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ANALYTICALLY PRICING EUROPEAN-STYLE OPTIONS UNDER THE MODIFIED BLACK-SCHOLES EQUATION WITH A SPATIAL-FRACTIONAL DERIVATIVE

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Abstract. This paper investigates the option pricing under the FMLS (finite moment log stable) model, which can effectively capture the leptokurtic feature observed in many financial markets. However, under the FMLS model, the option price is governed by a modified Black-Scholes equation with a spatial-fractional derivative. In comparison with standard derivatives of integer order, the fractional-order derivatives are characterized by their "globalness", i.e., the rate of change of a function near a point is affected by the property of the function defined in the entire domain of definition rather than just near the point itself. This has added an additional degree of difficulty not only when a purely numerical solution is sought but also when an analytical method is attempted. Despite this difficulty, we have managed to find an explicit closed-form analytical solution for European-style options after successfully solving the FPDE (fractional partial differential equation) derived from the FMLS model. After the validity of the put-call parity under the FMLS model is verified both financially and mathematically, we have also proposed an efficient numerical evaluation technique to facilitate the implementation of our formula so that it can be easily used in trading practice.

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1. Introduction. It is well documented in the literature that the BS (Black-Scholes) model usually underestimates the probability of underlying price moving significantly over small time steps [3]. For example, when analyzing the S&P 500 data, a "leptokurtic distribution" is observed, which has a higher peak and two heavier tails than those of the normal distribution. Numerous efforts have been made to develop alternative asset return models capable of capturing the leptokurtic feature observed in financial market data. Those models have either allowed the volatility to evolve stochastically, such as the Heston model [8], or added jumps to the process of underlying price, such as the Press model [18], Merton's jump diffusion model [16], and so on. One of the most popular approaches in the latter category is to assume that under an equivalent martingale measure, the underlying price stays within a family of Lévy processes, which include the standard Brownian motion, Poisson process and compound Poisson processes as the simplest form. In addition to the fat tails they have, Lévy distributions allow for long jumps as well, which are frequently observed in real markets.

Among all the Lévy processes, the maximally skewed LS (Lévy stable) process introduced in [3] has been studied by a number of authors. This special Lévy process gives rise to an interesting financial model known as the FMLS (finite moment log-stable) model [3], which can not only successfully capture the high-frequency empirical probability distribution of the S&P 500 data, but also fit simultaneously volatility smirks at different maturities. Most importantly, in contrast to many other models driven by different Lévy processes, the FMLS model guarantees that all moments of the underlying index level are finite, which ensures the existence of an equivalent martingale measure and the finiteness of option prices at all maturities. The current work is carried out under the framework of the FMLS model. The extension of our approach to other models driven by different Lévy processes (e.g., KoBol and CGMY mentioned in [6]) is quite promising.

Mathematically, to characterize the non-locality induced by the pure jumps under the FMLS model, the FPDE (fractional partial differential equation), which is a subset of the class of pseudo-differential equations, needs to be solved. We remark that in the new FPDE governing the option price under the FMLS model, the second-order spatial derivative involved in the standard BS equation is replaced by an α-order spatial derivative, with α being any real number belonging to $(1, 2]$. Due to its non-local nature, the fractional operator in fact weights information of the portfolio over a range of underlying values rather than looking at localized information.

In the quantitative finance area, two types of fractional derivatives are mainly documented: a time-fractional derivative and (or) a spatial-fractional derivative. Regarding a fractional derivative in time, Wyss [19] considered the pricing of option derivatives under the modified BS equation with a time-fractional derivative and derived a closed-form solution for European vanilla options. However, in his work, no plausible financial reason is provided to explain why a time-fractional derivative should be adopted. Later on, Cartea and Meyer-Brandis [7] proposed a model explicitly using information on the waiting time between trades. It is shown that their model can effectively capture the empirical waiting-time distribution under the data generating measure. This model was further investigated by Cartea in [5], where he found that the value of European-style derivatives satisfies a FPDE containing a non-local operator in time-to-maturity known as the Caputo fractional derivative. On the other hand, regarding the option pricing with a spatial-fractional derivative, Carr and Wu [3] introduced the FMLS model to the literature and showed its superior performance against several widely used alternatives. Under this model, many techniques have been developed to compute option values, as summarized in [14]. Substantial progress has been made by Cartea and del-Castillo-Negrete [6] who successfully connected the FMLS process (as well as KoBol or CGMY processes) with the spatial-fractional derivatives. They considered the pricing of barrier options under the FMLS (as well as KoBol or CGMY) model purely numerically by using a finite difference method. Cartea [4] also showed that the hedging strategies can be substantially improved once fractional derivatives are adopted.

In this paper, we consider systematically the pricing of option derivatives under the FMLS model. Despite a number of difficulties such as the non-local nature of the spatial-fractional derivative that prevents efficient numerical valuation for the option price, we have successfully derived an explicit closed-form analytical solution for European-style vanilla options under the FMLS model.[1] Upon using the newly derived analytical solution, the asymptotic behavior of the solution can be well examined, which provides further justification for adopting the FMLS model to price options. Moreover, we have also verified, from both financial and mathematical points of view, the validity of the put-call parity under the FMLS model. Another important point is that the implementation of our solution is not as straightforward as the case of the BS formula because of the appearance of the Fox functions in the kernel of integration in the current solution. However, we have proposed an efficient and accurate numerical evaluation technique to significantly facilitate the implementation of our formula so that the FMLS model can be easily used in trading practice.

The paper is organized as follows: In Section 2, we introduce the FPDE system that the price of European-style options must satisfy under the FMLS model. In Section 3, we derive a closed-form analytical solution from the established FPDE system and examine the asymptotic behavior of the solution. We also prove the validity of the put-call parity under the FMLS model. In Section 4, numerical examples and some quantitative analyses are presented. Concluding remarks are given in the last section.

2. FMLS model. Under the risk neutral measure $\mathbb{Q}$, the FMLS model assumes that the log value of the underlying i.e., $x_t = \ln S_t$, with dividend yield D follows a stochastic differential equation of the maximally skewed LS process:

$$dx_t = (r - D - \nu)dt + \sigma dL_t^{\alpha,-1}, \tag{2.1}$$

where r and D are the risk free interest rate and the dividend yield, respectively. t is the current time, and $\nu = -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}$ is a convexity adjustment. $L_t^{\alpha,-1}$ denotes the maximally skewed LS process, which is a special case of the Lévy-α-stable process $L_t^{\alpha,\beta}$, where $\alpha \in (0, 2]$ is the tail index describing the deviation of the LS process from the

[1] A solution written in terms of the inverse Fourier transform without the inversion being carried out analytically is still of closed form. However, since numerical inversion of Fourier transform should be handled carefully, such kinds of solutions are not truly "explicit" as far as the computation of the numerical values of an option is concerned.

Brownian motion, and $\beta \in [-1,1]$ is the skew parameter. To ensure that the underlying return has the support on the whole real line, the tail index α needs to be restricted to (1,2], as demonstrated in [3]. We remark that in the maximally skewed LS process ($\beta = -1$), the random variable x_t is maximally skewed to the left, meaning that the right tail of the distribution is fast decaying so that exponential moments exit. This setting of parameters implies that the FMLS only exhibits downward jumps while its upward movements have continuous paths [6], a feature that might not agree well with the empirical evidence. However, it should be pointed out that although this model is not perfect in modeling option prices, it can still be regarded as a springboard for future extensions which can capture finer properties of the option market, as suggested in [3]. In the following, we shall consider the pricing of European-style vanilla options under this model.

Let $V(x,t;\alpha)$ be the price of European-style options, with x being the log underlying price defined as $x = \ln S$ and α being the tail index. Cartea and del-Castillo-Negrete [6] showed that under the FMLS model, $V(x,t;\alpha)$ satisfies the following FPDE:

$$\begin{cases} \dfrac{\partial V}{\partial t} + (r + \dfrac{1}{2}\sigma^\alpha \sec\dfrac{\alpha\pi}{2})\dfrac{\partial V}{\partial x} - \dfrac{1}{2}\sigma^\alpha \sec\dfrac{\alpha\pi}{2}\,{}_{-\infty}D_x^\alpha V - rV = 0, \\ V(x,T;\alpha) = \Pi(x), \end{cases} \tag{2.2}$$

where $\Pi(x)$ is the payoff function, which is defined as $\max(e^x - K, 0)$ and $\max(K - e^x, 0)$ for European calls and puts, respectively, with K being the strike price. ${}_{-\infty}D_x^\alpha$ here is the one-dimensional Weyl fractional operator, which is defined as

$${}_{-\infty}D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{-\infty}^{x}\frac{f(y)}{(x-y)^{\alpha+1-n}}dy,\ n-1 \le \Re(\alpha) < n.$$

From this definition, it can be observed that when $\alpha \to 2$, the above one-dimensional Weyl fractional operator becomes the second-order differentiation, and consequently, (2.2) degenerates to the classical BS system for European options.

It should be remarked that (2.2) is fundamentally different from the FPDE system in [19], where the fractional derivative appears in the time direction and can be eliminated by the Laplace transform, whereas in our case, the Laplace transform would not help. Despite those difficulties, we have managed to derive an explicit closed-form analytical solution from (2.2), as will be shown in the next section.

3. Closed-form analytical solution. In this section, we derive a closed-form analytical solution for European-style options under the FMLS model. This section is further divided into three subsections, according to three important issues to be addressed. In the first subsection, the detailed derivation of our formula is provided, whereas in the second and last subsections, the asymptotic behavior of our solution and the put-call parity under the FMLS model are examined, respectively.

3.1. *Solution procedure.* To make analysis convenient, we shall first change the backward problem into a forward problem by introducing $\tau = -\frac{1}{2}\sigma^\alpha(\sec\frac{\alpha\pi}{2})(T-t)$. The

FPDE system (2.2) then becomes

$$\begin{cases} \dfrac{\partial V}{\partial \tau} = (\gamma - 1)\dfrac{\partial V}{\partial x} + {}_{-\infty}D_x^{\alpha} V - \gamma V = 0, \\ V(x, 0; \alpha) = \Pi(x), \end{cases} \tag{3.1}$$

where $\gamma = \dfrac{-2r}{\sigma^{\alpha}\sec(\frac{\alpha\pi}{2})}$ is the relative interest rate of the volatility with fractional order α to the risk-free interest rate. One can observe that if $\alpha = 2$, γ becomes $\dfrac{2r}{\sigma^2}$, a quantity identical to the relative interest rate of the BS model; while if α is smaller than 2, an additional factor $-\sec(\dfrac{\alpha\pi}{2})$ appears, which eliminates the arbitrage opportunities introduced by σ^{α}.

To solve (3.1) analytically, we shall start from the expression of V in the Fourier space, i.e., $\tilde{V}(\xi, \tau, \alpha) = F[V(x, \tau; \alpha)]$. It is shown in [6] that $\tilde{V}(\xi, \tau, \alpha)$ satisfies

$$\begin{cases} \dfrac{\partial \tilde{V}}{\partial \tau} = (\gamma - 1)i\xi\tilde{V} - |\xi|^{\alpha}\tilde{V} - \gamma\tilde{V} = 0, \\ \tilde{V}(\xi, 0; \alpha) = \tilde{\Pi}(\xi), \end{cases} \tag{3.2}$$

where $\tilde{\Pi}(\xi) = F[\Pi(x)]$. From a FPDE point of view, (3.2) can in fact be straightforwardly obtained after the Fourier transform is applied on (3.1). Upon solving (3.2), the option price in the Fourier space can be written as

$$\tilde{V}(\xi, \tau; \alpha) = e^{-\gamma\tau}\tilde{\Pi}(\xi)e^{-(1-\gamma)\tau i\xi - |\xi|^{\alpha}\tau}. \tag{3.3}$$

To obtain the option price $V(x, \tau; \alpha)$ in the original $x-$space, one still needs to carry out the Fourier inversion either numerically or analytically, a formidable process that often prevents this great technique being widely used to solve PDEs. In the following, we shall concentrate on carrying out (3.3) purely analytically.

According to the convolution theorem of the Fourier transform, it is clear that $V(x, \tau; \alpha) = e^{-\gamma\tau}V(x, 0; \alpha) * F^{-1}[e^{-(1-\gamma)\tau i\xi - |\xi|^{\alpha}\tau}]$, which can be further reduced to

$$V(x, \tau; \alpha) = e^{-\gamma\tau}V(x, 0; \alpha) * P(x - (1 - \gamma)\tau; \alpha) \tag{3.4}$$

after the shift theorem is applied, where $P(x; \alpha) = F^{-1}[e^{-|\xi|^{\alpha}\tau}]$.

Upon realizing that $e^{-|\xi|^{\alpha}}$ is nothing but the characteristic function of a centered and symmetric Lévy distribution [17], as well as the relationship between Fourier transform and the characteristic function of a probability density function, one can deduce that the Fourier inversion of $e^{-|\xi|^{\alpha}\tau}$ is equal to multiples of the closed-form representation of the Lévy stable density $f_{\alpha,0}$ [17], which is usually expressed in terms of the Fox function, i.e.,

$$P(x; \alpha) = \frac{1}{\tau^{1/\alpha}} f_{\alpha,0}\left(\frac{|x|}{\tau^{1/\alpha}}\right) = \frac{1}{\alpha\tau^{1/\alpha}} H_{2,2}^{1,1}\left[\frac{|x|}{\tau^{1/\alpha}} \middle| \begin{matrix} (1 - \frac{1}{\alpha}, \frac{1}{\alpha}) & (\frac{1}{2}, \frac{1}{2}) \\ (0, 1) & (\frac{1}{2}, \frac{1}{2}) \end{matrix} \right]. \tag{3.5}$$

Now, combining (3.4) and (3.5), we obtain

$$V(x, \tau; \alpha) = \int_{-\infty}^{+\infty} e^{-\gamma\tau}\Pi(\xi)\frac{1}{\tau^{\frac{1}{\alpha}}} f_{\alpha,0}\left(\frac{|x - \xi - (1 - \gamma)\tau|}{\tau^{\frac{1}{\alpha}}}\right)d\xi,$$

which can be written as

$$V(x,\tau;\alpha)=\int_{-\infty}^{+\infty} e^{-\gamma\tau}\Pi(x-(1-\gamma)\tau-\tau^{\frac{1}{\alpha}}m)f_{\alpha,0}(|m|)dm,$$

after the changing of integral variable technique is applied. Therefore, for European put options, we have

$$V_p(x,\tau;\alpha)=Ke^{-\gamma\tau}\int_{d_1}^{+\infty} f_{\alpha,0}(|m|)dm - e^x\int_{d_1}^{+\infty} e^{-\tau-\tau^{\frac{1}{\alpha}}m}f_{\alpha,0}(|m|)dm, \tag{3.6}$$

where $d_1=\dfrac{x-\ln K-(1-\gamma)\tau}{\tau^{\frac{1}{\alpha}}}$.

It should be remarked that our solution procedure may not be as versatile as the approach proposed by Carr and Madan [2]. Since analytical inversion of the Fourier transform is not always possible (performing Fourier inversion purely analytically is usually a very difficult task and may be limited to special forms of payoff functions), our approach may not work for other financial derivatives. Indeed, whether our solution procedure can be extended should be considered case by case, depending on the specific option taken into consideration.

On the other hand, one can never overlook the advantages if the Fourier inversion can be worked out analytically for some special cases like the one presented here. Firstly, the explicit closed-form solution clearly exhibits the relationship between the parameters and variables in the original underlying space, which may pave the way for further quantitative analysis of the FMLS model. Secondly, it is much easier to deal with the explicit closed-form analytical solution than those semi-analytical formulae (without the Fourier inversion being analytically carried out). It is found that the Fourier integrands may have poles in the complex plane for some cases, resulting in the value of corresponding Fourier integrals varying with the choice of contour [11]. Furthermore, those integrands may also exhibit oscillations, which pose considerable difficulties in numerical implementation [14]. Lastly, with the explicit closed-form analytical solutions, important hedging parameters (e.g., the Greeks) can be easily calculated, whereas approximation methods, such as the purely numerical schemes, sometime exhibit problems that greatly affect the accuracy, especially when they are adopted to determine those parameters [13]. Of course, one could also use an option pricing formula left in the integral form as a result of Fourier inversion to calculate the Greeks. But, the resulting Greeks are in the Fourier space, which may cause difficulties in numerical realization, as already stated.

3.2. *Asymptotic behavior of the closed-form solution.* One of the most efficient ways to check the validity of our closed-form solution (3.6) is to investigate its asymptotic behavior with parameters involved taken on some extreme values. Whether the observed asymptotic behavior agrees with the financial terms set for the corresponding option could be a necessary condition to verify the solution. Moreover, the examination of the asymptotic behavior of the option price will also reveal some essential properties of the pricing model used. In view of this, we shall conduct some asymptotic analyses on (3.6) in this section.

As pointed out previously, (3.1) becomes the classical BS system for European options as $\alpha\to 2$. Therefore, it is expected that (3.6) will approach the price of a European put option asymptotically as $\alpha\to 2$. This is indeed so as shown in the following theorem.

THEOREM 3.1. As $\alpha\to 2$, (3.6) degenerates to the BS formula for European puts, i.e.,

$$\lim_{\alpha\to 2} V_p(x,\tau;\alpha) = Ke^{-\gamma\tau}N(-d_2) - e^x N(-d_1),$$

where $d_1 = \dfrac{x-\ln K+(\gamma-1)\tau}{\sqrt{2\tau}}$, $d_2 = d_1 - \sqrt{2\tau}$, and $N(x)$ is the standard normal distribution function defined as $N(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{x} e^{-\frac{z^2}{2}}dx$.

Proof. According to the definition of $f_{\alpha,0}$, it is known that

$$\lim_{\alpha\to 2} f_{\alpha,0}(|m|) = \frac{1}{2}H_{2,2}^{1,1}\left[|m|\left| \begin{matrix} (\frac{1}{2},\frac{1}{2}) & (\frac{1}{2},\frac{1}{2}) \\ (0,1) & (\frac{1}{2},\frac{1}{2}) \end{matrix} \right.\right],$$

which can be simplified as

$$f_{2,0}(|m|) = \frac{1}{2}H_{1,1}^{1,0}\left[|m|\left| \begin{matrix} (\frac{1}{2},\frac{1}{2}) \\ (0,1) \end{matrix} \right.\right], \tag{3.7}$$

whose Mellin transform admits $\mathcal{M}[f_{2,0}(|m|)] = \dfrac{1}{2}\dfrac{\Gamma(s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}$. On the other hand, according to the property of the Gamma function, it is known that $\Gamma(\dfrac{1}{2}s)\Gamma(\dfrac{1}{2}+\dfrac{1}{2}s) = 2^{1-s}\sqrt{\pi}\Gamma(s)$, and thus

$$\frac{\Gamma(s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)} = \frac{(\frac{1}{2})^{-s}\Gamma(\frac{1}{2}s)}{2\sqrt{\pi}}. \tag{3.8}$$

Now, taking the inverse Mellin transform on (3.8), we obtain

$$\mathcal{M}^{-1}[\frac{\Gamma(s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}] = \mathcal{M}^{-1}[\frac{(\frac{1}{2})^{-s}\Gamma(\frac{1}{2}s)}{2\sqrt{\pi}}] = \frac{e^{-m^2/4}}{\sqrt{\pi}},$$

which, combined with (3.7) yields $f_{2,0}(|m|) = \dfrac{e^{-m^2/4}}{2\sqrt{\pi}}$, a function identical to the standard Gaussian density.

Now replacing the Lévy stable density function in (3.6) by the standard Gaussian density, the BS formula will be obtained after some simple algebraic manipulations, and Theorem 3.1 is thus proved. □

After investigating the relationship between our solution and the BS formula, we shall examine the asymptotic behavior of (3.6) for extreme values of the underlying. It is anticipated that the option price derived from any reasonable pricing model would certainly have suitable growth conditions at $x = \pm\infty$. This is achieved in the following theorem.

THEOREM 3.2. (i) $\lim_{x\to-\infty} V_p(x,\tau;\alpha) = Ke^{-\gamma\tau}$; (ii) $\lim_{x\to\infty} V_p(x,\tau;\alpha) = 0$.

Proof. Firstly, we shall prove that $\lim_{x\to-\infty} V_p(x,\tau;\alpha) = Ke^{-\gamma\tau}$. According to the definition of d_1, it is not difficult to show that $d_1 \to -\infty$ as $x \to -\infty$. By realizing that $f_{\alpha,0}$ is the Lévy stable density, it is clear that $\int_{-\infty}^{+\infty} f_{\alpha,0}(x)dx = 1$, which, combined with the fact that $f_{\alpha,0}(x)$ is symmetric in x, yields $\int_{-\infty}^{+\infty} f_{\alpha,0}(|x|)dx = 1$. Consequently, the first integral of (3.6) will approach $Ke^{-\gamma\tau}$ as $x \to -\infty$. On the other hand, because of the appearance of the exponential function e^x, the second integral of (3.6) will vanish as $x \to -\infty$. Taking both points mentioned above into consideration, it is clear that $\lim_{x\to-\infty} V_p(x,\tau;\alpha) = Ke^{-\gamma\tau}$.

To prove (ii) of Theorem 2, we notice that the first integral of (3.6) will definitely vanish as $x \to +\infty$. Therefore, we shall concentrate on showing that the second integral will also vanish as $x \to +\infty$. For the second integral involved in (3.6), we have

$$\lim_{x\to+\infty} e^x \int_{d_1}^{+\infty} e^{-\tau-\tau^{\frac{1}{\alpha}}m} f_{\alpha,0}(|m|)dm = \lim_{x\to+\infty} \frac{\int_{d_1}^{+\infty} e^{-\tau-\tau^{\frac{1}{\alpha}}m} f_{\alpha,0}(|m|)dm}{e^{-x}},$$

which is equal to $\lim_{x\to+\infty} e^{\gamma\tau}\tau^{\frac{1}{\alpha}} f_{\alpha,0}(|d_1|)$, after the L'hospital's rule is applied. According to the fact that any density function will vanish at infinity, it is clear that $\lim_{x\to+\infty} f_{\alpha,0}(|d_1|) = 0$, because $d_1 \to +\infty$ as $x \to +\infty$. Consequently, we obtain

$$\lim_{x\to+\infty} e^x \int_{d_1}^{+\infty} e^{-\tau-\tau^{\frac{1}{\alpha}}m} f_{\alpha,0}(|m|)dm = 0,$$

which shows that the second integral of (3.6) will vanish as $x \to +\infty$. Therefore, we have $\lim_{x\to+\infty} V_p(x,\tau;\alpha) = 0$. □

According to Theorem 2, it is clear that under the FMLS model, if the underlying becomes extremely small, a European put option would certainly be exercised at the expiry, and thus its current value is equal to the discounted strike price. On the other hand, if the underlying becomes infinitely large, a European put option is worthless now, because it is impossible for the option to become "in the money" within a finite period between now and the expiry. Clearly, the above two points agree well with the financial terms set for a European put option. In this sense, it is reasonable to adopt the FMLS model for the pricing of option derivatives, at least for European-style options.

3.3. *Put-call parity.* One of the most important principles in the option pricing field is the so-called put-call parity, which reveals the relationship between the prices of European vanilla options when they have the same maturity date and strike price. By using the put-call parity, the price of a European put or call option can be deduced directly from its European counterpart. In view of the importance of the put-call parity, in this section, we shall further verify, financially and mathematically, the validity of the put-call parity under the FMLS model.

Financially, because of the introduction of a convexity adjustment to the Lévy process, the risk-neutral measure exists under the FMLS model, as pointed out previously. The existence of the risk-neutral measure, on the other hand, implies that the "no arbitrage opportunity" assumption still holds under this particular model. The put-call parity can then be achieved by using the same portfolio analysis as the one adopted in the BS model [10].

On the other hand, through rigorous mathematical analysis, one can also show that the put-call parity holds under the FMLS model, as expected from the financial argument mentioned above. We conclude the mathematical proof of the put-call parity in the following theorem.

THEOREM 3.3. For any given $\alpha \in (1,2]$, the prices of a European call option and its European counterpart satisfy the put-call parity, assuming that they have the same maturity and strike price, i.e.,

$$V_c(x,\tau;\alpha) - V_p(x,\tau;\alpha) = e^x - Ke^{-\gamma\tau}.$$

Proof. Due to the linearity of the governing operator contained in (3.1), it is known that the value of longing a call while shorting a put satisfies

$$\begin{cases} \dfrac{\partial V_{c-p}}{\partial \tau} = (\gamma-1)\dfrac{\partial V_{c-p}}{\partial x} + {}_{-\infty}D_x^\alpha V_{c-p} - \gamma V_{c-p} = 0, \\ V_{c-p}(x,0;\alpha) = e^x - K, \end{cases}$$

where $V_{c-p}(x,\tau;\alpha) = V_c(x,\tau;\alpha) - V_p(x,\tau;\alpha)$. By using the approach adopted in section 3.1 in deriving $V(x,\tau;\alpha)$, we obtain

$$V_{c-p}(x,\tau;\alpha) = \underbrace{-Ke^{-\gamma\tau}\int_{-\infty}^{+\infty} f_{\alpha,0}(|m|)dm}_{I} + \underbrace{e^x\int_{-\infty}^{+\infty} e^{-\tau-\tau^{\frac{1}{\alpha}}m} f_{\alpha,0}(|m|)dm}_{II}.$$

From the above expression for $V_{c-p}(x,\tau;\alpha)$, one can easily deduce that I is equal to $-Ke^{-\gamma\tau}$, because $\int_{-\infty}^{+\infty} f_{\alpha,0}(|m|)dm = 1$, as pointed out in Section 3.2. On the other hand, according to the symmetric in the density function $f_{\alpha,0}(\cdot)$, it is not difficult to show that

$$\begin{aligned} II &= e^x \int_{-\infty}^{+\infty} e^{-\tau-\tau^{\frac{1}{\alpha}}m} f_{\alpha,0}(m)dm, \\ &= e^{x-\tau}\int_{-\infty}^{+\infty} e^{-i(-i\tau^{\frac{1}{\alpha}})m} f_{\alpha,0}(m)dm. \end{aligned}$$

It should be pointed out that II can in fact be viewed as the Fourier transform of $f_{\alpha,0}(m)$ at the point $(-i\tau^{\frac{1}{\alpha}})$. Consequently, by using the fact that $f_{\alpha,0}(m) = F^{-1}[e^{-|\xi|^\alpha}](m)$, we obtain

$$II = e^{x-\tau}F[f_{\alpha,0}(m)](-i\tau^{\frac{1}{\alpha}}) = e^{x-\tau-|-i\tau^{\frac{1}{\alpha}}|^\alpha} = e^{x-\tau+[i(-i\tau^{\frac{1}{\alpha}})]^\alpha} = e^x.$$

Now, combining the values of the integration I and II, we obtain $V_{c-p} = e^x - Ke^{-\gamma\tau}$, and complete the proof of the validity of the put-call parity under the FMLS model. □

We remark that the put-call parity has greatly facilitated the pricing of European vanilla options under the FMLS model, in the sense that the price of either a European call or put can be deduced straightforwardly from the parity once the price of its European counterpart is determined accurately from our closed-form analytical solution. However, due to the complexity of the Lévy density, the implementation of our solution in terms of finding specific numerical values from (3.6) may not be as straightforward as the BS formula. This issue will be illustrated in detail in the next section, where some numerical examples and useful discussions are provided.

4. Numerical examples and discussions. Once the closed-form analytical solution is obtained for the price of a particular option derivative, the main concern of market practitioners becomes its implementation. Whether the formula can be efficiently computed is one of the main criteria of assessing its practical usefulness. Therefore, in this section, the implementation of our formula (3.6) will be illustrated, together with some useful discussions.

Although (3.6) is written in a similar form as the classical BS formula, it is, however, not so straightforward as the latter, as far as the computing for numerical value is concerned. The difficulties mainly arise from the fact that the Lévy density $f_{\alpha,0}(x)$ has a rather slow convergence rate as $x\to\infty$, in comparison with the Gaussian density involved in the BS formula.

To determine $f_{\alpha,0}(x)$, we shall use the series representation, i.e.,

$$f_{\alpha,0}(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Gamma(1+n/\alpha)}{n!}\sin(\frac{\pi n}{2})(-x)^{n-1}, \tag{4.1}$$

rather than its usual form expressed in the Mellin space, as the latter will involve an additional inversion of the Mellin transform in the final calculation of the integral. However, from our numerical experiment, it is observed that (4.1) converges rather slowly when x becomes very large. Therefore, in order to speed up the calculation without unnecessarily sacrificing accuracy, the large asymptotic of $f_{\alpha,0}(x)$,

$$f_{\alpha,0}(x) \sim \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Gamma(1+n/\alpha)}{n!}\sin(\frac{\pi n\alpha}{2})|x|^{-1-n\alpha},$$

is adopted when x is beyond a critical value. Numerical experiments show that $x \approx 4.5$ is an appropriate critical value for all the numerical examples presented below.

On the other hand, it is not an easy task to deal with the integrals in semi-infinite domain involved (3.6) either. These integrals cannot be expressed in terms of standard built-in functions such as the normal distribution function appearing in the BS formula. Furthermore, the lower convergence rate at $x\to\infty$ of the Lévy density $f_{\alpha,0}(x)$, in comparison with the Gaussian density, has added the difficulty in numerically carrying out the quadrature involved in (3.6). In our numerical experiments, we evaluate these integrals by using the generalized Laguerre-Gauss quadrature, which is an efficient way to calculate integrals in semi-infinite domain. The detailed implementation of this scheme can be found in many textbooks regrading numerical methods [9] and is thus omitted here.

Having demonstrated the implementation details of our closed-form analytical solution, we shall now compare it with those written in terms of Fourier integrals. In particular, the Bates formula [1] will be adopted for comparison purposes. This is because in comparison with various semi-infinite Fourier integrals, the Bates formulation requires only a single integration with an integrand converging faster due to the quadratic term in the denominator [14]. However, as far as the computation of the numerical values

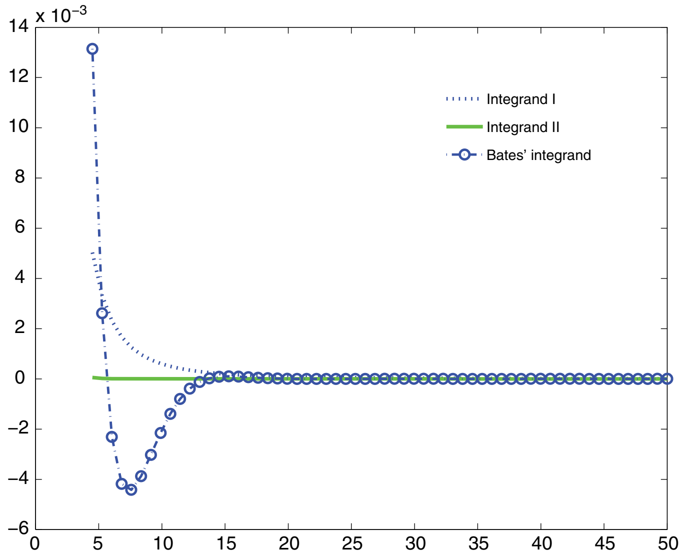
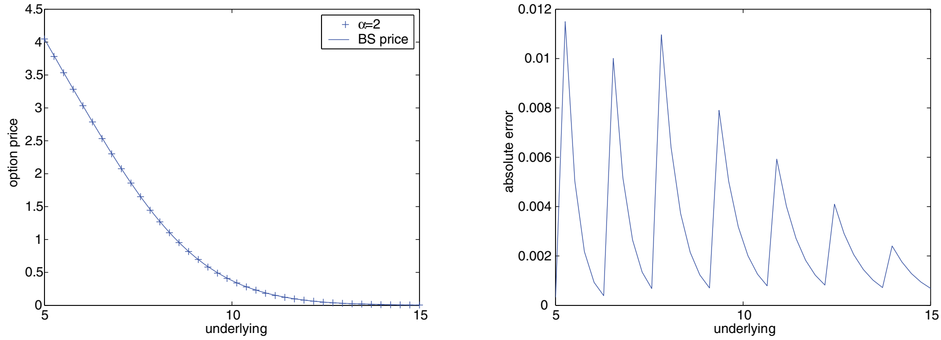


FIG. 1. Comparison of decay rates of different integrands. Model parameters are $K = \$10$, $r = 0.1$, $\sigma = 0.1929$, $\alpha = 1.75$, $T - t = 1$ (year).

of an option is concerned, the Bates solution is not truly "explicit", although it is also of closed form. A clear advantage of our formula against the Bates solution or various semi-infinite Fourier integrals is that ours has no need to work out the expression of the characteristic function in advance, which is, however, an essential part of the latter.

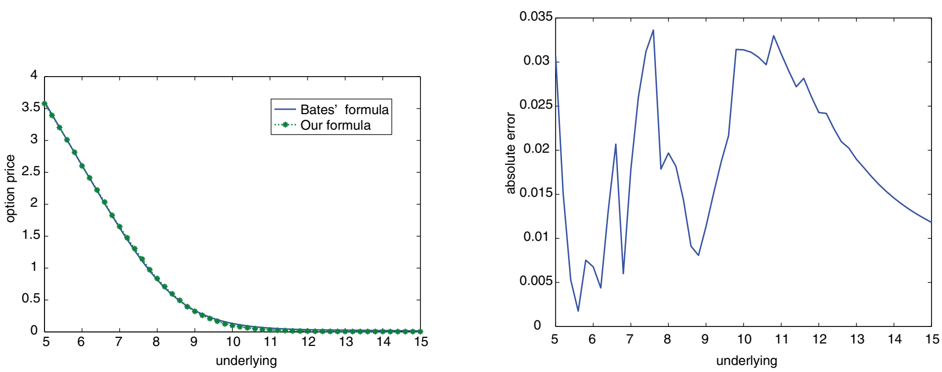
The computational efficiency of carrying out our formula is demonstrated through the comparison of the decay rates of the integrand involved in Bates' formula and those in our solution, as shown in Fig 1. From this figure, it is clear that our integrands converge to zero at almost the same rate as the one in Bates' formula, implying that our formula can be as efficiently carried out as the Bates solution, once the same quadrature rule is adopted for the implementation of the semi-infinite domain integrals involved in both formulae. Indeed, the calculation of our formula (3.6) can be completed within 0.7 seconds on a personal computer, where it takes almost the same amount of time to compute the Bates formula for a numerical value.

The best way to test the reliability of the proposed numerical evaluation technique for (3.6) is to calculate our solution at $\alpha = 2$ and compare it with the standard BS formula with the same parameter settings. Theoretically, when $\alpha = 2$, our solution is identical to the BS formula, if all the other parameters are the same, as shown in Theorem 3.1 already. The comparison is provided in Fig 2(a), where two sets of European put prices determined respectively from (3.6) with $\alpha = 2$ and the BS formula are displayed as a function of the underlying at a given time to maturity. The absolute differences between the two sets of prices are further shown in Fig 2(b). Furthermore, for $\alpha \neq 2$, we compared our solution with those calculated from the Bates formula with the generalized Laguerre-Gauss quadrature. Also, the comparison of the option prices is provided in Fig 3(a), whereas the absolute differences between the two sets of prices are shown in Fig 3(b). From these figures, one can clearly observe that our option prices agree perfectly well with those listed in the literature, with the maximum absolute error being no more than 1.2% and 3.5% for the case of $\alpha = 2$ and $\alpha = 1.85$, respectively.



(a) BS price vs. our price at $\alpha = 2$ (b) Option price differences

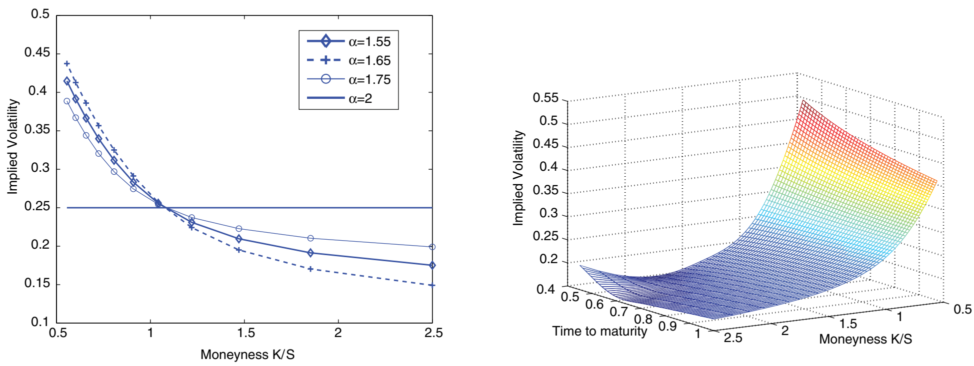
FIG. 2. Comparison of our solution at $\alpha = 2$ with the BS formula. Model parameters are $K = \$10$, $r = 0.1$, $\sigma = 0.2$, $T - t = 1$ (year).



(a) Bate's price vs. our price at $\alpha = 1.85$ (b) Option price differences

FIG. 3. Comparison of our solution at $\alpha = 1.85$ with the Bates formula. Model parameters are $K = \$10$, $r = 0.1$, $\sigma = 0.2$, $T - t = 1$ (year).

With confidence in the proposed implementation technique, we shall now investigate the effect of different tail index α on the prices of European puts. We remark that for comparison purposes, the volatilities are chosen such that the $\alpha-$stable distribution has the same quartiles as the BS distribution with a volatility σ_{BS}. Here we set $\sigma_{BS} = 0.25$, which corresponds to $\sigma_{(\alpha=1.55)} = 0.2299$, $\sigma_{(\alpha=1.65)} = 0.2380$, and $\sigma_{(\alpha=1.75)} = 0.2440$. The implied volatilities for these parameter settings are provided in Fig 4(a), where one can clearly observe that the implied volatilities for those at-the-money options are almost equal to $\sigma_{BS} = 0.25$. The implied volatility surface for a particular α value ($\alpha = 1.65$) is further shown in Fig 4(b), from which one can observe asymmetry volatility smiles for short maturities, but volatility skews as the times to maturity become longer.



(a) Implied volatilities for different α values, with $K = \$10$, $r = 0.1$, $T - t = 1$ (year).

(b) Implied volatility surface, with $\alpha = 1.65$, $K = \$10$, $r = 0.1$.

FIG. 4. Volatility smirk under FMLS model.

Depicted in Fig 5 is the comparison among several sets of European put option prices at different levels of α values, while all the other parameters (except the volatility σ) are set to be the same. It can be observed from this figure that once α increases up to 2, the option prices are gradually decreasing to the BS price. In other words, the BS formula tends to underprice European puts with underlying following a Lévy process. Moreover, the pricing bias of the BS formula gets larger as α becomes smaller. This is indeed reasonable and could be plausibly explained from a financial point of view as follows.

Compared to the Gaussian density of the underlying prices under the BS model, the Lévy density increases the probability of the stock price exhibiting large moments or jumps over small time steps [3]. Thus, the terminal distribution of the underlying modeled by the Lévy process would have fatter tails at both ends than the lognormal distribution of the BS model. Moreover, both tails will become fatter as α gets smaller, since the Lévy density satisfies the inverse power-law asymptotically at large underlying values, i.e., $f_{\alpha,0}(x) \sim \dfrac{1}{|x|^{1+\alpha}}$ [17].

Now, consider a European put that is significantly out of the money. It can be shown that this option will have a positive value only if there is a large decrease in the underlying price. Its value is therefore dependent only on the left tail of the terminal distribution of the asset. The fatter the left tail is, the more valuable the option would be. Consequently, the BS model tends to underprice those out-of-the-money puts, and the pricing bias becomes larger as the left tail becomes fatter, which corresponds to smaller α values.

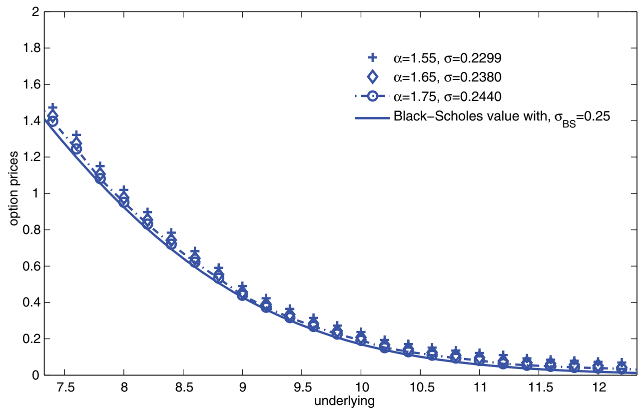


FIG. 5. European puts at different α values. Model parameters are $K = \$10$, $r = 0.1$, $T - t = 1$ (year).

To obtain the pricing biases for those in-the-money puts, we use the put-call parity verified in Section 3.3. From this parity, it can be identified that if a European call is out of the money, its European counterpart is in the money, and vise versa. Consequently, an in-the-money European put must exhibit the same pricing biases as an out-of-the-money European call. On the other hand, for an out-of-the-money European call, its value depends only on the right tail of the terminal underlying price, because this option will have positive value only if there is a large increase in the underlying price. Therefore, the fatter the right tail is, the more valuable the option becomes. Thanks to the relationship between in-the-money European puts and out-of-the-money European calls, it is now clear that the BS model tends to underprice those in-the-money puts as well, and the pricing bias becomes larger as the right tail becomes fatter.

Taking the above points regarding the pricing biases for out-of-the-money and in-the-money European puts into consideration, it is clear that the BS model underestimates the European puts, if the underlying follows a Lévy process. Moreover, the pricing biases tend to be larger as α becomes smaller. Similarly, it can be shown that the BS model overprices European calls with underlying subject to Lévy process, and the pricing biases are larger for smaller α values as well.

5. Conclusion. By solving the FPDE, an explicit closed-form analytical solution for European-style options under the FMLS model is successfully obtained for the first time. The asymptotic behavior of our solution is then examined, which confirms the reliability of the FMLS model. It is also shown, in the current paper, that the put-call parity holds under the FMLS model, which is of both theoretical and practical importance. On the other hand, for practical purposes, we propose an efficient numerical evaluation technique for the current formula as well. Through various numerical experiments, the correctness of our solution and the good performance of the corresponding evaluation technique are clearly presented. Finally, the influence of the tail index on the option prices is also discussed financially.

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