Hz-\frac{X_K U}{2\alpha}}}{2k\pi}\left[\chi\left(0, \frac{z}{\sqrt{4\alpha(t-KT)}}; \frac{X_K{}^2+y^2}{x^2}, \frac{z^2(U^2-4\alpha^2H^2)}{16\alpha^2}, \frac{Hz}{2}\right)\right] \tag{35}$$

If $t - KT > T/2$,

$$\theta_{\mathrm{r}2} = \theta_{12}^K + \frac{QHe^{Hz-\frac{Y_K U}{2\alpha}}}{2k\pi}\left[\chi\left(0, \frac{z}{\sqrt{4\alpha[t-(N+1/2)\ T]}}; \frac{Y_K{}^2+y^2}{x^2}, \frac{z^2(U^2-4\alpha^2H^2)}{16\alpha^2}, \frac{Hz}{2}\right)\right] \tag{36}$$

where $\theta_{12}^K$, $X_K$, $Y_K$ are obtained from $\theta_1^N$, $X_N$, $Y_N$ by changing $N$ to $K$. Then, the complete integral form of temperature field can be expressed as

$$\theta(x, y, z, t) = \theta_\mathrm{amb} + \sum_{N=0}^{K-1}\left[\left(\theta_{11}^N - \theta_{12}^N\right) - \left(\theta_{21}^N - \theta_{22}^N\right)\right] + (\theta_{\mathrm{r}1} - \theta_{\mathrm{r}2}) \tag{37}$$

where $\theta_{11}^{N}$, $\theta_{21}^{N}$ and $\theta_{r1}$ are the calculation expressions of the parts which satisfied ready-made functions in the integral solutions and they have the same form with $\theta_{1}^{N}$, $\theta_{2}^{N}$ and $\theta_{r}$ in Equation (17).

*4.3. Non-Dimensionalization and Parameter Discussion*

By introducing one more dimensionless parameter $\omega = (2\alpha H)/U$, the non-dimensionalized forms of each parts can be expressed as

$$\Theta_{amb} = \Theta_0\left[\mathrm{erf}\left(\frac{\varsigma}{\sqrt{2\kappa\beta}}\right) + e^{\left(\omega\varsigma+\frac{\kappa\beta\varsigma^2}{2}\right)}\mathrm{erfc}\left(\frac{\varsigma^2}{2\kappa\beta} + \omega\sqrt{\frac{\kappa\beta}{2}}\right)\right] \tag{38}$$

$$\begin{aligned}\Theta_{12}^{N} = &\sqrt{\pi}\omega e^{\omega\varsigma-\xi-\beta(N-\kappa)} \\ &\times\left[\begin{array}{l}\chi\left(0,\frac{\varsigma}{\sqrt{2\beta(\kappa-N)}};\frac{[\xi+\beta(\kappa-N)]^2+\mu^2}{\varsigma^2},\frac{(1-\omega^2)\varsigma^2}{4},\frac{\omega\varsigma}{2}\right) \\ -\chi\left(0,\frac{\varsigma}{\sqrt{2\beta[\kappa-(N+1/2)]}};\frac{[\xi+\beta(\kappa-N)]^2+\mu^2}{\varsigma^2},\frac{(1-\omega^2)\varsigma^2}{4},\frac{\omega\varsigma}{2}\right)\end{array}\right]\end{aligned} \tag{39}$$

$$\begin{aligned}\Theta_{22}^{N} = &\sqrt{\pi}\omega e^{\omega\varsigma-\xi-\beta(N-\kappa)} \\ &\times\left[\begin{array}{l}\chi\left(0,\frac{\varsigma}{\sqrt{2\beta[\kappa-(N+1/2)]}};\frac{[\xi+\beta[\kappa-(N+1)]]^2+\mu^2}{\varsigma^2},\frac{(1-\omega^2)\varsigma^2}{4},\frac{\omega\varsigma}{2}\right) \\ -\chi\left(0,\frac{\varsigma}{\sqrt{2\beta[\kappa-(N+1)]}};\frac{[\xi+[\kappa-(N+1)]]^2+\mu^2}{\varsigma^2},\frac{(1-\omega^2)\varsigma^2}{4},\frac{\omega\varsigma}{2}\right)\end{array}\right]\end{aligned} \tag{40}$$

If $\kappa - K \leq 1/2$,

$$\Theta_{r2} = \sqrt{\pi}\omega e^{\omega\varsigma-\xi-\beta(K-\kappa)}\left[\chi\left(0,\frac{\varsigma}{\sqrt{2\beta(\kappa-K)}};\frac{[\xi+\beta(\kappa-K)]^2+\mu^2}{\varsigma^2},\frac{(1-\omega^2)\varsigma^2}{4},\frac{\omega\varsigma}{2}\right)\right] \tag{41}$$

If $\kappa - K > 1/2$,

$$\begin{aligned}\Theta_{r2} = &\Theta_{12}^{K} + \sqrt{\pi}\omega e^{\omega\varsigma-\xi-\beta(K-\kappa)} \\ &\times\left[\chi\left(0,\frac{\varsigma}{\sqrt{2\beta[\kappa-(K+1/2)]}};\frac{[\xi+\beta[\kappa-(K+1)]]^2+\mu^2}{\varsigma^2},\frac{(1-\omega^2)\varsigma^2}{4},\frac{\omega\varsigma}{2}\right)\right]\end{aligned} \tag{42}$$

specifying when $K = 0$, $\Theta = \Theta_{amb} + (\Theta_{r1} - \Theta_{r2})$.

It is easy to see that $\beta$ characterizes the relative speed and motion range of the heat source, $\omega$ characterizes the relative intensity of boundary convective heat transfer, and $\Theta_0$ characterizes the difference between initial and ambient temperature. These three comprehensively describe the influence factors of temperature field from various angles.

Finally, two points are discussed about the temperature field.

(1) If the initial temperature is equal to the ambient temperature, which means $\Theta_{amb} = 0$, then

$$\Theta = \sum_{N=0}^{K-1}\left[\left(\Theta_{11}^{N} - \Theta_{12}^{N}\right) + \left(\Theta_{21}^{N} - \Theta_{22}^{N}\right)\right] + (\Theta_{r1} - \Theta_{r2}) \tag{43}$$

Hence, the initial temperature only has an influence on the component Θamb of Θ but has nothing to do with the other components.

(2) The situation of $\omega = 0$.

With the two conditions listed as
① $h = 0$, which means the adiabatic boundary. $\omega = 0$ leads $\Theta_{12}^{N} = \Theta_{22}^{N} = \Theta_{r2} = 0$.
② Neglecting the influence of initial temperature, $\Theta_{amb} = 0$.

Then, the expression of dimensionless surplus temperature degenerates into the form as

$$\Theta = \sum_{N=0}^{K-1} (\Theta_{11}^{N} + \Theta_{21}^{N}) + \Theta_{r1} \quad (44)$$

The equation is the surplus temperature solution in the region $0 < z < +\infty$ of an infinite medium induced by a $2Q$-strength point heat source moving along $x$ axis. Therefore, if $\Theta_{amb} = 0$, it can be inferred that the temperature field of a semi-infinite medium subjected to a periodic-motion point heat source is similar to that in an infinite medium. However, the former temperature values are lower than the latter, due to surface cooling.

## 5. Conclusions

In this study, analytical solutions to the various relative-scale media are derived, including infinite and semi-infinite ones. Dimensionless parameters are designed to reveal the temperature evolution regularities from the heating mechanism. Some conclusions are made below:

1. The temperature field evolution is the comprehensive result of dimensionless parameters $\gamma$, $\beta$, $\omega$ and $\Theta_0$. In general, the smaller the distance away from the heat source, the shorter the time intervals brought by larger velocity, the lower the relative intensity of boundary-convective heat transfer and the higher the initial temperature, the greater the benefit will be to heat accumulation and temperature rise. Among which, the most important influence factor can be $\gamma$, as the regions far away from the point heat source are little impacted.
2. The periodically reciprocating motion of point heat source results in surplus temperature appearing as a special feature, oscillating around the stable time-averaged quantity. For an infinite medium, the surplus is symmetrical about the horizontal coordinate at $\xi = \beta/4$. The maximum temperature value is obtained at the same location, but the peak value of fluctuation amplitude appears outside the abscissa interval $[0, \beta/2]$, the regions of maximum surplus temperature and most violent fluctuation amplitude are abhorrent.
3. The reduced parameter $\beta$ has a critical influence on temperature distribution, the amplitude of temperature fluctuation, the time to reach steady-state and the stable time-averaged quantity. In the regions of an infinite medium where temperature fluctuation is not obvious, surplus temperature reaches a steady state more quickly, and the stable time-averaged quantity is larger when the reduced parameter $\beta$ increases.
4. Only if some specific conditions are satisfied, such as adiabatic boundary and neglecting the influence of the initial temperature, the analytical solutions to the temperature field of the semi-infinite media degenerate into a similar form to these infinite media.

Studies on the temperature evolution mechanism help further enrich the moving heat source theory and have great potential to provide theoretical guidance for engineering practice, such as in the fields of welding, heat exchangers and so on.

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## Glossary

| | |
|---|---|
| **Nomenclature** | |
| $A$ | dimensionless fluctuation amplitude, (-) |
| $a, b, c$ | general expression of variables, (-) |
| $c$ | specific heat capacity of mass, (J/kg·K) |
| $g$ | heating rate operator, (-) |
| $H$ | prescribed form, $H = h/k$, (-) |
| $h$ | convective heat transfer coefficient, (W/m$^2$·°C) |
| $I(*)$ | initial condition, (-) |
| $K$ | integer of the ratio of time to period, (-) |
| $k$ | thermal conductivity coefficient, (W/m·K) |
| $N$ | sequence number of complete cycles, (-) |
| $O$ | origin coordinate, (-) |
| $Q$ | Heat power, (W) |
| $\mathbf{r}$ | position vector, (-) |
| $S$ | displacement, (m) |
| $T$ | motion period, (s);temperature distribution, (°C) |
| $t$ | temperature, (°C); time, (s) |
| $U$ | uniform motion velocity, (m/s) |
| $x$:$y$,$z$ | coordinate position, (m) |
| **Greek symbol** | |
| $\alpha$ | thermal diffusivity, (m$^2$/s) |
| $\theta$ | surplus temperature, (°C) |
| $\rho$ | medium density, (kg/m$^3$) |
| $\tau$ | time, (s) |
| $\delta(x - x')$ | distribution of physical quantities in space, (m) |
| **Superscripts** | |
| $K$ | the former k complete periods |
| $N$ | sequence number of complete cycles |
| **Subscripts** | |
| 1 | simplified situation;former half of the $k$th complete period |
| 2 | practical situation;latter half of the $k$th complete period |
| p | point heat source |
| u | uniform velocity motion |
| v | variable velocity motion |
| $\infty$ | environment |

## References

1. Cui, C.; Huang, X.Y.; Liu, C.Y. The Green function and its application to heat transfer in a low permeability porous channel. *J. Electron. Packag.* **2000**, *122*, 274–278. [CrossRef]
2. Povstenko, Y.; Kyrylych, T. Time-fractional heat conduction in an infinite plane containing an external crack under heat flux loading. *Comput. Math. Appl.* **2019**, *78*, 1386–1395. [CrossRef]
3. Berrone, L.R.; Tarzia, D.A.; Villa, L.T. Asymptotic behaviour of a non-classical heat conduction problem for a semi-infinite material. *Math. Method Appl. Sci.* **2000**, *23*, 1161–1177. [CrossRef]
4. Gembarovic, J.; Gembarovic, J., Jr. A comparison of DHW algorithm for temperature distribution calculation with Fourier's algorithm for transmission of heat between discrete bodies. *Int. J. Thermophys.* **2007**, *28*, 891–905. [CrossRef]
5. Zhao, C.B.; Valliappan, S. Mapped transient infinite elements for heat transfer problems in infinite media. *Comput. Methods Appl. Mech. Eng.* **1993**, *108*, 119–131. [CrossRef]
6. Sheng, X.; Zhang, J.R.; Lu, Y.Q.; Lu, X. Temperature evolution on infinite/finite- length cylindrical solids subjected to reciprocating motion heat source. *Case Stud. Therm. Eng.* **2021**, *28*, 101559. [CrossRef]

7. Winczek, J. Modeling of Temperature field during multi-pass GMAW surfacing or rebuilding of steel elements taking into account the heat of the deposit metal. *Appl. Sci.* **2017**, *7*, 6. [CrossRef]
8. Winczek, J.; Gawronska, E.; Gucwa, M.; Sczygiol, N. Theoretical and experimental investigation of temperature and phase transformation during SAW overlaying. *Appl. Sci.* **2019**, *9*, 1472. [CrossRef]
9. Levin, P. A general solution of 3-D quasi-steady-state problem of a moving heat source on a semi-infinite solid. *Mech. Res. Commun.* **2008**, *35*, 151–157. [CrossRef]
10. Zubair, S.M.; Chaudhry, M.A. Temperature solutions due to continuously operating spherical surface-heat sources in an infinite medium. *Int. Commun. Heat Mass Transf.* **1991**, *18*, 805–811. [CrossRef]
11. Mebarek-Oudina, F. Numerical modeling of the hydrodynamic stability in vertical annulus with heat source of different lengths. *Eng. Sci. Technol.* **2017**, *20*, 1324–1333. [CrossRef]
12. BniLam, N.; Al-Khoury, R. Transient heat conduction in an infinite medium subjected to multiple cylindrical heat sources: An application to shallow geothermal systems. *Renew. Energ.* **2016**, *97*, 145–154. [CrossRef]
13. Wang, Y.J.; Fu, P.F.; Guan, Y.J.; Lu, Z.J.; Wei, Y.T. Research on modeling of heat source for electron beam welding fusion-solidification zone. *Chin. J. Aeronaut.* **2013**, *26*, 217–223. [CrossRef]
14. Azar, A.S.; As, S.K.; Akselsen, O.M. Analytical modeling of weld bead shape in dry hyperbaric GMAW using Ar-He chamber gas mixtures. *J. Mater. Eng. Perform.* **2013**, *22*, 673–680. [CrossRef]
15. Zubair, S.M.; Chaudhry, M.A. Temperature solutions due to steady, periodic-type, moving-point-heat sources in an infinite medium. *Int. Commun. Heat Mass Transf.* **1994**, *21*, 207–215. [CrossRef]
16. Chaudhry, M.A.; Zubair, S.M. Analytic study of temperature solutions due to gamma-type moving point-heat source. *Int. J. Heat Mass Transf.* **1993**, *36*, 1633–1637. [CrossRef]
17. Mendez, P.F.; Lu, Y.; Wang, Y. Scaling analysis of a moving point heat source in steady-state on a semi-infinite soild. *J. Heat Trans.-ASME* **2018**, *140*, 081301. [CrossRef]
18. Winczek, J. Analytical solution to transient temperature field in a half-infinite body caused by moving volumetric heat source. *Int. J. Heat Mass Transf.* **2010**, *53*, 5774–5781. [CrossRef]
19. Honkonen, J. Contour-ordered Green's functions in stochastic field theory. *Theor. Math. Phys.* **2013**, *175*, 827–834. [CrossRef]
20. Sheng, X.; Lu, X.; Zhang, J.R.; Lu, Y.Q. An analytical solution to temperature field distribution in a thick rod subjected to periodic-motion heat sources and application in ball screws. *Eng. Optim.* **2021**, *53*, 2144–2163. [CrossRef]
21. Arif, M.; Kuman, P.; Kuman, W.; Mostafa, Z. Heat transfer analysis of radiator using different shaped nanoparticles water-based ternary hybrid nanofluid with applications: A fractional model. *Case Stud. Therm. Eng.* **2022**, *31*, 101837. [CrossRef]
22. Hu, H.P. *Theory of Heat Conduction*; USTC Press: Hefei, China, 2010.
23. Özisik, M.N. *Heat Conduction*; John Wiley and Sons: New York, NY, USA, 1980.
24. Fujii, F. Theory of the steady laminar natural convection above a horizontal line heat source and a point heat source. *Int. J. Heat Mass Transf.* **1963**, *6*, 597–606. [CrossRef]