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Analytical Stability Conditions on Interconnected Nonlinear Systems With Delays

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ABSTRACT This paper is concerned with stability of interconnected systems with time delays. We develop a self-contained approach to stability analysis for linear and nonlinear systems in a unified framework. New lemmas are established on matrix properties and used as the key to make negativity of the derivative of the Lyapunov function. The scalar and simple analytical stability conditions are given. Unlike the majority of the literature on stability of delay systems, no matrix equations/inequalities are involved in our conditions, which is true even for large-scale systems and nonlinear subsystems with delayed interconnections. They are applicable to the more general nonlinear, time-varying, and/or interconnected systems than the relevant results reported in the literature. The examples are presented for illustration of the new results.

INDEX TERMS Stability, interconnected systems, time delay, Lur'e Postnikov, arrow form matrix, aggregation technique.

I. INTRODUCTION

In this paper, the stability of interconnected nonlinear systems with delays is analyzed from a new perspective

$$y_j^{(n)}(t) + \sum_{i=0}^{n-1} f_{j,i}\,(.)\, y_j^{(i)}(t) + \sum_{i=0}^{n-1} g_{j,i}(.) y_j^{(i)}(t - \tau_j(t)) = 0, \quad (1)$$

with the initial time at $t = t_0$ and the initial conditions:

$$y_j^{(i)}(t) = \phi_{j,i}(t),\ t \in [t_0 - \tau_m, t_0], \quad i = 0, \ldots, n-1,$$

where $y_j(t)$ is the system output, and $\tau_j(t)$ is the time delay in the system. In practice, the time delay may be unknown and can vary over time in a certain interval. It is thus assumed that $\tau_j(t)$ is continuous and differentiable over $[0, +\infty]$, and has an upper limit τ_m. $f_{j,i}(.)$, $g_{j,i}(.)$: $\mathcal{D} \times \Omega \times \Omega \to R$, $i = 0, \ldots, n-1$, are the nonlinear functions of the time, output, its $(n-1)$ derivatives, and delayed output and its $(n-1)$ derivatives, where $\mathcal{D} = [-\tau_m, \infty]$, and Ω is a connected domain of R^n containing a neighbourhood of 0. It is assumed that $f_{j,i}$ and $g_{j,i}$, $i = 0, \ldots, n-1$, are such that the system (1) has a unique continuous solution for the given initial condition.

The investigation of interconnected systems have been substantially studied [1]–[6] in recent years due to their wide application in several domains such as physics, mechanics, economics, chemistry, biology, telecommunication, Networked Control Systems (NCS), Internet of Things (IoT), etc. [7]–[12]. Great attention has been also paid to the problems of modeling, characterization, structural properties, control analysis, optimization and feedback design strategies of these systems [13]–[19]. Despite the existence of several analytical and designs methods, the choice of the adequate technique remains quite open. In this context, engineering intuition and human experience should be relied on in order to assess modeling, measure information as well as plant structure and apply analytical and design processes accordingly. In most existing approaches, a complex system is treated as large scale if it is decoupled or divided into a set of interconnected subsystems or the so-called small-scale systems for computational and practical reasons [20]–[24]. Such treatment occurs when controlling strongly-related interconnected power systems, water systems broadly distributed in space as well as traffic systems having numerous external signals or large-space flexible structures. In this case, the controlled systems are too large and the problems that should be resolved are so complex. Therefore, many research studies have tried to divide the process of analyzing and synthesizing the whole system into practically independent sub-problems to deal with the incomplete information on the system, handle

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uncertainties and accommodate delays. One major approach consists in treating the complexity as an important and dominating problem in the systems theory and practice [25].

As the interconnected Lur'e Postnikov systems with time delays is one of the most important connected systems, we investigate, in this paper, their stability analysis. We also present, for the first time, the stability conditions under analytical form dependent on arbitrary-chosen parameters named α_i, in case of single nonlinear systems, and $\alpha_{j,i}$ in the case of interconnected systems. This analytical form can be tuned by judicious choice of these parameters even if the system or the subsystems is unstable. It should be noted that the stability analysis of interconnected systems is not easy even for small order linear systems where the stability of each isolated subsystem does not imply the stability of the global system as the magnitude of the interconnections affects the stability of the composite system and difficulties will be greater when delays appear in the interconnections between the different subsystems. Moreover, the basic control feedback problem consists in finding a control input vector on the basis of the a priori knowledge of the plant described by its design model in the presence of a class of nonlinearities. The control goal is usually given in the form of the design requirements and the a posteriori information about the outputs and the reference signals; that is why this problem is not easy to solve in interconnected systems. In fact, the controller receives all sensor data available from the subsystems and determines all input signals of the plant where all information is assumed to be available for a single unit that designs and applies the controller to the plant. Unfortunately, this procedure becomes incorrect when there is time delay in the interconnection links. In fact, the transmission of information from one subsystem to another may produce a delay, as revealed in [26], [27]. Significant delays may also be caused by the sensors, the actuators and the computing time required for control [28]–[33]. In this case, the presence of time delays may result in complex behaviors such as oscillations, instability, etc. [34], [35]. Thus, the task of well controlling a nonlinear system with interconnections and time-delay remains one of the most challenging control problems [36]. To deal with issue, interconnection terms between the various subsystems should be handled using various approaches. For instance, in [37]–[40], the variable structure control approach was used to control an interconnected system. Several other researchers [41]–[43] presented robust decentralized variable structure control for such systems.

Obviously, the above-mentioned approaches did not consider time-delay in the interconnections. Besides, many stability criteria were proposed on delay systems by using aggregation techniques and radially unbounded Lyapunov functions [44]–[49]. It is well known that the majority of the literature on stability of delay systems studied stability conditions in terms of linear matrix inequalities (LMIs) [50]–[53]. This observation remains true until now in a huge volume of publications and the size of LMIs increases with order/complexity of the systems. It is desirable to have a very few number of stability conditions, regardless of order/complexity of a delay system. We present, in this work, a self-contained approach of stability analysis for linear and nonlinear systems with delay in a unified framework. New lemmas are established on matrix properties and used to make the derivative of the Lyapunov function negative. The scalar and simple analytical stability conditions are obtained. Unlike the majority of the literature on stability of delay systems, no matrix equations/inequalities are involved in our conditions, which is true even for large scale systems and nonlinear subsystems with delayed interconnections. The contributions are highlighted with regards to the relevant literature as follows:

1) The Lyapunov function used in this paper is not the same as that applied in [54].
2) The condition of making the derivative of the Lyapunov in [54] negative is based on M-matrices and related properties [54]–[56]; whereas, in this study, the two elements are not utilized. We develop two brand new lemmas to solve this problem of negativity.
3) Theorem 1, applied in our paper, enables stability analysis for a general system, where all the elements of an arrow-form matrix can be nonlinear or time-varying including both the system's coefficient functions, f_i and g_i; and artificially introduced parameters, α_i. This greatly generalizes the results presented in [54], [58]–[61] where only one row or one column of the arrow form matrix could be so and its diagonal elements must be constant. The new capacities are illustrated in our examples.
4) In case of constant delays (then its time derivative is zero), the first condition of Theorem 1 is always satisfied and the stability condition of Theorem 1 is reduced to its second condition only, regardless of delay. Furthermore, in the case of variable delays, the condition on delay depends only on the coefficients $\delta_{j,i}(.)$. These results were not given in [54].
5) This paper presents the modelling of a system consisting of two Lur'e Postnikov plants with delayed inter-connections and feedbacks, and its stability conditions (Theorem 2), while such a system was not considered in [54].

Notations: Throughout this paper, let $R = (-\infty, +\infty)$ and R^n be an $n-$dimensional linear vector space over the reals with the norm $\|.\|$. Let $C_n = C([-\tau_m, 0]; \Omega)$ be the Banach space of continuous functions mapping the interval $[-\tau_m, 0]$ into $\Omega \subset R^n$ with the topology of uniform convergence. For a given $\phi \in C_n$, we define $\||\phi\|| = \sup_{-\tau_m \leq \theta \leq 0} \|\phi(\theta)\|$. The notations $\|.\|$ refers to the Euclidean vector norm or the induced matrix norm, as appropriate. If their dimensions are not explicitly stated, matrices, are assumed to have compatible dimensions. Let $\sup_{\mathcal{D}} |f(.)|$ be the supremum of $f(.)$ calculated over $\mathcal{D} \times \Omega \times \Omega$, where $f(.)$ can be any of f_i and g_i and their algebraic combination. Finally, we denote the right Dini derivative of a function $V(t)$ with respect to time t by $D^+V(t)$.

The rest of the paper is organized as follows: Section 2 presents stability analysis for single systems, while Section 3 extends to interconnected Lur'e systems. Section 4 concludes the paper.

II. SINGLE NONLINEAR SYSTEMS

Consider a class of nonlinear systems in the form of

$$y^{(n)}(t)+\sum_{i=0}^{n-1} f_i\,(.)\,y^{(i)}(t)+\sum_{i=0}^{n-1} g_i(.)y^{(i)}(t-\tau(t))=0, \quad (2)$$

with the initial time at $t=t_0$ and the initial conditions:

$$y^{(i)}(t)=\phi_i(t),\ t\in[t_0-\tau_m,t_0], \quad i=0,\ldots,n-1,$$

It should be pointed out that several systems can be modeled by (2), see [54], [55] and the references therein. One example is the well known Lur'e Postnikov system [44], [46], [47], [54]–[56], [61], [62]. Stability of such systems is difficult to analyze, even for second-order systems with a constant delay and single nonlinear coefficient [63].

Define the state variables:

$$x_{i+1}(t)=y^{(i)}(t), \quad i=0,\ldots,n-1,$$

which lead to

$$\dot{x}_i(t)=x_{i+1}(t), \quad i=0,\ldots,n-1.$$

Let $x(t)=[x_1(t),\ldots,x_n(t)]^T\in R^n$.

The system (2) is then described by the following state space representation,

$$\begin{aligned}\dot{x}(t)&=A(.)x(t)+\tilde{B}^1(.)x(t-\tau(t)),\\ x(t)&=\phi(t),\quad \forall t\in[t_0-\tau_m,\ t_0],\end{aligned} \quad (3)$$

where

$$A(.)=\begin{bmatrix}0 & 1 & \cdots & 0\\ 0 & 0 & \ddots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & 1\\ -f_0(.) & -f_1(.) & \cdots & -f_{n-1}(.)\end{bmatrix},$$

and

$$\tilde{B}^1(.)=\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & 0\\ -g_0(.) & -g_1(.) & \ldots & -g_{n-1}(.)\end{bmatrix}.$$

Apply the following state transformation,

$$x=Pz,\ \text{where } P=\begin{bmatrix}1 & 1 & \cdots & 1 & 0\\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 0\\ \vdots & \vdots & \ldots & \vdots & \vdots\\ \alpha_1^{n-2} & \alpha_2^{n-2} & \cdots & \alpha_{n-1}^{n-2} & 0\\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_{n-1}^{n-1} & 1\end{bmatrix}, \quad (4)$$

with $\alpha_i\neq\alpha_k$, $\forall i,k=1,\ldots,n-1$, and $\alpha_i^j, j=1,\ldots,n-1$, denote power j of α_i. The system (3) becomes

$$\begin{aligned}\dot{z}(t)&=F(.)z(t)+\check{B}^1(.)z(t-\tau(t)),\\ z(t)&=P^{-1}\phi(t),\quad \forall t\in[t_0-\tau_m,\ t_0],\end{aligned} \quad (5)$$

where

$$\begin{aligned}F(.)&=P^{-1}A(.)P\\ &=\begin{bmatrix}\alpha_1 & 0 & \ldots & 0 & \beta_1\\ 0 & \alpha_2 & \ddots & \vdots & \beta_2\\ \vdots & \ddots & \ddots & 0 & \vdots\\ 0 & \cdots & 0 & \alpha_{n-1} & \beta_{n-1}\\ \gamma_1(.) & \cdots & \cdots & \gamma_{n-1}(.) & \gamma_n(.)\end{bmatrix},\\ \check{B}^1(.)&=P^{-1}\tilde{B}^1(.)P\\ &=\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & 0\\ \delta_1(.) & \delta_1(.) & \cdots & \delta_n(.)\end{bmatrix},\end{aligned}$$

for any $i=1,\ldots,n-1$, $\beta_i=\prod_{k\neq i}^{n-1}(\alpha_i-\alpha_k)^{-1}$, $\gamma_i(.)=-(\alpha_i^n+\sum_{j=0}^{n-1} f_j(.)\alpha_i^j)=-p_{F(.)}(\alpha_i)$, $\delta_i(.)=-\sum_{j=0}^{n-1} g_j(.)\alpha_i^i=-p_{A_1(.)}(\alpha_i)$, and

$$\begin{aligned}\gamma_n(.)&=-f_{n-1}(.)-\sum_{j=1}^{n-1}\alpha_j,\\ \delta_n(.)&=-g_{n-1}(.).\end{aligned} \quad (6)$$

Define the matrix $\Gamma(.)$ as follows,

$$\Gamma(.)=\begin{bmatrix}\alpha_1 & 0 & \ldots & 0 & \tilde{\gamma}_1(.)\\ 0 & \alpha_2 & \ddots & \vdots & \tilde{\gamma}_2(.)\\ \vdots & \ddots & \ddots & 0 & \vdots\\ 0 & \cdots & 0 & \alpha_{n-1} & \tilde{\gamma}_{n-1}(.)\\ |\beta_1| & \cdots & \cdots & |\beta_{n-1}| & \tilde{\gamma}_n(.)\end{bmatrix} \quad (7)$$

where, $\tilde{\gamma}_i(.)=|\gamma_i(.)|+\sup_{\mathcal{D}}(|\delta_i(.)|)$, $\forall i=1,\ldots,n-1$, $\tilde{\gamma}_n(.)=\gamma_n(.)+\sup_{\mathcal{D}}(|\delta_n(.)|)$. Let $h(z)$ be a function of z on the domain $\mathcal{D}$. $H(z)\triangleq\frac{|h(z)|}{\sup_{\mathcal{D}}|h(z)|}$ is well defined if $h(z)\neq 0$, for some $z\in\mathcal{D}$; otherwise, $H(z)=0$ if $h(z)\equiv 0$, for all $z\in\mathcal{D}$.

We are now in the position to state the main result of this section.

Theorem 1: The time-delay system (5) is asymptotically stable if there exist distinct real numbers, $\alpha_i<0$, $i=1,\ldots,n-1$, such that the following inequalities hold true,

$$\dot{\tau}(t)+\max_{1\le j\le n}\left\{\frac{|\delta_j(.)|}{\sup_{\mathcal{D}}|\delta_j(.)|}\right\}\le 1, \quad (8)$$

$$\tilde{\gamma}_n(.)-\sum_{i=1}^{n-1}\frac{\tilde{\gamma}_i(.)|\beta_i|}{\alpha_i}<0. \quad (9)$$

Note first that the advantage of Theorem 1 is its simple and scalar conditions, where no Linear Matrix Inequality is present to be solved. It accommodates the parameter uncertainties. It allows great freedoms of a judicious choice of α_i, $i = 1, \ldots, n-1$. The proof of Theorem 1 needs two new lemmas. Define a Λ-matrix with the following special form:

$$\Lambda = \begin{bmatrix} \lambda_{1,1} & 0 & \ldots & 0 & \lambda_{1,n} \\ 0 & \lambda_{2,2} & \ddots & \vdots & \lambda_{2,n} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1,n-1} & \lambda_{n-1,n} \\ \lambda_{n,1} & \lambda_{n,2} & \cdots & \lambda_{n,n-1} & \lambda_{n,n} \end{bmatrix}, \quad (10)$$

where $\lambda_{i,n},\ \lambda_{n,i} > 0$, $i = 1, \ldots, n-1$; and $\lambda_{i,i} < 0$, $i = 1, \ldots, n$.

Lemma 1: Given a constant Λ-matrix and any constant vector $\eta < 0$, there is $\rho > 0$ for $\Lambda\rho < \eta$, if

$$\lambda_{n,n} - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}\lambda_{i,n}}{\lambda_{i,i}} < 0. \quad (11)$$

Proof: We first show existence of a solution ρ for $\Lambda\rho < \eta$. One evaluates

$$\det(\Lambda) = \psi \prod_{j=1}^{n-1} \lambda_{j,j},\ \psi = \lambda_{n,n} - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}\lambda_{i,n}}{\lambda_{i,i}},$$

which is not zero under the assumed condition: $\psi < 0$. Thus, Λ^{-1} exists. Then for all $\eta < 0$, construct $\rho \triangleq \Lambda^{-1}(\eta - \epsilon)$, for $\epsilon > 0$, with $\epsilon \in R^n$. It follows that $\Lambda\rho = \Lambda\Lambda^{-1}(\eta - \epsilon) = \eta - \epsilon < \eta$. Next, we show the positivity of ρ for $\Lambda\rho < \eta$. Consider the inequality, $\Lambda\rho < \eta$, for any $\eta < 0$. We split it into two parts:

$$\lambda_{i,i}\rho_i + \lambda_{i,n}\rho_n < \eta_i,\ i = 1, \ldots, n-1, \quad (12)$$

$$\sum_{i=1}^{n-1} \lambda_{n,i}\rho_i + \lambda_{n,n}\rho_n < \eta_n. \quad (13)$$

Equation (12) is equivalent to

$$\begin{bmatrix} \rho_1 \\ \vdots \\ \rho_{n-1} \end{bmatrix} > \begin{bmatrix} \lambda_{1,1} & & \\ & \ddots & \\ & & \lambda_{n-1,n-1} \end{bmatrix}^{-1} \times \left\{ \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-1} \end{bmatrix} - \begin{bmatrix} \lambda_{1,n} \\ \vdots \\ \lambda_{n-1,n} \end{bmatrix} \rho_n \right\}, \quad (14)$$

where the inequality direction has been reversed since $\lambda_{i,i} < 0$. Due to $\lambda_{n,j} > 0, j = 1, \ldots, n-1$, the above is changed to

$$-\begin{bmatrix} \lambda_{n,1} \\ \vdots \\ \lambda_{n,n-1} \end{bmatrix}^T \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_{n-1} \end{bmatrix} < -\begin{bmatrix} \lambda_{n,1} \\ \vdots \\ \lambda_{n,n-1} \end{bmatrix}^T \begin{bmatrix} \lambda_{1,1} & & \\ & \ddots & \\ & & \lambda_{n-1,n-1} \end{bmatrix}^{-1} \times \left\{ \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-1} \end{bmatrix} - \begin{bmatrix} \lambda_{1,n} \\ \vdots \\ \lambda_{n-1,n} \end{bmatrix} \rho_n \right\}.$$

It follows from (13) that

$$\lambda_{n,n}\rho_n < -\sum_{i=1}^{n-1} \lambda_{n,i}\rho_i + \eta_n \quad (15)$$

$$= -\begin{bmatrix} \lambda_{n,1} \\ \vdots \\ \lambda_{n,n-1} \end{bmatrix}^T \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_{n-1} \end{bmatrix} + \eta_n. \quad (16)$$

Using (15), (16) is re-written as

$$\lambda_{n,n}\rho_n < -\sum_{i=1}^{n-1} \frac{\lambda_{n,i}\eta_i}{\lambda_{i,i}} + \sum_{i=1}^{n-1} \frac{\lambda_{n,i}\lambda_{i,n}}{\lambda_{i,i}}\rho_n + \eta_n, \quad (17)$$

that is,

$$\left(\lambda_{n,n} - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}\lambda_{i,n}}{\lambda_{i,i}}\right)\rho_n < \eta_n - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}\eta_i}{\lambda_{i,i}}. \quad (18)$$

It is easy to verify from signs of relevant elements in (18) that the right-hand-side of (18) is strictly negative. This along with condition (11) implies that

$$\rho_n > \frac{\eta_n - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}\eta_i}{\lambda_{i,i}}}{\left(\lambda_{n,n} - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}\lambda_{i,n}}{\lambda_{i,i}}\right)} > 0. \quad (19)$$

It then follows from (14) that $\rho_i > 0, i = 1, \ldots, n-1$, which together with (19) indicates $\rho > 0$. The proof is completed. ∎

Lemma 2: Given a $\Lambda(.)$-matrix with uncertain elements and any $\eta < 0$, there is $\bar{\rho} > 0$ for

$$\Lambda(.)\bar{\rho} < \eta(.), \quad (20)$$

if

$$\lambda_{n,n}(.) - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}(.)\lambda_{i,n}(.)}{\lambda_{i,i}(.)} < 0. \quad (21)$$

Proof: We want to show that (12) and (13) hold with ρ replaced by $\bar{\rho}$. Let

$$\bar{\rho}_n > \sup_{\mathcal{D}} \left\{ \frac{\eta_n - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}(.)\eta_i}{\lambda_{i,i}(.)}}{\lambda_{n,n}(.) - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}(.)\lambda_{i,n}(.)}{\lambda_{i,i}(.)}} \right\} \geq \frac{\eta_n - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}(.)\eta_i}{\lambda_{i,i}(.)}}{\lambda_{n,n}(.) - \sum_{i=1}^{n-1} \frac{\lambda_{n,i}(.)\lambda_{i,n}(.)}{\lambda_{i,i}(.)}}, \quad (22)$$

and

$$\begin{bmatrix}\bar{\rho}_1\\ \vdots \\ \bar{\rho}_{n-1}\end{bmatrix} > \sup_{\mathcal{D}}\left\{\begin{bmatrix}\lambda_{1,1}(.) & & \\ & \ddots & \\ & & \lambda_{n-1,n-1}(.)\end{bmatrix}^{-1} \times\left(\begin{bmatrix}\eta_1\\ \vdots\\ \eta_{n-1}\end{bmatrix}-\begin{bmatrix}\lambda_{1,n}(.)\\ \vdots\\ \lambda_{n-1,n}(.)\end{bmatrix}\right)\bar{\rho}_n\right\}$$
$$\geq \left\{\begin{bmatrix}\lambda_{1,1}(.) & & \\ & \ddots & \\ & & \lambda_{n-1,n-1}(.)\end{bmatrix}^{-1} \times\left(\begin{bmatrix}\eta_1\\ \vdots\\ \eta_{n-1}\end{bmatrix}-\begin{bmatrix}\lambda_{1,n}(.)\\ \vdots\\ \lambda_{n-1,n}(.)\end{bmatrix}\right)\bar{\rho}_n\right\}. \quad (23)$$

Equation (22) implies that

$$\bar{\rho}_n > \frac{\eta_n - \sum_{i=1}^{n-1}\frac{\lambda_{n,i}(.)\eta_i}{\lambda_{i,i}(.)}}{\left(\lambda_{n,n}(.) - \sum_{i=1}^{n-1}\frac{\lambda_{n,i}(.)\lambda_{i,n}(.)}{\lambda_{i,i}(.)}\right)}, \quad (24)$$

which, by following the proof of Lemma 1, yields (13). Similarly, (23) will give (12). ■

Proof of Theorem 1: Since α_i, $i = 1, \ldots, n-1$, are arbitrary, we choose $\alpha_i < 0$ with $\alpha_i \neq \alpha_k, \forall i, k = 1, \ldots, n-1$, so that $\Gamma(.)$ is a Λ-matrix, indeed. Thus, it follows from Lemma 2 that if

$$\widetilde{\gamma}_n(.) - \sum_{i=1}^{n-1}\frac{\widetilde{\gamma}_i(.)|\beta_i|}{\alpha_i} < 0, \quad (25)$$

there exists a constant vector $\rho > 0$ such that $\Gamma(.)\rho < \eta$ holds true for $\eta < 0$. Hence, we choose the radially unbounded, positive definite Lyapunov function candidate as

$$V(t) = \sum_{i=1}^{n}\rho_i|z_i(t)| + \rho_n\sum_{i=1}^{n}\sup_{\mathcal{D}}|\delta_i(.)|\int_{t-\tau(t)}^{t}|z_i(\upsilon)|\,d\upsilon. \quad (26)$$

Because $\rho > 0$, $V(t) > 0$. The initial condition for the solution of system (5) is given by $z_{t_0} = P^{-1}x_{t_0}(\theta) = P^{-1}x(t_0+\theta) = P^{-1}\phi(\theta)$, $-\tau_m \leq \theta \leq 0$. Then, we have

$$V(t_0) \leq \|P^{-1}\|\left\{\sum_{i=1}^{n}\rho_i|x_i(t_0)| + \rho_n\sum_{i=1}^{n}\sup_{\mathcal{D}}|\delta_i(.)|\,\|\phi\|\tau_m\right\} < \infty.$$

The right Dini derivative of $V(t,z)$ under the solution of (5) is given by

$$\begin{aligned} D^+V(t,z(t)) &= \sum_{i=1}^{n-1}\rho_i\frac{d^+|z_i(t)|}{dt^+} + \rho_n\frac{d^+|z_n(t)|}{dt^+} \\ &\quad+\rho_n\sum_{i=1}^{n}\sup_{\mathcal{D}}|\delta_i(.)|\left[|z_i(t)| - (1-\dot{\tau}(t))\,|z_i(t-\tau(t))|\right]. \end{aligned} \quad (27)$$

It is seen that

$$\begin{aligned}\frac{d^+|z_i(t)|}{dt^+} &= \dot{z}_i(t)sign(z_i(t)) \\ &= (\alpha_i z_i(t) + \beta_i z_n(t))\,sign(z_i(t)) \\ &\leq \alpha_i z_i(t)sign(z_i(t) + |\beta_i||z_n(t)| \\ &= \alpha_i|z_i(t)| + |\beta_i||z_n(t)|,\end{aligned}$$

and

$$\begin{aligned}&\frac{d^+|z_n(t)|}{dt^+} \\ &= \left(\sum_{i=1}^{n}\gamma_i(.)z_i(t) + \sum_{i=1}^{n}\delta_i(.)z_i(t-\tau(t))\right)sign(z_n(t)) \\ &\leq \gamma_n(.)z_n(t)sign(z_n(t)) + \sum_{i=1}^{n-1}|\gamma_i(.)||z_i(t)| \\ &\quad+\sum_{i=1}^{n}|\delta_i(.)||z_i(t-\tau(t))| \\ &= \gamma_n(.)|z_n(t)| + \sum_{i=1}^{n-1}|\gamma_i(.)||z_i(t)| + \sum_{i=1}^{n}|\delta_i(.)||z_i(t-\tau(t))|.\end{aligned} \quad (28)$$

Using (8), it follows that

$$-(1-\dot{\tau}(t)) \leq -\max_{1\leq j\leq n-1}\left\{\frac{|\delta_j(.)|}{\sup_{\mathcal{D}}|\delta_j(.)|}\right\} \leq -\frac{|\delta_i(.)|}{\sup_{\mathcal{D}}|\delta_i(.)|}, \quad (29)$$

for $i = 1, \ldots, n-1$. Therefore, it follows from (29) that

$$\begin{aligned}&\frac{d^+|z_n(t)|}{dt^+} + \sum_{i=1}^{n}\sup_{\mathcal{D}}|\delta_i(.)|\,[|z_i(t)| - (1-\dot{\tau}(t))\,|z_i(t-\tau(t))|] \\ &\leq \left(\gamma_n(.) + \sup_{\mathcal{D}}|\delta_n(.)|\right)|z_n(t)| \\ &\quad+\sum_{i=1}^{n-1}\left(|\gamma_i(.)| + \sup\nolimits_{\mathcal{D}}|\delta_i(.)|\right)|z_i(t)|.\end{aligned} \quad (30)$$

Substituting (30) to (27) yields

$$\begin{aligned}D^+V(t,z(t)) &\leq \sum_{i=1}^{n-1}\rho_i\left(\alpha_i|z_i(t)| + |\beta_i||z_n(t)|\right) \\ &\quad+\rho_n\left[\left(\sup_{\mathcal{D}}|\delta_n(.)| + \gamma_n(.)\right)|z_n(t)|\right.\end{aligned}$$

$$+\sum_{i=1}^{n-1}\left(|\gamma_i(.)|+\sup_{\mathcal{D}}|\delta_i(.)|\right)|z_i(t)|\Bigg]$$
$$=\sum_{i=1}^{n-1}\left[\rho_i\alpha_i+\rho_n\widetilde{\gamma}_i(.)\right]|z_i(t)|$$
$$+\left[\rho_n\widetilde{\gamma}_n(.)+\sum_{i=1}^{n-1}\rho_i|\beta_i|\right]|z_n(t)|$$
$$=|z(t)|^T\Gamma(.)\rho<|z(t)|^T\eta<0, \quad (31)$$

since $\eta<0$. The proof is completed. ■

Remark 1: Theorem 1 enables stability analysis for the systems with time delay, where all the elements of $\Gamma(.)$ could be nonlinear or time-varying, including both the system's coefficient functions, f_i and g_i; and artificially introduced parameters, α_i, $i=1,\ldots,n-1$. This greatly generalizes all the results based on the arrow form matrix where the studied system is without delay in [58]–[60] and the references therein, or with delay but constant α_i in [54]–[57] and [61].

The conditions of Theorem 1 can be simplified in certain cases.

Corollary 1: If there exist distinct $\alpha_i<0, i=1,\ldots,n-1$, such that $\gamma_i(.)\beta_i>0$, the stability conditions of Theorem 1 reduce to

$$\dot{\tau}(t)+\max_{1\le j\le n}\left\{\frac{|\delta_j(.)|}{\sup_{\mathcal{D}}|\delta_j(.)|}\right\}\le 1,$$
$$\sup_{\mathcal{D}}(|\delta_n(.)|)-\sum_{i=1}^{n-1}\frac{\sup_{\mathcal{D}}|\delta_i(.)||\beta_i|}{\alpha_i}<\frac{p_{F(.)}(0)}{\prod_{j=1}^{n-1}(-\alpha_j)}.$$

Proof: Note that the first condition of Theorem 1 remains same as in Corollary 1. The simplification is in the second condition which is to be shown now. Take the partial fraction expansion of $\frac{p_{F(.)}(s)}{\prod_{j=1}^{n-1}(s-\alpha_j)}$:

$$\frac{p_{F(.)}(s)}{\prod_{j=1}^{n-1}(s-\alpha_j)}=s+f_{n-1}(.)+\sum_{i=1}^{n-1}\alpha_i+\sum_{i=1}^{n-1}\frac{R_i(.)}{s-\alpha_i}, \quad (32)$$

where $R_i(.)$, $i=1,\ldots,n-1$, are given by

$$R_i(.)=\left[\frac{p_{F(.)}(s)(s-\alpha_i)}{\prod_{j=1}^{n-1}(s-\alpha_j)}\right]_{s=\alpha_i}=-\gamma_i(.)\beta_i.$$

Knowing from (6) that $\gamma_n(.)=-f_{n-1}(.)-\sum_{i=1}^{n-1}\alpha_i$, then (32) becomes

$$\frac{p_{F(.)}(s)}{\prod_{j=1}^{n-1}(s-\alpha_j)}=-\gamma_n(.)-\sum_{i=1}^{n-1}\frac{\gamma_i(.)\beta_i}{s-\alpha_i},$$

which leads to

$$\frac{p_{F(.)}(0)}{\prod_{j=1}^{n-1}(-\alpha_j)}=-\gamma_n(.)+\sum_{i=1}^{n-1}\frac{\gamma_i(.)\beta_i}{\alpha_i}.$$

Then, if $\gamma_i(.)\beta_i>0$, the inequality (9) becomes the 2nd condition in this corollary. The proof is completed. ■

If the roots of $p_{\breve{B}^1}(.)$ are all real, negative and distinct, we can choose the α_i to be equal to these roots.

Corollary 2: If $p_{\breve{B}^1(.)}(\alpha_i)=0$, $\alpha_i<0$, $i=1,\ldots,n-1$, where $\alpha_i\neq\alpha_j$ for all $i\neq j=1,\ldots,n-1$, such that $\gamma_i(.)\beta_i>0$, then the stability conditions of Theorem 1 become

$$\dot{\tau}(t)+\frac{|\delta_n(.)|}{\sup_{\mathcal{D}}|\delta_n(.)|}\le 1, \quad (33)$$
$$\sup_{\mathcal{D}}(|\delta_n(.)|)<g_{n-1}(.)\frac{p_{F(.)}}{p_{\breve{B}^1(.)}}. \quad (34)$$

Proof: If $p_{\breve{B}^1(.)}(\alpha_i)=0$, we obtain $\delta_i(.)=-p_{\breve{B}^1(.)}(\alpha_i)=0$, $i=1,\ldots,n-1$, $p_{\breve{B}^1(.)}(0)=g_{n-1}(.)\prod_{j=1}^{n-1}(-\alpha_j)$. Since $\gamma_i(.)\beta_i>0$ is assumed, Corollary 1 is then applicable and its conditions become those in this corollary after using the above relations. The proof is completed. ■

III. INTERCONNECTED NONLINEAR SYSTEMS

Consider two coupled Lur'e Postnikov systems with delayed interconnections, as shown in Fig. 1. Each open-loop plant without interconnections ($r_{j-(-1)^j,j}=0$) is described by its state space representation:

$$\mathcal{S}_j:\begin{cases}\dot{x}_j(t)=A_jx_j(t)+B_ju_j(t), j=1,2,\\ u_j(t)=\varphi_j(\varepsilon_j(t));\ \varphi_j(0)=0,\\ \varepsilon_j(t)=-C_jx(t-d_j),\end{cases} \quad (35)$$

where

$$x_j(t)=\left[y_j(t),\ \dot{y}_j(t),\ \ldots,y_j^{(n-1)}(t)\right]^T,$$
$$A_j=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ \vdots & \ddots & 1 & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ 0 & \cdots & \cdots & 0 & 1\\ -\frac{a_0^j}{d_n^j} & -\frac{a_1^j}{d_n^j} & -\frac{a_2^j}{d_n^j} & \cdots & -\frac{a_{n-1}^j}{d_n^j}\end{bmatrix},$$
$$B_j=\begin{bmatrix}0 & \ldots & 0 & b_j\end{bmatrix}^T,\ b_j=\frac{k_j}{d_n^j},$$
$$C_j=\begin{bmatrix}c_0^j & \ldots & c_{n-2}^j & c_{n-1}^j\end{bmatrix}^T$$

τ_j, d_j : are time delays,

$r_{j,j-(-1)^j}$: are the interconnection variables.

Using the Mean Value Theorem, one gets

$$\varphi_j(\varepsilon_j(t))-\varphi_j(0)=\frac{\partial\varphi_j(\varepsilon_j)}{\partial\varepsilon_j}\left(\varepsilon_j(t)-0\right),$$

which gives

$$\varphi_j(\varepsilon_j(t))=-\tilde{\varphi}_j(.)C_jx(t-d_j).$$

Therefore, $\tilde{\varphi}_j(.)$ can be interpreted, in certain cases, as the instantaneous gain at any point of the characteristic of the nonlinearity φ_j. Then (35) becomes

$$\dot{x}_j(t)=A_jx_j(t)+\tilde{B}_j^1(.)x_j(t-d_j), \quad (36)$$

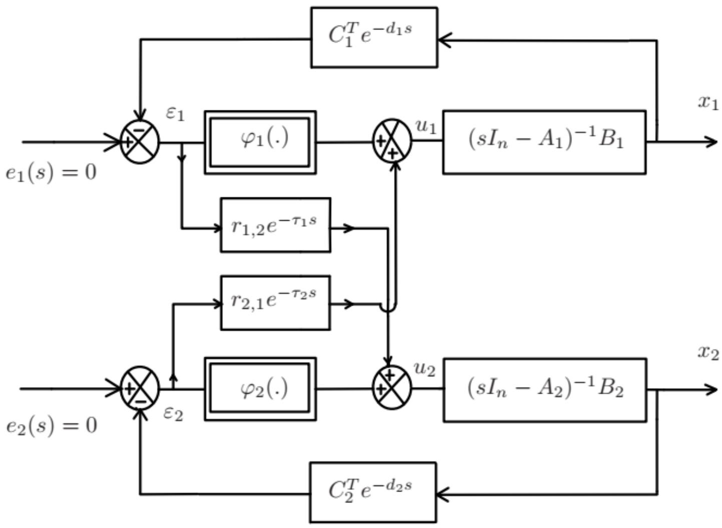

FIGURE 1. **Two coupled Lur'e Postnikov systems with delayed interconnections.**

with

$$\tilde{B}_j^1(.) = -B_j\tilde{\varphi}_j(.)C_j^T = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ -\tilde{b}_0^{1,j}(.) & \cdots & -\tilde{b}_{n-2}^{1,j}(.) & -\tilde{b}_{n-1}^{1,j}(.) \end{bmatrix}.$$

From the equation (36), we notice that the system $\mathcal{S}_j$ is a particular form of (1).

Now, we consider the interconnection between $\mathcal{S}_j$ and $\mathcal{S}_{j-(-1)^j}$ given by Fig.1 with $r_{j,j-(-1)^j} \neq 0$. Then, we obtain

$$u_j(t) = \tilde{\varphi}_j(.)\varepsilon_j(t) + r_{j-(-1)^j,j}\varepsilon_{j-(-1)^j}(t - \tau_{j-(-1)^j}),$$

and the system (36) changes to

$$\dot{x}_j(t) = A_jx_j(t) + \tilde{B}_j^1(.)x_j(t - d_j) + \tilde{B}_{j-(-1)^j}^2 x_{j-(-1)^j}(t - (d_{j-(-1)^j} + \tau_{j-(-1)^j})), \quad (37)$$

where

$$\tilde{B}_j^2 = -B_jr_{j-(-1)^j,j}C_{j-(-1)^j}^T = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ -\tilde{b}_0^{2,j} & \cdots & -\tilde{b}_{n-2}^{2,j} & -\tilde{b}_{n-1}^{2,j} \end{bmatrix},$$

with $\tilde{b}_i^{1,j}(.) = b_jc_i^j\tilde{\varphi}_j(.)$ and $\tilde{b}_i^{2,j} = r_{j,j-(-1)^j}b_{j-(-1)^j}c_i^j$.

Apply the state transformation:

$$X(t) = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} Z(t), \quad (38)$$

where

$$X(t) = \left[x_1^T(t),\ x_2^T(t)\right]^T$$
$$Z(t) = \left[z_1^T(t),\ z_2^T(t)\right]^T,$$

and

$$P_j = \begin{bmatrix} 1 & \cdots & 1 & 0 \\ (\alpha_{j,1}) & \cdots & (\alpha_{j,n-1}) & \vdots \\ (\alpha_{j,1})^2 & \cdots & (\alpha_{j,n-1})^2 & \vdots \\ \vdots & \vdots & \vdots & 0 \\ (\alpha_{j,1})^{n-1} & \cdots & (\alpha_{j,n-1})^{n-1} & 1 \end{bmatrix}, \ j = 1, 2,$$
$$\alpha_{j,i} \neq \alpha_{j,k}, \ i, \ k = 1, \ 2, \ \cdots, \ n-1.$$

The whole system is given by

$$\dot{Z}(t) = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} Z(t) + \begin{bmatrix} \check{B}_1^1 & 0 \\ 0 & \check{B}_2^1 \end{bmatrix} Z(t - \bar{d}_1) + \begin{bmatrix} 0 & \check{B}_1^2 \\ \check{B}_2^2 & 0 \end{bmatrix} Z(t - \bar{d}_2), \quad (39)$$

where

$$Z(t - \bar{d}_1) = \left[z_1^T(t - d_1),\ z_2^T(t - d_2)\right]^T,$$
$$Z(t - \bar{d}_2) = \left[z_1^T(t - (\tau_1 + d_1)),\ z_2^T(t - (\tau_2 + d_2))\right]^T,$$
$$F_j = P_j^{-1}A_jP_j = \begin{bmatrix} \alpha_{j,1} & 0 & \ldots & 0 & \beta_1^j \\ 0 & \alpha_{j,2} & \ddots & \vdots & \beta_2^j \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \alpha_{j,n-1} & \beta_{n-1}^j \\ \gamma_{j,1} & \cdots & \cdots & \gamma_{j,n-1} & \gamma_{j,n} \end{bmatrix},$$
$$\check{B}_j^1 = P_j^{-1}\tilde{B}_j^1(.)P_j$$

$$= \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \delta^1_{j,1}(.) & \cdots & \delta^1_{j,n-1}(.) & \delta^1_{j,n}(.) \end{bmatrix},$$

$$\breve{B}^2_j = P_j^{-1}\tilde{B}^2_j P_{j-(-1)^j}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \delta^2_{j,1} & \cdots & \delta^2_{j,n-1} & \delta^2_{j,n} \end{bmatrix},$$

with, $j = 1, 2,$

$$\begin{aligned} \gamma_{j,i} &= -p_{A_j}(\alpha_{j,i}), \\ \delta^1_{j,i}(.) &= -p_{\tilde{B}^1_j(.)}(\alpha_{j,i}), \\ \delta^2_{j,i} &= -p_{\tilde{B}^2_j}(\alpha_{j-(-1)^j,i}), \\ & \quad i = 1, \ldots, n-1, \\ \gamma_{j,n} &= -\frac{a^j_{n-1}}{a^j_n} - \sum_{i=1}^{n-1}\alpha_{j,i}, \\ \delta^1_{j,n}(.) &= -\tilde{b}^{1,j}_{n-1}(.), \\ \delta^2_{j,n} &= -\tilde{b}^{2,j}_{n-1}, \\ p_{A_j}(s) &= s^n + \sum_{i=0}^{n-1}\frac{a^j_i}{a^j_n}s^i, \\ p_{\tilde{B}^1_j(.)}(s) &= \sum_{i=0}^{n-1}\tilde{b}^{1,j}_i(.)s^i, \\ p_{\tilde{B}^2_j}(s) &= \sum_{i=0}^{n-1}\tilde{b}^{2,j}_i s^i. \end{aligned}$$

Applying two permutations to the state vector: the first is between $n-$th and $(2n-1)$-th rows, the second between n-th and $(2n-1)$-th columns. The new state space representation (while we retain Z as the state) is given by

$$\dot{Z}(t) = \Upsilon Z(t) + \Xi_1(.)Z(t-\bar{d}_1) + \Xi_2 Z(t-\bar{d}_2), \quad (40)$$

where

$$\Upsilon = \begin{bmatrix} \Upsilon_{1,1} & 0 & \Upsilon_{1,3} \\ 0 & \Upsilon_{2,2} & \Upsilon_{2,3} \\ \Upsilon_{3,1} & \Upsilon_{3,2} & \Upsilon_{3,3} \end{bmatrix},$$

$$\Xi_j(.) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Xi^j_{3,1}(.) & \Xi^j_{3,2}(.) & \Xi^j_{3,3}(.) \end{bmatrix}, \quad j = 1, 2,$$

with

$$\Upsilon_{1,1} = \mathrm{diag}\begin{pmatrix}\alpha_{1,1} & \alpha_{1,2} & \ldots & \alpha_{1,n-1}\end{pmatrix},$$

$$\Upsilon_{2,2} = \mathrm{diag}\begin{pmatrix}\alpha_{2,n-1} & \alpha_{2,1} & \ldots & \alpha_{2,n-2}\end{pmatrix},$$

$$\Upsilon_{1,3} = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix}^T,$$

$$\Upsilon_{2,3} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \beta_{2,n-1} & \beta_{2,1} & \cdots & \beta_{2,n-2} \end{bmatrix}^T,$$

$$\Upsilon_{3,1} = \begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\Upsilon_{3,2} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \gamma_{2,n-1} & \gamma_{2,1} & \cdots & \gamma_{2,n-2} \end{bmatrix},$$

$$\Upsilon_{3,3} = \begin{bmatrix} \gamma_{1,n} & 0 \\ 0 & \gamma_{2,n} \end{bmatrix},$$

$$\Xi^1_{3,1}(.) = \begin{bmatrix} \delta^1_{1,1}(.) & \delta^1_{1,2}(.) & \cdots & \delta^1_{1,n-1}(.) \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\Xi^1_{3,2}(.) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \delta^1_{2,n-1}(.) & \delta^1_{2,1}(.) & \cdots & \delta^1_{2,n-2}(.) \end{bmatrix},$$

$$\Xi^1_{3,3}(.) = \begin{bmatrix} \delta^1_{1,n}(.) & 0 \\ 0 & \delta^1_{2,n}(.) \end{bmatrix},$$

$$\Xi^2_{3,1} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \delta^2_{1,1} & \delta^2_{1,2} & \cdots & \delta^2_{1,n-1} \end{bmatrix},$$

$$\Xi^2_{3,2} = \begin{bmatrix} \delta^2_{2,n-1} & \delta^2_{2,1} & \cdots & \delta^2_{2,n-2} \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\Xi^2_{3,3} = \begin{bmatrix} 0 & \delta^2_{2,n} \\ \delta^2_{1,n} & 0 \end{bmatrix}.$$

Theorem 2: The closed-loop system (37) is asymptotically stable if there exist distinct $\alpha_{j,i} < 0,\ j = 1, 2,\ i = 1, \ldots, n-1$, such that there holds the following,

$$\max\left(\bar{\gamma}_{1,n}(.), \bar{\gamma}_{2,n}(.)\right) - \sum_{j=1}^{2}\sum_{i=1}^{n-1}\frac{\bar{\gamma}_{j,i}(.)\left|\beta_{j,i}\right|}{\alpha_{j,i}} < 0. \quad (41)$$

Proof: Consider the matrix $\overline{\Gamma}(.)$ given in (42), as shown at the bottom of the next page, with $\bar{\gamma}_{j,i}(.) = \gamma_{j,i} + |\sup_{\mathcal{D}}\delta^1_{j,i}(.)| + |\delta^2_{j,i}|,\ \forall i = 1, \ldots, n-1,\ j = 1, 2,\ \bar{\gamma}_{j,n}(.) = \gamma_{j,n} + |\sup_{\mathcal{D}}\delta^1_{j,n}(.)| + |\delta^2_{j,n}|$. Since $\alpha_{k,i}, k, i = 1, \ldots, n-1$, are arbitrary, we choose $\alpha_{k,i} < 0$ with $\alpha_{k,i} \neq \alpha_{\ell,j},\ \forall k, \ell = 1, 2$, $\forall i, k = 1, \ldots, n-1$, such that $\overline{\Gamma}(.)$ is a Λ-matrix. Thus, it follows from Lemma 2 that if (41) holds true, there exists a constant vector $\overline{\rho} > 0$ such that $\overline{\Gamma}(.)\overline{\rho} < \overline{\eta}$ for $\overline{\eta} < 0$. Hence, we choose the radially unbounded, positive definite Lyapunov function candidate as

$$\overline{V}(t) = \sum_{i=1}^{2n-2}\overline{\rho}_i|Z_i(t)| + \overline{\rho}_{2n-1}\left(v_{2n-1}(t) + v_{2n}(t)\right), \quad (43)$$

with

$$\begin{aligned} v_{2n-1}(t) = |Z_{2n-1}(t)| &+ \sum_{i=1}^{n-1}\sup_{\mathcal{D}}\left|\delta^1_{1,i}(.)\right|\int_{t-\bar{d}_1}^{t}|Z_i(\nu)|d\nu \\ &+ \sup_{\mathcal{D}}\left|\delta^1_{1,n}(.)\right|\int_{t-\bar{d}_1}^{t}|Z_{2n-1}(\nu)|d\nu \\ &+ \left|\delta^2_{2,n-1}\right|\int_{t-\bar{d}_2}^{t}|Z_n(\nu)|d\nu \\ &+ \sum_{i=1}^{n-2}\left|\delta^2_{2,i}\right|\int_{t-\bar{d}_2}^{t}|Z_{n+i}(\nu)|d\nu \\ &+ \left|\delta^2_{2,n}\right|\int_{t-\bar{d}_2}^{t}|Z_{2n}(\nu)|d\nu, \end{aligned}$$

and

$$v_{2n}(t) = |Z_{2n}(t)| + \sup_{\mathcal{D}} \left|\delta^1_{2,n-1}(.)\right| \int_{t-\bar{d}_1}^{t} |Z_n(\nu)| d\nu + \sum_{i=1}^{n-2} \sup_{\mathcal{D}} \left|\delta^1_{2,i}(.)\right| \int_{t-\bar{d}_1}^{t} |Z_{n+i}(\nu)| d\nu + \sup_{\mathcal{D}} \left|\delta^1_{2,n}(.)\right| \int_{t-\bar{d}_1}^{t} |Z_{2n}(\nu)| d\nu + \sum_{i=1}^{n-1} \left|\delta^2_{1,i}\right| \int_{t-\bar{d}_2}^{t} |Z_i(\nu)| d\nu + \left|\delta^2_{1,n}\right| \int_{t-\bar{d}_2}^{t} |Z_{2n-1}(\nu)| d\nu.$$

The right Dini derivative of $\overline{V}(z(t))$, along the solution of (40), is given by

$$D^+\overline{V}(t) = \sum_{i=1, i\neq n}^{2n-2} \overline{\rho}_i \frac{d^+|Z_i(t)|}{dt^+} + \overline{\rho}_n \frac{d^+|Z_n(t)|}{dt^+} + \overline{\rho}_{2n-1}\left(\frac{d^+ v_{2n-1}(t)}{dt^+} + \frac{v_{2n}(t)}{dt^+}\right).$$

It is seen that

$$\begin{aligned} \frac{d^+|Z_i(t)|}{dt^+} &= sign\,(Z_i(t))\, \frac{d^+ Z_i(t)}{dt^+} \\ &= sign\,(Z_i(t))\left(\alpha_{1,i} Z_i(t) + \beta_{1,i} Z_{2n-1}(t)\right) \\ &\leq \alpha_{1,i}\,|Z_i(t)| + \left|\beta_{1,i}\right| |Z_{2n-1}(t)| \\ &\qquad i = 1, \ldots, 2n-2,\ i \neq n, \end{aligned} \tag{44}$$

$$\begin{aligned} \frac{d^+|Z_n(t)|}{dt^+} &= sign(Z_n(t))\left(\alpha_{2,n-1} Z_n(t) + \beta_{2,n-1} Z_{2n}(t)\right) \\ &\leq \alpha_{2,n-1}\,|Z_n(t)| + \left|\beta_{2,n-1}\right| |Z_{2n}(t)|, \end{aligned} \tag{45}$$

$$\begin{aligned} \frac{d^+ v_{2n-1}(t)}{dt^+} &= \frac{d^+\,|Z_{2n-1}(t)|}{dt^+} + \sum_{i=1}^{n-1} \sup_{\mathcal{D}} \left|\delta^1_{1,i}(.)\right| (|Z_i(t)| \\ &\quad - \left|Z_i(t-\bar{d}_1)\right|) + \sup_{\mathcal{D}} \left|\delta^1_{1,n}(.)\right| (|Z_{2n-1}(t)| \\ &\quad - \left|Z_{2n-1}(t-\bar{d}_1)\right|) + \left|\delta^2_{2,n-1}\right| (|Z_n(t)| \\ &\quad - \left|Z_n(t-\bar{d}_2)\right|) + \sum_{i=1}^{n-2} \left|\delta^2_{2,i}\right| (|Z_{n+i}(t)| \\ &\quad - \left|Z_{n+i}(t-\bar{d}_2)\right|) + \left|\delta^2_{2,n}\right| (|Z_{2n}(t)| \\ &\quad - \left|Z_{2n}(t-\bar{d}_2)\right|), \end{aligned} \tag{46}$$

$$\begin{aligned} \frac{d^+ v_{2n}(t)}{dt^+} &= \frac{d^+\,|Z_{2n}(t)|}{dt^+} + \sup_{\mathcal{D}} \left|\delta^1_{2,n-1}(.)\right| (|Z_n(t)| \\ &\quad - \left|Z_n(t-\bar{d}_1)\right|) + \sum_{i=1}^{n-2} \sup_{\mathcal{D}} \left|\delta^1_{2,i}(.)\right| (|Z_{n+i}(t)| \\ &\quad - \left|Z_{n+i}(t-\bar{d}_1)\right|) + \sup_{\mathcal{D}} \left|\delta^1_{2,n}(.)\right| (|Z_{2n}(t)| \\ &\quad - \left|Z_{2n}(t-\bar{d}_1)\right|) + \sum_{i=1}^{n-1} \left|\delta^2_{1,i}\right| (|Z_i(t)| \\ &\quad - \left|Z_i(t-\bar{d}_2)\right|) + \left|\delta^2_{1,n}\right| (|Z_{2n-1}(t)| \\ &\quad - \left|Z_{2n-1}(t-\bar{d}_2)\right|). \end{aligned} \tag{47}$$

One sees that

$$\begin{aligned} &\frac{d^+\,|Z_{2n-1}(t)|}{dt^+} \\ &= \Bigg(\sum_{\ell=1}^{n-1} \gamma_{1,\ell} Z_\ell(t) + \gamma_{1,n} Z_{2n-1}(t) \\ &\quad \sum_{\ell=1}^{n-1} \delta^1_{1,\ell}(.) Z_\ell(t-\bar{d}_1) + \delta^1_{1,n}(.) Z_{2n-1}(t-\bar{d}_1) \\ &\quad + \delta^2_{2,n-1} Z_n(t-\bar{d}_2) + \sum_{\ell=1}^{n-2} \delta^2_{2,\ell} Z_{n+\ell}(t-\bar{d}_2) \\ &\quad + \delta^2_{2,n} Z_{2n}(t-\bar{d}_2)\Bigg)\, sign(Z_{2n-1}(t)) \end{aligned} \tag{48}$$

$$\leq \sum_{\ell=1}^{n-1} |\gamma_{1,\ell}||Z_\ell(t)| + \gamma_{1,n}|Z_{2n-1}(t)|$$

$$\overline{\Gamma}(.) = \begin{bmatrix} \alpha_{1,1} & 0 & \cdots & \cdots & \cdots & 0 & \left|\beta_{1,1}\right| \\ 0 & \ddots & & & & \vdots & \vdots \\ \vdots & \alpha_{1,n-1} & \ddots & & & \vdots & \left|\beta_{1,n-1}\right| \\ \vdots & \ddots & \alpha_{2,n-1} & \ddots & & \vdots & \left|\beta_{2,n-1}\right| \\ \vdots & & & \alpha_{2,1} & \ddots & \vdots & \left|\beta_{2,1}\right| \\ \vdots & & \ddots & & \ddots & 0 & \vdots \\ & & & \ddots & & & \\ 0 & \cdots & \cdots & \cdots & \cdots\ \ 0 & \alpha_{2,n-2} & \left|\beta_{2,n-2}\right| \\ \overline{\gamma}_{1,1}(.) & \cdots\ \ \overline{\gamma}_{1,n-1}(.) & \overline{\gamma}_{2,n-1}(.) & \cdots & \overline{\gamma}_{2,n-2}(.) & \overline{\gamma}_{2,1}(.) & \max\left\{\overline{\gamma}_{1,n}(.), \overline{\gamma}_{2,n}(.)\right\} \end{bmatrix} \tag{42}$$

$$\sum_{\ell=1}^{n-1}|\delta_{1,\ell}^1(.)||Z_\ell(t-\bar{d}_1)|+|\delta_{1,n}^1(.)||Z_{2n-1}(t-\bar{d}_1)|$$
$$+|\delta_{2,n-1}^2||Z_n(t-\bar{d}_2)|+\sum_{\ell=1}^{n-2}|\delta_{2,\ell}^2||Z_{n+\ell}(t-\bar{d}_2)|$$
$$+|\delta_{2,n}^2||Z_{2n}(t-\bar{d}_2)|, \quad (49)$$

and

$$\frac{d^+|Z_{2n}(t)|}{dt^+} = \Bigg(\gamma_{2,n-1}Z_n(t)+\sum_{\ell=1}^{n-2}\gamma_{2,\ell}Z_\ell(t) +\gamma_{2,n}Z_{2n}(t)+\delta_{2,n-1}^1(.)Z_n(t-\bar{d}_1) +\sum_{\ell=1}^{n-2}\delta_{2,\ell}^1(.)Z_{n+\ell}(t-\bar{d}_1)+\delta_{2,n}^1(.)Z_{2n}(t-\bar{d}_1) +\sum_{\ell=1}^{n-2}\delta_{1,\ell}^2Z_\ell(t-\bar{d}_2)+\delta_{1,n}^2Z_{2n-1}(t-\bar{d}_2)\Bigg) \times sign(Z_{2n}(t))$$
$$\leq|\gamma_{2,n-1}||Z_n(t)|+\sum_{\ell=1}^{n-2}|\gamma_{2,\ell}||Z_\ell(t)|+\gamma_{2,n}|Z_{2n}(t)|$$
$$|\delta_{2,n-1}^1(.)||Z_n(t-\bar{d}_1)|+\sum_{\ell=1}^{n-2}|\delta_{2,\ell}^1(.)||Z_{n+\ell}(t-\bar{d}_1)|$$
$$+\sum_{\ell=1}^{n-2}|\delta_{1,\ell}^2||Z_\ell(t-\bar{d}_2)|+|\delta_{2,n}^1(.)||Z_{2n}(t-\bar{d}_1)|$$
$$+|\delta_{1,n}^2||Z_{2n-1}(t-\bar{d}_2)|. \quad (50)$$

The use of (48) in (46) yields

$$\frac{d^+v_{2n-1}(t)}{dt^+}\leq\sum_{i=1}^{n-1}\left(|\gamma_{1,i}|+\sup_{\mathcal{D}}\left|\delta_{1,i}^1(.)\right|\right)|Z_i(t)| +\left(\gamma_{1,n}+\sup_{\mathcal{D}}\left|\delta_{1,n}^1(.)\right|\right)|Z_{2n-1}(t)| +\left|\delta_{2,n-1}^2\right||Z_n(t)|+\sum_{i=1}^{n-2}\left|\delta_{2,i}^2\right||Z_{n+i}(t)| +\left|\delta_{2,n}^2\right||Z_{2n}(t)|, \quad (51)$$

and the use of (50) in (47) yields

$$\frac{d^+v_{2n}(t)}{dt^+}\leq\sum_{i=1}^{n-2}\left(|\gamma_{2,i}|+\sup_{\mathcal{D}}\left|\delta_{2,i}^1(.)\right|\right)|Z_{n+i}(t)| +\left(|\gamma_{2,n-1}|+\sup_{\mathcal{D}}\left|\delta_{2,n-1}^1(.)\right|\right)|Z_n(t)| +\sum_{i=1}^{n-2}\left|\delta_{1,i}^2\right||Z_i(t)|+\left|\delta_{1,n}^2\right||Z_{2n-1}(t)| +\left(\gamma_{2,n}+\sup_{\mathcal{D}}\left|\delta_{2,n}^1(.)\right|\right)|Z_{2n}(t)|. \quad (52)$$

Let $\overline{Z} = \left[|Z_1|, \ldots, |Z_{2n-2}|, (|Z_{2n-1}|+|Z_{2n}|)\right]^T$. It then follows from (44), (45), (51) and (52) that

$$D^+\overline{V}(t)\leq\overline{Z}^T(t)\overline{\Gamma}(.)\overline{\rho}<\overline{Z}^T(t)\overline{\eta}<\sum_{i=1}^{2n-1}\overline{Z}_i\overline{\eta}_i<0.$$

The proof is completed. ■

When no delay exists in the feedback in Fig. 1, that is, $d_j=0, j=1,2$, Theorem 2 can be much simplified.

Corollary 3: The closed-loop system defined by (37) is asymptotically stable if there exist distinct $\alpha_{j,i}<0$, $j=1,2$, $i=1,\ldots,n-1$, such that there holds the following,

$$\max_{1\leq j\leq 2}\{\bar{\gamma}_{j,n}(.)\}-\sum_{j=1}^{2}\sum_{i=1}^{n-1}\frac{|\bar{\gamma}_{j,i}(\varepsilon_j)|\,|\beta_{j,i}|}{\alpha_{j,i}}<0. \quad (53)$$

Proof: From Fig.1, when there is no delay in the feedback of each subsystem, the system (35) changes to

$$\dot{x}_j(t)=\tilde{A}_j(\varepsilon_j)x_j(t)+\tilde{B}_{j-(-1)^j}x_{j-(-1)^j}(t-\tau_{j-(-1)^j}) \quad (54)$$

where

$$\tilde{A}_j(.)=\left[A_j-B_j\tilde{\varphi}_j(.)C_j^T\right] = \begin{bmatrix}0&1&0&0\\ \vdots&\ddots&\ddots&0\\ 0&\cdots&0&1\\ -\tilde{a}_0^j(.)&\cdots&-\tilde{a}_{n-2}^j(.)&-\tilde{a}_{n-1}^j(.)\end{bmatrix}, \quad (55)$$

$$\tilde{B}_j=B_jr_{j-(-1)^j,j}C_{j-(-1)^j}^T = \begin{bmatrix}0&\cdots&0&0\\ \vdots&&\vdots&\vdots\\ 0&\cdots&0&0\\ -\tilde{b}_0^j&\cdots&-\tilde{b}_{n-2}^j&-\tilde{b}_{n-1}^j\end{bmatrix}, \quad (56)$$

with $\tilde{a}_i^j(.)=\frac{d_i^j}{d_n^j}+b_jc_i^j\tilde{\varphi}_j(.)$ and $\tilde{b}_i^j=r_{j,j-(-1)^j}b_{j-(-1)^j}c_i^j$.

Apply the same state transformation as in (38), the whole system becomes

$$\dot{Z}(t)=\begin{bmatrix}F_1(.)&0\\0&F_2(.)\end{bmatrix}Z(t) +\begin{bmatrix}0&\check{B}_1\\\check{B}_2&0\end{bmatrix}Z(t-\tau), \quad (57)$$

where

$$Z(t-\tau)=\left[z_1^T(t-\tau_1),\ z_2^T(t-\tau_2)\right]^T,$$
$$F_j(.)=P_j^{-1}\tilde{A}_j(.)P_j$$

$$= \begin{bmatrix} \alpha_{j,1} & 0 & \dots & 0 & \beta_1^j \\ 0 & \alpha_{j,2} & \ddots & \vdots & \beta_2^j \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \alpha_{j,n-1} & \beta_{n-1}^j \\ \gamma_{j,1}(.) & \dots & \dots & \gamma_{j,n-1}(.) & \gamma_{j,n}(.) \end{bmatrix},$$

$$\check{B}_j = P_j^{-1}\tilde{B}_j P_{j-(-1)^j}$$

$$= \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ \delta_{j,1} & \dots & \delta_{j,n-1} & \delta_{j,n} \end{bmatrix},$$

with, $j = 1, 2,$

$$\begin{aligned} \gamma_{j,i}(.) &= -p_{A_j(.)}(\alpha_i), \\ \delta_{j,i} &= -p_{B_j}(\alpha_i), \\ & \quad i = 1, \dots, \ n-1; \end{aligned}$$

$$\begin{aligned} \gamma_{j,n}(.) &= -\tilde{a}_{n-1}^j(.) - \sum_{i=1}^{n-1} \alpha_{j,i}, \\ \delta_{j,n} &= -\tilde{b}_{n-1}^j, \\ p_{A_j(.)}(s) &= s^n + \sum_{i=0}^{n-1} \tilde{a}_i^j(\varepsilon_j) s^i, \\ p_{B_j}(s) &= \sum_{i=0}^{n-1} \tilde{b}_i^j s^i. \end{aligned}$$

The new state space representation is given by

$$\dot{Z}(t) = \Upsilon(\varepsilon_1, \varepsilon_2) Z(t) + \Xi Z(t-\tau), \tag{58}$$

where

$$\Upsilon(.) = \begin{bmatrix} \Upsilon_{1,1} & 0 & \Upsilon_{1,3} \\ 0 & \Upsilon_{2,2} & \Upsilon_{2,3} \\ \Upsilon_{3,1}(\varepsilon_1) & \Upsilon_{3,2}(\varepsilon_2) & \Upsilon_{3,3}(\varepsilon_1, \varepsilon_2) \end{bmatrix},$$

$$\Xi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Xi_{3,1} & \Xi_{3,2} & \Xi_{3,3} \end{bmatrix},$$

with

$$\begin{aligned} \Upsilon_{1,1} &= \text{diag}\left(\alpha_{1,1} \quad \alpha_{1,2} \quad \dots \alpha_{1,n-1}\right), \\ \Upsilon_{2,2} &= \text{diag}\left(\alpha_{2,n-1} \quad \alpha_{2,1} \quad \dots \quad \alpha_{2,n-2}\right) \\ \Upsilon_{1,3} &= \begin{bmatrix} \beta_1^1 & \beta_2^1 & \dots & \beta_{n-1}^1 \\ 0 & 0 & \dots & 0 \end{bmatrix}^T, \\ \Upsilon_{2,3} &= \begin{bmatrix} 0 & 0 & \dots & 0 \\ \beta_{n-1}^2 & \beta_1^2 & \dots & \beta_{n-2}^2 \end{bmatrix}^T, \\ \Upsilon_{3,1}(.) &= \begin{bmatrix} \gamma_{1,1}(\varepsilon_1) & \gamma_{1,2}(\varepsilon_1) & \dots & \gamma_{1,n-1}(\varepsilon_1) \\ 0 & 0 & \dots & 0 \end{bmatrix}, \\ \Upsilon_{3,2}(.) &= \begin{bmatrix} 0 & 0 & \dots & 0 \\ \gamma_{2,n-1}(\varepsilon_2) & \gamma_{2,1}(\varepsilon_2) & \dots & \gamma_{2,n-2}(\varepsilon_2) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \Upsilon_{3,3}(.) &= \begin{bmatrix} \gamma_{1,n}(\varepsilon_1) & 0 \\ 0 & \gamma_{2,n}(\varepsilon_2) \end{bmatrix}, \\ \Xi_{3,1} &= \begin{bmatrix} 0 & 0 & \dots & 0 \\ \delta_{1,1} & \delta_{1,2} & \dots & \delta_{1,n-1} \end{bmatrix}, \\ \Xi_{3,2} &= \begin{bmatrix} \delta_{2,n-1} & \delta_{2,1} & \dots & \delta_{2,n-2} \\ 0 & 0 & \dots & 0 \end{bmatrix}, \\ \Xi_{3,3} &= \begin{bmatrix} 0 & \delta_{2,n} \\ \delta_{1,n} & 0 \end{bmatrix}. \end{aligned}$$

In this case, we get

$$\begin{aligned} \bar{\gamma}_{j,i}(.) &= \left|\gamma_{j,i}(\varepsilon_j)\right| + \left|\delta_{j,i}\right|, \ \forall i = 1, \dots, n-1, \\ \bar{\gamma}_{j,n}(.) &= \gamma_{j,n}(\varepsilon_j) + \left|\delta_{j-(-1)^j,n}\right|, \ j = 1, 2. \end{aligned}$$

We choose $\alpha_{k,i} < 0$ with $\alpha_{k,i} \neq \alpha_{\ell,j}$, $\forall k, \ell = 1, 2$, $\forall i, k = 1, \dots, n-1$, such that (53) holds true. Then $\overline{\Gamma}(.)$ is a Λ-matrix, and Lemma 2 is invoked. The rest of the proof follows the similar arguments to the proof of Theorem 2 and is omitted. The proof is completed. ■

IV. EXAMPLES

We illustrate, in this section, some numerical examples together with their simulations in order to show the validity and enumerate the benefits of the developed approaches.

Example 1: Consider the example in [64] (page 69) with added complexity: the system have time variable delay $\tau(t)$

$$\begin{aligned} \dot{x}(t) = &\begin{bmatrix} -2 + g(t) & g(t) \\ g(t) & -0.9 + g(t) \end{bmatrix} x(t) \\ &+ \beta \begin{bmatrix} -1 + g(t) & 0 \\ -1 & -1 - g(t) \end{bmatrix} x(t - \tau(t)), \end{aligned} \tag{59}$$

where $x(t) \in R^2$, $\beta \in R$ and $|g(t)| \leq 0.1$.

The condition (7) of Theorem 1 becomes

$$\dot{\tau}(t) \leq 1 - \max\left\{\frac{|1|}{1.1}, \frac{|1 + g(t)|}{1.1}\right\}.$$

In this case, the matrix $\Gamma(.)$ is evaluated as

$$\Gamma(.) = \begin{bmatrix} \alpha(.) & |g(t)| + |\beta| \\ |g(t)| & -0.9 + g(t) + 1.1|\beta| \end{bmatrix}$$

where $\alpha(.) = -2 + g(t) + 1.1|\beta|$. By Theorem 1, the stability condition is given by

$$-0.9 + g(t) + 1.1|\beta| - \frac{|g(t)|^2 + |g(t)||\beta|}{-2 + g(t) + 1.1|\beta|} < 0. \tag{60}$$

Knowing that $|g(t)| \leq 0.1$ then (60) holds true if $|\beta| < \frac{1.9}{1.1}$. In fact, we can verify that $\alpha(.) = -2 + g(t) + 1.1|\beta| < 0$, and (60) becomes

$$|\beta| \leq 0.667.$$

However, the authors, in [64], obtained by solving LMIs a more restrictive result: the system is asymptotically stable and independent of delays for $|\beta| \leq 0.644$.

Example 2: Consider the system in Fig.1 with second-order dynamics for each plant,

$$A_j = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\mu_{j,1}\mu_{j,2}} & -\frac{\mu_{j,1}+\mu_{j,2}}{\mu_{j,1}\mu_{j,2}} \end{bmatrix},$$
$$B_j = \begin{bmatrix} 0 \\ \frac{k_j}{\mu_{j,1}\mu_{j,2}} \end{bmatrix} = \begin{bmatrix} 0 \\ b_j \end{bmatrix},$$
$$C_j^T = \begin{bmatrix} \lambda_j & 1 \end{bmatrix}, \ \lambda_j > 0,$$
$$C_j^T(sI_2 - A_j)^{-1}B_j = \frac{k_j(s+\lambda_j)}{(\mu_{j,1}s+1)(\mu_{j,2}s+1)},$$
$$\tilde{B}_j^1(.) = -\tilde{\varphi}_j(.)b_j \begin{bmatrix} 0 & 0 \\ \lambda_j & 1 \end{bmatrix},$$
$$\tilde{B}_j^2 = -r_{j-(-1)^j,j}b_j \begin{bmatrix} 0 & 0 \\ \lambda_{j-(-1)^j} & 1 \end{bmatrix}.$$

The choice of $\alpha_{1,1} = -\lambda_1$ and $\alpha_{2,1} = -\lambda_2$, leads to

$$F_j = \begin{bmatrix} -\lambda_j & 1 \\ -p_{A_j}(-\lambda_j) & \lambda_j - \frac{\mu_{j,1}+\mu_{j,2}}{\mu_{j,1}\mu_{j,2}} \end{bmatrix},$$
$$\check{B}_j^1(.) = \begin{bmatrix} 0 & 0 \\ 0 & -b_j\tilde{\varphi}_j(\varepsilon_j) \end{bmatrix},$$
$$\check{B}_j^2 = \begin{bmatrix} 0 & 0 \\ 0 & -r_{j-(-1)^j,j}b_j \end{bmatrix},$$
$$\overline{\Gamma}(.) = \begin{bmatrix} -\lambda_1 & 0 & 1 \\ 0 & -\lambda_2 & 1 \\ |p_{A_1}(-\lambda_1)| & |p_{A_2}(-\lambda_2)| & \max\limits_{1\le j\le 2}\{\overline{\gamma}_{j,2}(.)\} \end{bmatrix},$$

where

$$\overline{\gamma}_{j,2}(.) = \lambda_j - \frac{\mu_{j,1}+\mu_{j,2}}{\mu_{j,1}\mu_{j,2}} + \sup_{\mathcal{D}}|b_j\tilde{\varphi}_j(\varepsilon_j)| + |-r_{j-(-1)^j,j}b_j|.$$

By Theorem 2, the stability condition in this particular case is given by

$$\max_{1\le j\le 2}\{\overline{\gamma}_{j,2}(.)\} - \frac{|p_{A_1}(-\lambda_1)|}{-\lambda_1} - \frac{|p_{A_2}(-\lambda_2)|}{-\lambda_2} < 0. \quad (61)$$

For simulation, we consider a numerical case with

$$C_1^T(sI_2 - A_1)^{-1}B_1 = \frac{0.2(s+0.75)}{(1+1.2s)(1+0.2s)},$$
$$C_2^T(sI_2 - A_2)^{-1}B_2 = \frac{0.5(s+1)}{(1+1.5s)(1+0.3s)},$$
$$r_{1,2} = 0.5, \ r_{2,1} = -1.$$

Then, (61) becomes

$$\max\left\{-4.25 + 0.83\sup_{\mathcal{D}}|\tilde{\varphi}_1(.)|, -2.44 + 1.11\sup_{\mathcal{D}}|\tilde{\varphi}_2(.)|\right\} - \frac{|-0.3542|}{-0.75} - \frac{|0.7778|}{-1} < 0,$$

that is,

$$\max\left\{-4.25 + 0.83\sup_{\mathcal{D}}|\tilde{\varphi}_1(.)|, -2.44 + 1.11\sup_{\mathcal{D}}|\tilde{\varphi}_2(.)|\right\} + 1.25 < 0.$$

Let

$$-4.25 + 0.83\sup_{\mathcal{D}}|\tilde{\varphi}_1(.)| < -2.44 + 1.11\sup_{\mathcal{D}}|\tilde{\varphi}_2(.)| < 0.$$

This yields the stability region determined by

$$\begin{cases} 2.1667 < \sup_{\mathcal{D}}|\tilde{\varphi}_1(.)| < 3.60, \\ -1.6250 + 0.75\sup_{\mathcal{D}}|\tilde{\varphi}_1(.)| < \sup_{\mathcal{D}}|\tilde{\varphi}_2(.)| < 2.2. \end{cases}$$

These inequalities define the stability region ($\sup_{\mathcal{D}}|\tilde{\varphi}_1(.)|, _{\mathcal{D}}|\tilde{\varphi}_2(.)|$) which are illustrated in Fig.2.

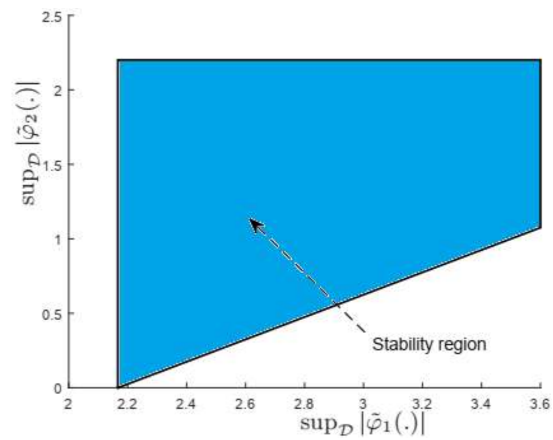

FIGURE 2. Stability region of the nonlinearities ($\sup_{\mathcal{D}}|\tilde{\varphi}_1(.)|$, $\sup_{\mathcal{D}}|\tilde{\varphi}_2(.)|$) for particular values of $\alpha_{1,1} = -0.75$ and $\alpha_{2,1} = -1$.

Example 3: Consider the system in Fig.1 with second-order dynamics for each plant,

$$A_j = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\mu_{j,1}\mu_{j,2}} & -\frac{\mu_{j,1}+\mu_{j,2}}{\mu_{j,1}\mu_{j,2}} \end{bmatrix},$$
$$B_j = \begin{bmatrix} 0 \\ \frac{k_j}{\mu_{j,1}\mu_{j,2}} \end{bmatrix} = \begin{bmatrix} 0 \\ b_j \end{bmatrix},$$
$$C_j^T = \begin{bmatrix} \lambda_j & 1 \end{bmatrix}, \ \lambda_j > 0,$$

$$C_j^T(sI_2 - A_j)^{-1}B_j = \frac{k_j(s+\lambda_j)}{(\mu_{j,1}s+1)(\mu_{j,2}s+1)},$$
$$\tilde{A}_j(.) = A_j - B_j\tilde{\varphi}_j(.)C_j^T,$$
$$\tilde{B}_j = -r_{j-(-1)^j,j}b_j \begin{bmatrix} 0 & 0 \\ \lambda_{j-(-1)^j} & 1 \end{bmatrix}.$$

The choice of $\alpha_{1,1} = -\lambda_1$ and $\alpha_{2,1} = -\lambda_2$, leads to

$$A_j(.) = \begin{bmatrix} -\lambda_j & 1 \\ -p_{A_j}(-\lambda_j) & \lambda_j - \frac{(\mu_{j,1}+\mu_{j,2})+k_j\tilde{\varphi}_j(\varepsilon_j)}{\mu_{j,1}\mu_{j,2}} \end{bmatrix},$$
$$\check{B}_j = \begin{bmatrix} 0 & 0 \\ 0 & -r_{j-(-1)^j,j}b_j \end{bmatrix},$$
$$\overline{\Gamma}(.) = \begin{bmatrix} -\lambda_1 & 0 & 1 \\ 0 & -\lambda_2 & 1 \\ |p_{A_1}(-\lambda_1)| & |p_{A_2}(-\lambda_2)| & \max\limits_{1\le j\le 2}\{\overline{\gamma}_{j,2}(.)\} \end{bmatrix},$$

where

$$\overline{\gamma}_{j,2}(.) = |-r_{j-(-1)^j,j}b_j| + \lambda_j - \frac{(\mu_{j,1}+\mu_{j,2}) + k_j\tilde{\varphi}_j(\varepsilon_j)}{\mu_{j,1}\mu_{j,2}}.$$

By Corollary 3, the stability condition in this particular case is given by

$$\max_{1\le j\le 2}\left\{\overline{\gamma}_{j,2}(.)\right\} - \frac{|p_{A_1}(-\lambda_1)|}{-\lambda_1} - \frac{|p_{A_2}(-\lambda_2)|}{-\lambda_2} < 0. \quad (62)$$

For simulation, we consider two numerical cases.

Case 1. Suppose two stable subsystems,

$$C_1^T(sI_2-A_1)^{-1}B_1 = \frac{s+1.5}{(1+s)(1+0.5s)},$$
$$C_2^T((sI_2-A_2)^{-1}B_2 = \frac{0.5(s+3)}{(1+1.25s)(1+0.25s)},$$
$$r_{1,2}=5,\ r_{2,1}=-1.$$

Then, (62) becomes

$$\max\{0.5-2\tilde{\varphi}_1(.), 6.2-1.6\tilde{\varphi}_2(.)\} - \frac{|-0.25|}{-1.5} - \frac{|-2.2|}{-3} < 0,$$

that is,

$$\max\{0.5-2\tilde{\varphi}_1(.), 6.2-1.6\tilde{\varphi}_2(.)\} + 0.9 < 0.$$

Let

$$0.5-2\tilde{\varphi}_1(.) < 6.2-1.6\tilde{\varphi}_2(.) < 0.$$
$$\begin{cases}\tilde{\varphi}_1(.) > 0.7,\\ 4.4375 < \tilde{\varphi}_2(.) < 3.5625+1.25\tilde{\varphi}_1(.),\end{cases}$$

This yields the stability region $(\tilde{\varphi}_1(.), \tilde{\varphi}_2(.))$ which are illustrated in Fig.3.

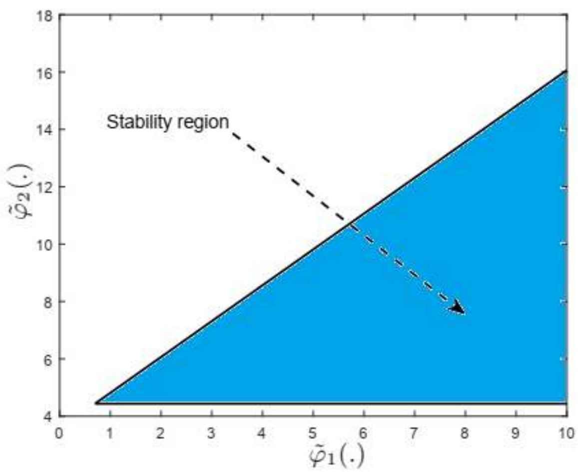

FIGURE 3. Stability region of the nonlinearities $(\tilde{\varphi}_1(.), \tilde{\varphi}_2(.))$ for particular values of $\alpha_{1,1} = -1.5$ and $\alpha_{2,1} = -3$.

Case 2. Suppose one of the subsystems is unstable,

$$C_1^T(sI_2-A_1)^{-1}B_1 = \frac{10(s+2.5)}{(1-s)(1+s)},$$
$$C_2^T(sI_2-A_2)^{-1}B_2 = \frac{1.5(s+1.25)}{(1+2s)(1+1.5s)},$$
$$r_{1,2}=5,\ r_{2,1}=-2.$$

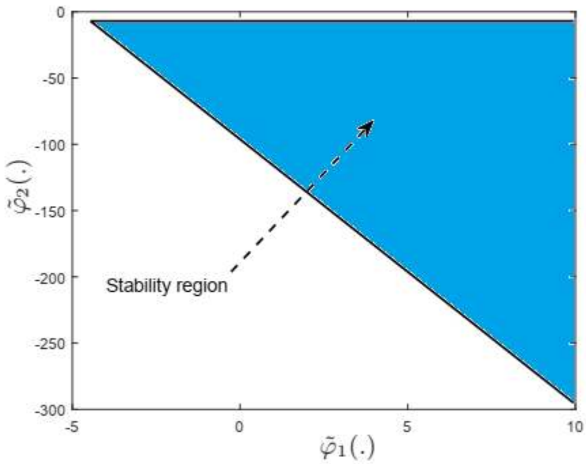

FIGURE 4. Stability region of the nonlinearities $(\tilde{\varphi}_1(.), \tilde{\varphi}_2(.))$ for particular values of $\alpha_{1,1} = -2.5$ and $\alpha_{2,1} = -1.25$.

Then, (62) becomes

$$\max\{-47-10\tilde{\varphi}_1(.), 1.0883+0.5\tilde{\varphi}_2(.)\} + 2.45 < 0.$$

Let

$$-47-10\tilde{\varphi}_1(.) < 1.0883+0.5\tilde{\varphi}_2(.) < 0.$$

This yields the stability region determined by

$$\begin{cases}\tilde{\varphi}_1(.) > -4.455,\\ -96.1766-20\tilde{\varphi}_1(.) < \tilde{\varphi}_2(.) < -7.0766,\end{cases}$$

which are illustrated in Fig.4. *Example 4.* Consider the example in [2] with added complexities: two plants, Σ_1 and Σ_2, have time delay τ and some plant parameters, w_1 and w_2, are nonlinear:

$$\Sigma_1 : \dot{x}_1 = \frac{-x_1+x_2}{1+x_1^2} + w_1(.)x_1(t-\tau),$$
$$\Sigma_2 : \dot{x}_2 = -2x_2 + \frac{x_1^2}{1+x_1^2} + w_2(.)x_2(t-\tau)).$$

In this case, the matrix $\overline{\Gamma}(.)$ is evaluated as

$$\overline{\Gamma}(.) = \begin{pmatrix} \alpha(.) & \dfrac{1}{1+x_1^2} \\ \dfrac{|x_1|}{1+x_1^2} & \sup_{\mathcal{D}}|w_2(.)| - 2 \end{pmatrix}, \quad (63)$$

where $\alpha(.) = \sup_{\mathcal{D}}|w_1(.)| - \dfrac{1}{1+x_1^2}$. By Theorem 2, the stability condition is given by

$$\sup_{\mathcal{D}}|w_2(.)| - 2 - \frac{|x_1|}{(1+x_1^2)^2\alpha(.)} < 0. \quad (64)$$

Note that (64) holds true if $\sup_{\mathcal{D}}|w_1(.)| < \dfrac{(1-|x_1|)^2}{(1+x_1^2)^2}$ and $\sup_{\mathcal{D}}|w_2(.)| < \dfrac{3}{2}$. In fact, we can verify that

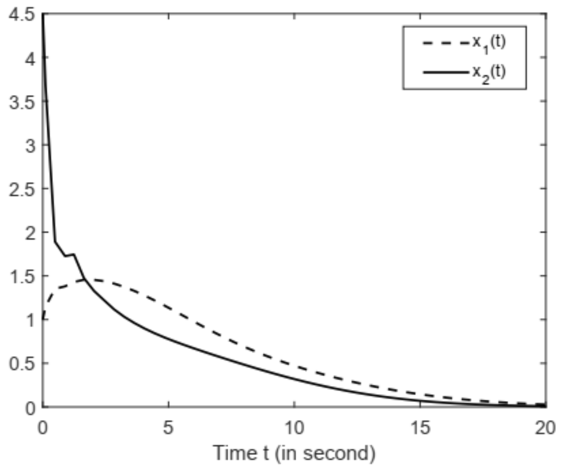

FIGURE 5. State response under conditions $w_1(.) = \frac{(1-|x_1|)^2}{(2+x_1^2)^2}$, $w_2 = 1$ and $\tau = 0.8$ *seconds.*

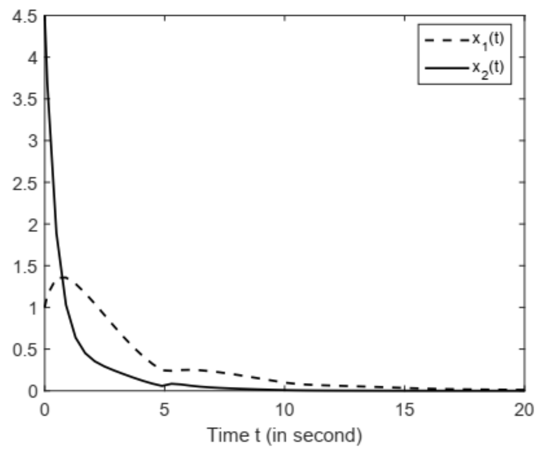

FIGURE 6. State response under conditions $w_1(.) = \frac{(1-|x_1|)^2}{(2+x_1^2)^2}$, $w_2(.) = \frac{1.3}{1+t^2}$ and $\tau = 5$ *seconds.*

$\alpha(.) = \sup_{\mathcal{D}} |w_1(.)| - \frac{1}{1+x_1^2} < \frac{-2|x_1|}{(1+x_1^2)^2} < 0$, and (64) becomes $\sup_{\mathcal{D}} |w_2(.)| - 2 - \frac{|x_1|}{(1+x_1^2)^2\alpha(.)} < \sup_{\mathcal{D}} |w_2(.)| - 2 + \frac{1}{2} < 0$. which yields $\sup_{\mathcal{D}} |w_2(.)| < \frac{3}{2}$. The state variable responses are shown in Fig.5 and Fig.6. *Example 5.* Consider the example in [5] with added complexities: the interconnections between two plants, Σ_1 and Σ_2, have a time delay τ and nonlinearities with γ_1 and γ_2:

$$\Sigma_1 : \dot{x}_1 = -x_1 - x_1^3 + \gamma_1(.)x_2(t-\tau),$$
$$\Sigma_2 : \dot{x}_2 = -x_2 - x_2^3 + \gamma_2(.)x_1(t-\tau).$$

In this case, the matrix $\overline{\Gamma}(.)$ is evaluated as

$$\overline{\Gamma}(.) = \begin{pmatrix} -1-x_1^2 & \sup_{\mathcal{D}} |\gamma_2(.)| \\ \sup_{\mathcal{D}} |\gamma_1(.)| & -1-x_2^2 \end{pmatrix}. \tag{65}$$

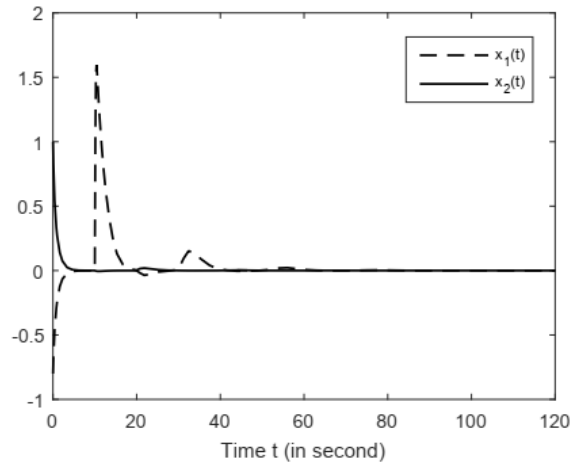

FIGURE 7. State response under conditions $\gamma_1 = 10$, $\gamma_2 = 0.02$ and $\tau = 10$ *seconds.*

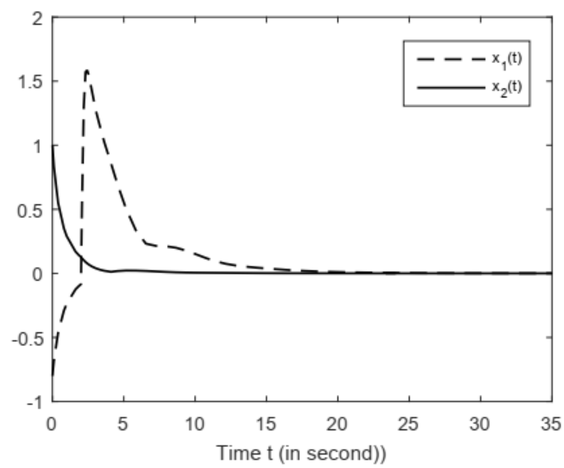

FIGURE 8. State response under conditions $\gamma_1 = 10$, $\gamma_2 = 0.02$ and $\tau = 2$ *seconds.*

By Theorem 2, the stability condition is given by

$$-1 - x_2^2 + \frac{\sup_{\mathcal{D}} |\gamma_1(.)| \sup_{\mathcal{D}} |\gamma_2|(.)}{1+x_1^2} < 0. \tag{66}$$

It can be verified that (66) holds true if

$$\sup_{\mathcal{D}} |\gamma_1(.)| \sup_{\mathcal{D}} |\gamma_2(.)| < 1.$$

The state variable responses are shown in Fig.7 and Fig.8.

V. CONCLUSION

In this work, we have presented the new stability conditions for interconnected nonlinear systems with delay. The conditions are explicit, scalar and easy to check. Indeed, the application of the proposed method to delayed interconnection between two Lur'e Postnikov system shows simplicity and effectiveness. Moreover, our approach is self-contained, and systematic, and it does not go through the M-matrix and the arrow form. Our theorems can deal with time delays, non-linearity and interconnections, and thus have more general applicability than those in the related literature.

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