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Analytically Calculating Membrane Displacement and the Equivalent Circuit Model of a Circular CMUT Cell

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***Abstract*—A small-signal equivalent circuit model and FEM often guide CMUT design. The small-signal model is usually derived using a combination of numerical and FEM analysis. A strictly analytical approach to CMUT design is desired because it provides design intuition and efficient numerical analysis. In this paper, we show that the mass-spring-damper model used for many MEMS structures accurately captures the behavior of a CMUT with a circular plate. We provide equations for the CMUT's equivalent mass-spring-damper parameters, pull-in point, and equivalent circuit parameters. Comparison with FEM shows that the model accurately captures the CMUT's behavior for a wide range of designs. Using this model, we can derive simple design equations, calculate the small-signal model for frequency response simulations, and simulate the CMUT's large-signal transient behavior.**

I. Deriving the Small-Signal Equivalent Circuit Model

For a CMUT with a circular plate (Fig. 1), we can calculate an equivalent spring constant, mass, and damping coefficient that accurately capture the plate's mechanical properties over its entire range of stable detection. To calculate these parameters, we assume a uniform pressure P deflects the plate. The pressure P given by (1) includes the electrical force, F_e, resulting from a voltage applied to the CMUT and the force from atmospheric pressure, P_{atm}.

$$P_0 = P_{atm} + \frac{F_e}{\pi a^2} \tag{1}$$

For a uniform pressure and the assumed membrane geometry, basic plate theory gives (2) for the plate's deflection as a function of radial position, r, plate radius, a, and the plate material's flexural rigidity, D [1].

$$w(r) = \frac{P_0 a^4}{64D}(1 - \frac{r^2}{a^2})^2 = w_{pk}(1 - \frac{r^2}{a^2})^2 \tag{2}$$

Flexural rigidity is given by (3), where t is the plate thickness, and E and ν are the plate material's Young's modulus and Poisson ratio, respectively.

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{3}$$

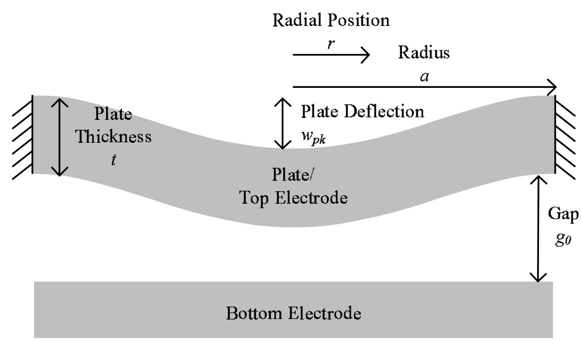


Fig. 1. An ideal circular-plate CMUT.

From (2), we see that the maximum plate deflection, w_{pk}, which occurs at the plate's center ($r = 0$), is given by (4).

$$w_{pk} = \frac{P_0 a^4}{64D} \tag{4}$$

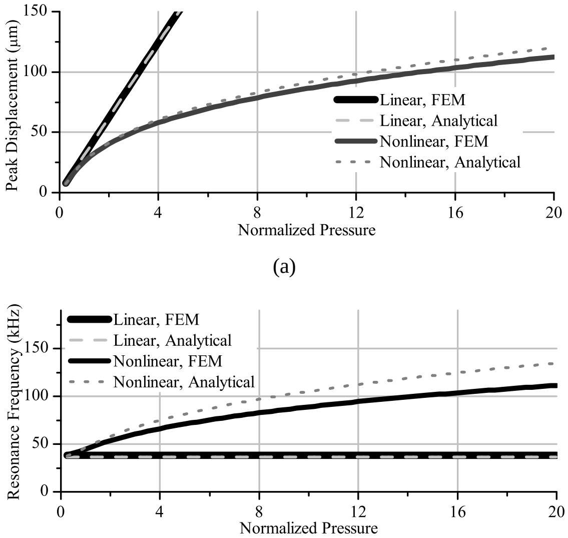
Averaging the deflection over the entire plate area shows that the average plate deflection equals $1/3$ of the peak deflection.

$$w_{avg} = \frac{\int_0^a 2\pi r w(r) dr}{\pi a^2} = \frac{P_0 a^4}{192D} = \frac{w_{pk}}{3} \tag{5}$$

Lohfink and Eccardt [2] refer to the equation for the plate's deflection as the shape function. We assume that the shape function given by (2) for a uniform pressure P holds for all stable deflections despite the nonuniformity of the electrical force—the electrical force is strongest at the plate's center, where the top and bottom electrodes are closest. This assumption simplifies deriving equivalent mechanical parameters and accurately predicts the plate displacement due to an applied voltage.

From the shape function and the CMUT gap, g_0, we can find the CMUT's electrical capacitance, C, as a function of plate displacement.

$$C = \int_0^a \frac{2\pi r \varepsilon_0}{g_0 - w_{pk}(1 - \frac{r^2}{a^2})^2} dr = \frac{\varepsilon_0 \pi a^2 \operatorname{arctanh}(\sqrt{\frac{w_{pk}}{g_0}})}{\sqrt{g_0 w_{pk}}} \tag{6}$$



(a)

(b)

Fig. 2. plate displacement and resonance frequency for different applied uniform pressures ($P_0 = 101$ kPa). (a) When midplane stretching is ignored, the plate displacement increases linearly with the applied pressure. A nonlinear spring constant, k_3, accurately captures the effects of large deflections. (b) The plate's natural resonance frequency. Because the plate's shape changes for large deflections, the effective mass changes resulting in some error in the resonance frequency prediction.

The first and second derivatives of C with respect to w_{avg} are given by (7) and (8).

$$\frac{dC}{dw_{avg}} = C' = \frac{\varepsilon_0 \pi a^2}{2g_0 w_{avg}(1 - \frac{3w_{avg}}{g_0})} - \frac{C}{2w_{avg}} \qquad (7)$$

$$\frac{d^2C}{dw_{avg}^2} = C'' = \frac{3\varepsilon_0 \pi a^2}{2g_0^2 w_{avg}(1 - \frac{3w_{avg}}{g_0})^2} - \frac{\varepsilon_0 \pi a^2}{2g_0 w_{avg}^2(1 - \frac{3w_{avg}}{g_0})} + \frac{C}{2w_{avg}^2} - \frac{1}{2w_{avg}}\frac{dC}{dw_{avg}} \qquad (8)$$

Because the plate's average displacement varies linearly with applied force, we can write the average plate displacement in terms of a linear spring constant, k_1.

$$w_{avg} = P_0 \frac{a^4}{192D} = F_m \frac{a^2}{192\pi D} = \frac{1}{k_1} F_m \qquad (9)$$

$$k_1 = \frac{192\pi D}{a^2} \qquad (10)$$

For deflections small relative to the plate's thickness, the plate's displacement is proportional to the applied force. However, for larger deflections, stretching of the plate's midplane results in a nonlinear relationship between displacement and force. We can capture this midplane stretching using a spring constant that is proportional to the cube of the plate displacement [1].

$$k_3 = D\frac{-24\pi(-896585 - 529610v + 342831v^2)}{29645a^2t^2} \qquad (11)$$

The force produced by the equivalent spring acts on an equivalent mass, m. We can calculate the equivalent mass from the resonance frequency [3].

$$\omega_0 = \sqrt{\frac{k_1}{m}} = \frac{10.22}{a^2\sqrt{\rho t/D}} \qquad (12)$$

$$m = \frac{k_1}{\omega_0^2} = 1.84\pi a^2 t\rho \qquad (13)$$

The final component of the mass-spring-damper system is the damping constant, R_d. The damping constant represents a force that is proportional to velocity; in addition it represents energy loss and mechanical noise. For this paper, we assume the damping resistance is equal to the plane-wave radiation impedance which assumes the transducer is large relative to a wavelength. For a smaller transducer, a complex radiation impedance must be considered [2].

$$R_b = Z_{rad}\pi a^2 \qquad (14)$$

II. Quality Factor and Resonance Frequency

From a design perspective, we often specify transducer requirements in terms of center frequency and bandwidth. From the expressions for k, m, and R_b, we can derive expressions for the CMUT's natural resonance frequency, damped resonance frequency, and quality factor. The natural resonance frequency, given by (12), equals the plate's resonance frequency in the absence of damping. When the quality factor, given by (15), is greater than 0.5, we use the damped resonance frequency, given by (16). If the quality factor is less than 0.5, the system is overdamped and the resonance frequency is undefined. For those designs, we can use the quality factor and natural resonance frequency as starting points and use the small-signal equivalent circuit model to evaluate the frequency response.

$$Q = \frac{m\omega_0}{R_b} = 1.84\frac{t\rho}{R_{med}}\omega_0 \qquad (15)$$

$$\omega_d = \omega_0\sqrt{1 - \frac{R_b^2}{4k_1 m}} \qquad (16)$$

We can use the quality factor, Q, to describe the CMUT's bandwidth. In the frequency domain, $1/Q$, gives the CMUT's fractional bandwidth, which equals the 3-dB bandwidth divided by the center frequency.

III. Voltage Actuation and Pull-In

A voltage applied between the CMUT's top and bottom electrodes, regardless of its polarity, deflects the plate towards the bottom electrode. If the applied voltage is less than the pull-in voltage, the plate deflects to a stable position. Applying a voltage greater than the pull-in voltage causes the plate to snap in contact with the bottom electrode. For CMUTs, the pull-in voltage is often referred to as the collapse voltage. However, for consistency with general MEMS literature, this paper uses the term pull-in.

Using the principal of virtual work, we can calculate the force on the plate created by an applied voltage. This electrical force, F_e, is given by (17) where (7) gives $\frac{dC}{dw_{avg}}$ for a circular plate.

$$F_e = \frac{dU_e}{dw_{avg}} = \frac{1}{2}V^2\frac{dC}{dw_{avg}} \tag{17}$$

The applied voltage usually consists of a dc bias voltage added to an ac excitation voltage. The dc bias voltage increases the CMUT's transmit and receive sensitivity, as described in Section IV, and results in static plate deflection. An ac excitation voltage creates plate motion.

To calculate the plate's static deflection, we find the deflection for which the electrical force equals the mechanical force. If the CMUT's cavity is vacuum sealed, then finding the static plate deflection first requires numerically solving (18) for the atmospheric plate deflection, w_{atm}.

$$\pi a^2 P_{atm} = k_1 w_{atm} + k_3 w_{atm}^3 \tag{18}$$

Using (19), we can find the mechanical restoring force of the displaced plate.

$$F_m = k_1(w_{avg} - w_{atm}) + k_3(w_{avg}^3 - w_{atm}^3) \tag{19}$$

Numerically solving (20) yields the static deflection.

$$F_m(w_{avg}) - F_e(w_{avg}) = 0 \tag{20}$$

If the applied dc voltage is greater than the pull-in voltage, (20) has no solution. A general methodology for finding the pull-in point is given in [4]. Using this methodology, we can find the pull-in point using the expression for the total energy stored in the CMUT's capacitance and equivalent spring.

$$U_{tot} = U_e + U_m \tag{21}$$

$$= \frac{1}{2}CV^2 + \frac{1}{2}k_1(w_{avg} - w_{atm})^2 + \frac{1}{4}k_3(w_{avg} - w_{atm})^4 \tag{22}$$

For a given dc voltage, V_{dc}, the plate deflects to a local minimum in the energy versus deflection curve defined by (21). We can find this minimum energy point by finding the local minimum where the first derivative of (21) with respect to w_{avg} equals zero—note that finding this energy minimum is equivalent to finding the deflection at which the electrical and mechanical forces are equal. For voltages greater than the pull-in voltage, no stable solution exists and the first derivative of U_{tot} never equals zero. As described in [4], when the applied voltage equals the pull-in voltage the energy curve has an inflection point rather than a local energy minimum. We can find the deflection at the pull-in point by solving (23) for w_{avg}, which can rewritten in terms of mass-spring-damper parameters as given by (24).

$$\frac{dU_m}{dw_{avg}}\frac{d^2C}{dw_{avg}^2} - \frac{d^2U_m}{dw_{avg}^2}\frac{dC}{dw_{avg}} = 0 \tag{23}$$

$$-F_m\frac{d^2C}{dw_{avg}^2} + k_{eff}\frac{dC}{dw_{avg}} = 0 \tag{24}$$

From the deflection that satisfies (23), w_{PI}, we use (25) to find the pull-in voltage, V_{PI}.

$$V_c = \sqrt{\frac{2\frac{dU_m(w_{pi})}{dw_{avg}}}{\frac{dC(w_{pi})}{dw_{avg}}}} = \sqrt{\frac{2F_m(w_{pi})}{\frac{dC(w_{pi})}{dw_{avg}}}} \tag{25}$$

By numerically solving (24) and (25), we can find the pull-in voltage and the plate deflection at the pull-in voltage. With some simplifying assumptions, we can also find closed-form analytical solutions for the pull-point. For example, for high-frequency immersion devices we can neglect k_3 and w_{atm}. With these assumptions, we find that at the pull-in point the peak plate deflection equals 46% of the gap and that (29) gives the pull-in voltage.

$$\frac{w_{PI,pk}}{g_0}\,|_{k_3=0,\,P_{atm}=0} = 0.46 \tag{26}$$

$$V_{PI}\,|_{k_3=0,\,P_{atm}=0} = 0.39\sqrt{\frac{g_0^3 k_1}{A\varepsilon_0}} = 0.39\sqrt{\frac{g_0^3 Q\omega_0 R_b}{A\varepsilon_0}} \tag{27}$$

Assuming the expressions for the circular plate pull-in parameters have the same form as the parallel-plate actuator parameters helps derive expressions for the pull-in deflection and pull-in voltage that account for atmospheric pressure.

$$\frac{w_{pk}}{g_0}\,|_{k_3=0} \approx 0.46(1 + 3.55\frac{F_{atm}/k_1}{g_0}) \tag{28}$$

$$V_{PI}\,|_{k_3=0} = \sqrt{\frac{2(k_1 w_{PI} - F_{atm})}{C'(w_{PI})}} \tag{29}$$

By solving (24) and (25) numerically we can find the pull-in point considering both atmospheric deflection and the nonlinear spring constant. A nonzero k_3 increases the pull-in deflection and the pull-in voltage.

IV. Deriving the Small-Signal Equivalent Circuit Model

From equations for the mechanical energy U_m and electrical energy U_e, we can derive an equivalent linearized small-signal equivalent circuit model of the CMUT. A small-signal circuit model provides useful insight about the CMUTs transmit and receive sensitivity and its frequency response. In addition, it provides a convenient way of simulating the small-signal response with tools such as SPICE.

To derive the model, we define the standard two-port model [5]. For small linear variations, the two ports are related as described by (30).

$$\begin{bmatrix} \delta V \\ \delta F \end{bmatrix} = AB = \begin{bmatrix} \left.\frac{\delta V}{\delta Q}\right|_{g=0} & \left.\frac{\delta V}{\delta g}\right|_{Q=0} \\ \left.\frac{\delta F}{\delta Q}\right|_{g=0} & \left.\frac{\delta F}{\delta g}\right|_{Q=0} \end{bmatrix} \begin{bmatrix} \delta Q \\ \delta g \end{bmatrix} \tag{30}$$

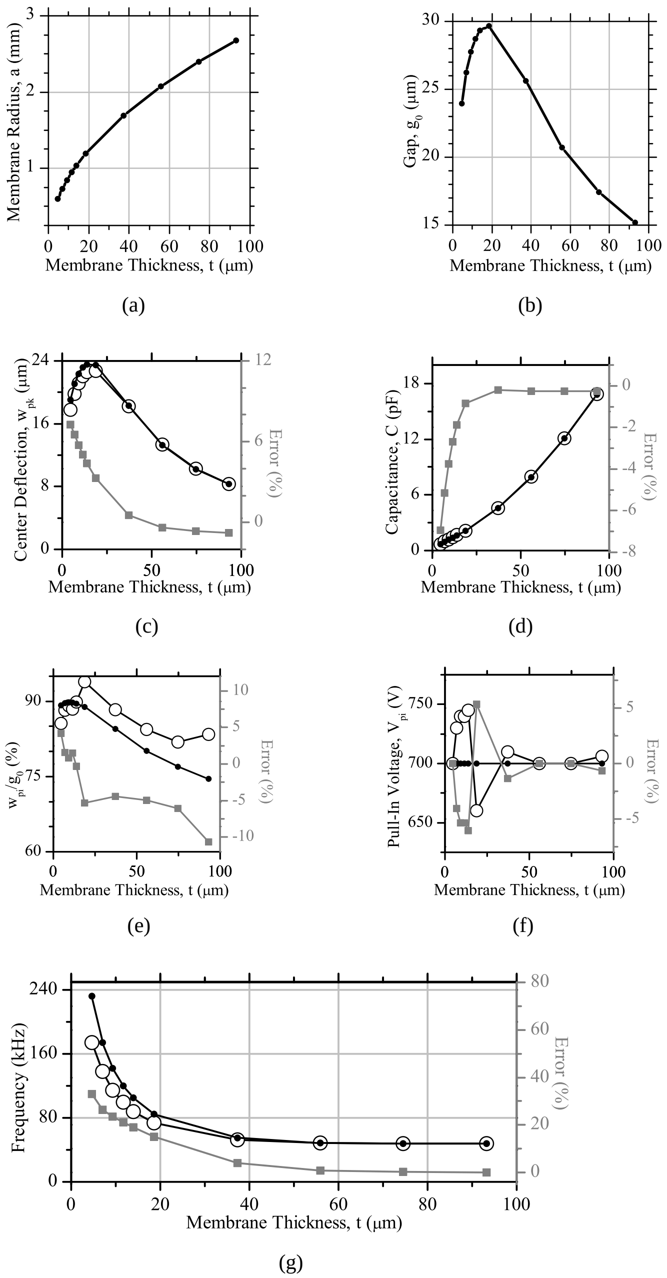


Fig. 3. Comparison of FEM (circles) and spring-mass-damper (dots) model results.

We can write each element of the matrix in terms of the equations for the CMUT's capacitance and spring constant.

$$A_{11} = \frac{\delta V}{\delta Q}\bigg|_g = \frac{d}{dQ}(\frac{Q}{C}) = \frac{1}{C} \tag{31}$$

$$A_{12} = A_{21} = \frac{dV}{dg} = \frac{dF}{dQ} = \frac{Q}{C^2}C' \tag{32}$$

$$A_{22} = \frac{\delta F}{\delta g}\bigg|_{Q=0} = k_{eff} - \frac{dF_e}{w_{avg}} = k + \frac{1}{2}Q^2(\frac{2C'^2}{C^3} - \frac{C''}{C^2}) \tag{33}$$

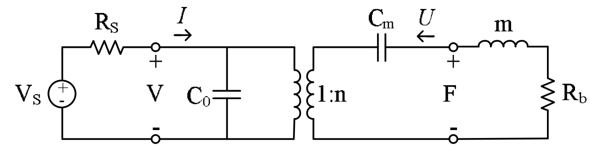


Fig. 4. Equivalent circuit of an electrostatic transducer.

The equivalent circuit in Fig. 4 captures the same behavior as (30). The circuit elements are defined by (34)-(38).

$$C_0 = C \tag{34}$$

$$n = \frac{A_{12}}{A_{11}} = VC' \tag{35}$$

$$k_e^2 = \frac{A_{11}^2}{A_{12}A_{22}} = \frac{n^2}{CA_{22}} \tag{36}$$

$$\frac{1}{C_m} = k_{eq} = A_{22}(1 - k_e^2) \tag{37}$$

$$L_m = m \tag{38}$$

Note that the capacitance C_m in the equivalent circuit model captures the spring-softening effect; as the applied voltage approaches V_{PI}, the equivalent spring constant approaches zero.

V. Conclusion

The spring-mass-damper model accurately predicts the CMUT's behavior for a wide range of designs. We can use the model to develop a set of simple hand calculations that can guide transducer design. In addition, we can use it to derive the small-signal equivalent circuit which is useful for simulating the transducer's frequency response and interaction with electronics. With tools such as Matlab Simulink, we can also use the model to make large-signal transient simulations.

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