Journal of Thermal Stresses
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713723680

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Online Publication Date: 01 October 2009

To cite this Article Alves, Margareth S., Muñoz Rivera, Jaime E., Sepúlveda, Mauricio and Vera Villagrán, Octavio P.(2009)'Analyticity of Semigroups Associated with Thermoviscoelastic Mixtures of Solids',Journal of Thermal Stresses,32:10,986 — 1004

To link to this Article: DOI: 10.1080/01495730903103028

URL: http://dx.doi.org/10.1080/01495730903103028

Journal of Thermal Stresses, 32: 986–1004, 2009

ISSN: 0149-5739 print/1521-074X online
DOI: 10.1080/01495730903103028

ANALYTICITY OF SEMIGROUPS ASSOCIATED WITH THERMOVISCOELASTIC MIXTURES OF SOLIDS

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In this paper we investigate the asymptotic behavior of solutions to the initial boundary value problem for a one-dimensional theory of mixtures of thermoviscoelastic solids. Our main goal is to present conditions which insure the analyticity and the lack of analyticity of the corresponding semigroup.

Keywords: Analyticity; Coupled system; C_0-semigroup; Thermoviscoelastic mixtures

INTRODUCTION

Thermoviscoelastic mixtures of solids is a subject which has deserved much attention in the recent years. The first works on the continuum theory of mixtures were the contributions by Truesdell and Toupin [1], Green and Naghdi [2, 3] and Bowen and Wiese [4]. Extensive reviews of the subject can be found in the works [5–8]. The first theory for a mixture of elastic solids based on the Lagrangian description has been presented by Bedford and Stern [9].

The theory of viscoelastic mixtures has been investigated by several authors (see [10–12] and references therein). In [10, 11], the authors derive the basic equations of a nonlinear theory of heat conducting viscoelastic mixtures in Lagrangian description. They assume that the constituents have a common temperature and that every thermodynamical process which takes place in the mixture satisfies the Clausius–Duhem inequality.

In this paper we want to emphasize the study of analyticity for C_0-semigroups associated with an one-dimensional linear theory of mixtures of

Received 28 November 2008; accepted 11 February 2009.
This MS has been supported by FONDECYT project 1070694, FONDAP and BASAL projects CMM, Universidad de Chile, and CI^{2}MA, Universidad de Concepción. OVV has been supported by Postdoctoral Fellowship of LNCC (National Laboratory of Scientific Computing), Brazil.
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two thermoviscoelastic solids. The model considered has been treated by Ieşan and Quintanilla [10] and Ieşan and Nappa [11]. In what follows, we briefly describe this model.

We consider a mixture of two interacting continua that occupies the interval $(0, L)$. The displacements of typical particles at time t are u and w, where $u = u(x, t)$, $w = w(y, t)$, $x, y \in (0, L)$. We assume that the particles under consideration occupy the same position at time $t = 0$, such that $x = y$ (see, e.g., [9]). The temperature deviation (difference to a fixed constant reference temperature) in each point x and the time t is given by $\theta = \theta(x, t)$. We denote by ρ_1 and ρ_2 the mass densities of the two constituents at time $t = 0$. T, S the partial stresses associated with the constituents, P the internal diffusive force, Θ the entropy density, Q the heat flux vector and T_0 is the absolute temperature in the reference configuration. In the absence of body forces and heat sources the system of equations which governs the linear theory consists of the equations of motion

$$\rho_1 u_{tt} = T_x - P, \quad \rho_2 w_{tt} = S_x + P$$

the energy equation

$$\rho T_0 \Theta_t = Q_x$$

where $\rho = \rho_1 + \rho_2$, and the constitutive equations. From the one-dimensional linear theory established in [10], it results that in the absence of porosity, the constitutive equations are the following

$$\begin{aligned}
T &= a_{11} u_x + a_{12} w_x + b_{11} u_{xt} + b_{12} w_{xt} - b_1 \theta \\
S &= a_{12} u_x + a_{22} w_x + b_{21} u_{xt} + b_{22} w_{xt} - b_2 \theta \\
P &= \alpha(u - w) + \alpha_1(u_t - w_t) + \alpha_2 \theta_x \\
\rho\Theta &= b_1 u_x + b_2 w_x + c\theta \\
Q &= K\theta_x + K_1(u_t - w_t)
\end{aligned}$$

where $c, \alpha, K, K_1, \alpha_i, \beta_i, a_{ij}, b_{ij} (i, j = 1, 2)$ are constitutive coefficients. The Clausius–Duhem inequality reduces to

$$b_{11}x^2 + (b_{12} + b_{21})xy + b_{22}y^2 + \alpha_1 z^2 + (\alpha_2 + K_1 T_0^{-1})z\ell + K T_0^{-1}\ell^2 \geq 0$$

for all x, y, z and ℓ. This inequality and the above constitutive equations can be found also in [11]. If we assume that $b_{12} = b_{21}$ and $\alpha_2 + K_1 T_0^{-1} = 0$ and substitute the constitutive equations into the motion equations and the energy equation, we obtain the system of field equations

$$\begin{aligned}
&\rho_1 u_{tt} - a_{11} u_{xx} - a_{12} w_{xx} - b_{11} u_{xxt} - b_{12} w_{xxt} + \alpha(u - w) + \alpha_1(u_t - w_t) + \beta_1 \theta_x = 0 \\
&\rho_2 w_{tt} - a_{12} u_{xx} - a_{22} w_{xx} - b_{12} u_{xxt} - b_{22} w_{xxt} - \alpha(u - w) - \alpha_1(u_t - w_t) + \beta_2 \theta_x = 0 \\
&c\theta_t - \kappa\theta_{xx} + \beta_1 u_{xt} + \beta_2 w_{xt} = 0
\end{aligned} \tag{1}$$

with $0 < x < L, t > 0, \kappa = KT_0^{-1}$. We assume that $\rho_1, \rho_2, c, \kappa$, and α are positive constants, $\alpha_1 \geq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$. The matrix $A = (a_{ij})$ is symmetric and positive definite and $B = (b_{ij}) \neq 0$ is symmetric and non negative definite, that is,

$$a_{11} > 0, \quad a_{11}\, a_{22} - a_{12}^2 > 0$$
$$b_{11} \geq 0, \quad b_{11}\, b_{22} - b_{12}^2 \geq 0$$

We study the system (1) with the following initial conditions:

$$u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad w(.,0) = w_0, \quad w_t(.,0) = w_1, \quad \theta(.,0) = \theta_0 \tag{2}$$

and the boundary conditions:

$$u(0,t) = u(L,t) = w(0,t) = w(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0 \quad \text{in } (0,\infty) \tag{3}$$

Our purpose in this work is to investigate the analyticity of the semigroup associated with the system (1)–(3). The exponential stability and analyticity of the semigroups associated with dissipative systems have been studied by many authors. We refer to the book of Liu and Zheng [13] for a general survey on these topics. However, the exponential stability for the case of the thermoelastic mixtures ($B = 0, \alpha_1 = 0$) has been only studied at [14, 15]. In [14], the authors prove (generically) the asymptotic stability. In [15], the authors prove that the semigroup associated is exponentially stable if and only if

$$\beta_2(\beta_1\rho_2 a_{11} + \beta_2\rho_1 a_{12}) \neq \beta_1(\beta_2\rho_1 a_{22} + \beta_1\rho_2 a_{12})$$

and

$$\frac{n^2\pi^2}{L^2} \neq \frac{\alpha((\rho_1\beta_2^2 - \rho_2\beta_1^2) + \beta_1\beta_2(\rho_1 - \rho_2))}{\beta_1\beta_2(\rho_2 a_{11} - a_{22}\rho_1) - a_{12}(\beta_1^2\rho_2 - \beta_2^2\rho_1)}$$

holds for all $n \in \mathbb{N}$.

We recall that very few contributions have been addressed to study the time behavior of the solutions of nonclassical elastic theories. Our main result is to establish conditions on the matrix B, which guarantee the analyticity of the corresponding semigroup. We show that the semigroup is analytic if and only if B is non singular.

This paper is organized as follows: Section two, outlines briefly the notation and the well-posed of the system is established. In section three, we show the analyticity of the corresponding semigroup provided B is positive definite. In section four, we show the non analyticity when B is singular.

Finally, throughout this paper C is a generic constant, not necessarily the same at each occasion (it will change from line to line), which depends in an increasing way on the indicated quantities.

EXISTENCE AND UNIQUENESS OF THE SOLUTIONS

In this section we study the setting of the semigroup and we establish the well-posed of the system. The initial boundary value problem (1)–(3) can be reduced to the following abstract initial value problem for a first-order evolution equation

$$\frac{d}{dt}U(t) = \mathscr{A}U(t), \quad U(0) = U_0, \ \forall t > 0 \tag{4}$$

where $U(t) = (u, w, u_t, w_t, \theta)^T$, $U_0 = (u_0, w_0, u_1, w_1, \theta_0)^T$ and

$$\mathscr{A}\begin{pmatrix} u \\ w \\ v \\ \eta \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ \eta \\ \frac{1}{\rho_1}(a_{11}u + a_{12}w + b_{11}v + b_{12}\eta)_{xx} - \frac{\alpha}{\rho_1}(u-w) - \frac{\alpha_1}{\rho_1}(v-\eta) - \frac{\beta_1}{\rho_1}\theta_x \\ \frac{1}{\rho_2}(a_{12}u + a_{22}w + b_{12}v + b_{22}\eta)_{xx} + \frac{\alpha}{\rho_2}(u-w) + \frac{\alpha_1}{\rho_2}(v-\eta) - \frac{\beta_2}{\rho_2}\theta_x \\ -\frac{\beta_1}{c}v_x - \frac{\beta_2}{c}\eta_x + \frac{\kappa}{c}\theta_{xx} \end{pmatrix} \tag{5}$$

We define $L^2_*(0, L) = \{\theta \in L^2(0, L) : \int_0^L \theta\, dx = 0\}$ the Hilbert space with the usual inner product and norm of $L^2(0, L)$ and consider

$$\mathscr{H} = H^1_0(0, L) \times H^1_0(0, L) \times L^2(0, L) \times L^2(0, L) \times L^2_*(0, L)$$

equipped with the inner product given by

$$\begin{aligned}\big\langle (u, w, v, \eta, \theta), (\tilde{u}, \tilde{w}, \tilde{v}, \tilde{\eta}, \tilde{\theta})\big\rangle_{\mathscr{H}} &= \int_0^L (a_{11}u_x\bar{\tilde{u}}_x + a_{12}(u_x\bar{\tilde{w}}_x + w_x\bar{\tilde{u}}_x) + a_{22}w_x\bar{\tilde{w}}_x)dx \\ &\quad + \alpha\int_0^L (u-w)(\overline{\tilde{u}-\tilde{w}})dx + \rho_1\int_0^L v\bar{\tilde{v}}\,dx \\ &\quad + \rho_2\int_0^L \eta\bar{\tilde{\eta}}\,dx + c\int_0^L \theta\bar{\tilde{\theta}}\,dx\end{aligned}$$

and the norm induced $\|\cdot\|_{\mathscr{H}}$. We can show that the norm $\|\cdot\|_{\mathscr{H}}$ is equivalent to usual norm of $\mathscr{H}$. We also consider the Hilbert space $V = \{\varphi \in H^2(0, L) \cap L^2_*(0, L) : \varphi_x \in H^1_0(0, L)\}$ with norm $\|\varphi\|_V = \|\varphi_{xx}\|_{L^2(0,L)}$.

Instead of dealing with (1)–(3) we will consider (4) in the Hilbert space $\mathscr{H}$, with the domain of the operator $\mathscr{A}$:

$$\begin{aligned}\mathscr{D}(\mathscr{A}) = \big\{&U = (u, w, v, \eta, \theta) \in \mathscr{H} : v, \eta \in H^1_0(0, L), \theta \in V \\ &a_{11}u + a_{12}w + b_{11}v + b_{12}\eta,\ a_{12}u + a_{22}w + b_{12}v + b_{22}\eta \in H^2(0, L)\big\}\end{aligned}$$

Firstly, we show that the operator $\mathscr{A}$ generates a C_0-semigroup of contractions on the space $\mathscr{H}$.

Proposition 1. *The operator $\mathscr{A}$ generates a C_0-semigroup $S_{\mathscr{A}}(t)$ of contractions on the space $\mathscr{H}$.*

Proof. We will show that $\mathscr{A}$ is a dissipative operator and 0 belongs to the resolvent set of $\mathscr{A}$, denoted by $\rho(\mathscr{A})$. Then our conclusion will follow using the well known the Lumer–Phillips theorem (see Pazy [16]). We observe that the operator $\mathscr{A}$ is densely defined from $\mathscr{D}(\mathscr{A})$ to $\mathscr{H}$ and if $U = (u, w, v, \eta, \theta) \in \mathscr{D}(\mathscr{A})$ then

$$\begin{aligned}
\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = {} & a_{11}\int_0^L v_x\bar{u}_x\,dx + a_{12}\int_0^L v_x\bar{w}_x\,dx + \alpha\int_0^L v\bar{u}\,dx - \alpha\int_0^L v\bar{w}\,dx \\
& + a_{12}\int_0^L \eta_x\bar{u}_x\,dx + a_{22}\int_0^L \eta_x\bar{w}_x\,dx - \alpha\int_0^L \eta\bar{u}\,dx + \alpha\int_0^L \eta\bar{w}\,dx \\
& - \int_0^L [a_{11}u_x + a_{12}w_x + b_{11}v_x + b_{12}\eta_x]\bar{v}_x\,dx - \int_0^L \beta_1\theta_x\bar{v}\,dx \\
& - \int_0^L [a_{12}u_x + a_{22}w_x + b_{12}v_x + b_{22}\eta_x]\bar{\eta}_x\,dx - \int_0^L \beta_2\theta_x\bar{\eta}\,dx - \alpha\int_0^L u\bar{v}\,dx \\
& + \alpha\int_0^L w\bar{v}\,dx + \alpha\int_0^L u\bar{\eta}\,dx - \alpha\int_0^L w\bar{\eta}\,dx - \int_0^L (\beta_1 v_x + \beta_2\eta_x - \kappa\theta_{xx})\bar{\theta}\,dx \\
& - \alpha_1\int_0^L |v-\eta|^2 dx.
\end{aligned}$$

Performing straightforward calculations we obtain

$$\begin{aligned}
Re\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = {} & -\kappa\|\theta_x\|^2_{L^2(0,L)} - b_{11}\|v_x\|^2_{L^2(0,L)} - b_{22}\|\eta_x\|^2_{L^2(0,L)} \\
& - 2b_{12}Re\int_0^L v_x\bar{\eta}_x\,dx - \alpha_1\|v-\eta\|^2_{L^2(0,L)}
\end{aligned} \tag{6}$$

Case I: The matrix B is positive definite. In this case we get

$$\begin{aligned}
Re\langle \mathscr{A}U, U\rangle_{\mathscr{H}} \le {} & -\kappa\|\theta_x\|^2_{L^2(0,L)} - \frac{\det B}{2b_{22}}\|v_x\|^2_{L^2(0,L)} - \frac{\det B}{2b_{11}}\|\eta_x\|^2_{L^2(0,L)} \\
& - \alpha_1\|v-\eta\|^2_{L^2(0,L)} \le 0
\end{aligned} \tag{7}$$

Therefore the operator $\mathscr{A}$ is dissipative.

Case II: The matrix B is singular. Now,

(a) $b_{11} > 0$ implies $b_{22} = b_{12}^2/b_{11}$. Then in (6) we have

$$Re\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = -\kappa\|\theta_x\|^2_{L^2(0,L)} - \frac{1}{b_{11}}\|b_{11}v_x + b_{12}\eta_x\|^2_{L^2(0,L)} - \alpha_1\|v-\eta\|^2_{L^2(0,L)} \le 0 \tag{8}$$

(b) $b_{11} = 0$ implies $b_{12} = 0$. Then in (6) we have

$$Re\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = -\kappa\|\theta_x\|^2_{L^2(0,L)} - b_{22}\|\eta_x\|^2_{L^2(0,L)} - \alpha_1\|v-\eta\|^2_{L^2(0,L)} \le 0 \tag{9}$$

Thus $\mathscr{A}$ is also dissipative.

On the other hand, we show that $0 \in \rho(\mathcal{A})$. In fact, given $F = (f, g, h, p, q) \in \mathcal{H}$, we must show that there exists a unique $U = (u, w, v, \eta, \theta)$ in $\mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = F$, that is,

$$v = f \quad \text{in } H_0^1(0, L) \tag{10}$$

$$\eta = g \quad \text{in } H_0^1(0, L) \tag{11}$$

$$(a_{11}u + a_{12}w + b_{11}v + b_{12}\eta)_{xx} - \alpha(u - w) - \alpha_1(v - \eta) - \beta_1\theta_x = \rho_1 h \quad \text{in } L^2(0, L) \tag{12}$$

$$(a_{12}u + a_{22}w + b_{12}v + b_{22}\eta)_{xx} + \alpha(u - w) + \alpha_1(v - \eta) - \beta_2\theta_x = \rho_2 p \quad \text{in } L^2(0, L) \tag{13}$$

$$-\beta_1 v_x - \beta_2\eta_x + \kappa\theta_{xx} = cq \quad \text{in } L_*^2(0, L) \tag{14}$$

Replacing (10), (11) in (14) we have

$$\kappa\theta_{xx} = cq + \beta_1 f_x + \beta_2 g_x \in L_*^2(0, L) \tag{15}$$

It is known that there is a unique $\theta \in V$ satisfying (15).

On the other hand, given the continuous and coercive sesquilinear form

$$\begin{aligned} M((u, w), (\varphi, \psi)) = {} & a_{11}\int_0^L u_x\bar{\varphi}_x\,dx + a_{12}\int_0^L u_x\bar{\psi}_x\,dx + a_{12}\int_0^L w_x\bar{\varphi}_x \\ & + a_{22}\int_0^L w_x\bar{\psi}_x\,dx + \alpha\int_0^L (u - w)\overline{(\varphi - \psi)}dx \end{aligned}$$

for $(u, w), (\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L)$ and the functional $G : H_0^1(0, L) \times H_0^1(0, L) \to \mathbb{C}$

$$\begin{aligned} G(\varphi, \psi) = {} & -\int_0^L (b_{11}v + b_{12}\eta)_x\bar{\varphi}_x\,dx - \rho_1\int_0^L h\bar{\varphi}\,dx - \beta_1\int_0^L \theta_x\bar{\varphi}\,dx \\ & - \int_0^L (b_{12}v + b_{22}\eta)_x\bar{\psi}_x\,dx - \rho_2\int_0^L p\bar{\psi}\,dx - \beta_2\int_0^L \theta_x\bar{\psi}\,dx \\ & - \alpha_1\int_0^L (v - \eta)\overline{(\varphi - \psi)}dx \end{aligned}$$

it follows using the Lax-Milgram theorem that there exists a unique vector function (u, w) in $H_0^1(0, L) \times H_0^1(0, L)$ such that

$$M((u, w), (\varphi, \psi)) = G(\varphi, \psi), \quad \forall(\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L)$$

Thus

$$\begin{aligned} & \int_0^L (a_{11}u_x + a_{12}w_x + b_{11}v_x + b_{12}\eta_x)\bar{\varphi}_x\,dx \\ & \quad + \alpha\int_0^L (u - w)\bar{\varphi}\,dx + \beta_1\int_0^L \theta_x\bar{\varphi}\,dx + \alpha_1\int_0^L (v - \eta)\bar{\varphi}\,dx = -\rho_1\int_0^L h\bar{\varphi}\,dx \\ & \int_0^L (a_{12}u_x + a_{22}w_x + b_{12}v_x + b_{22}\eta_x)\bar{\psi}_x\,dx \\ & \quad - \alpha\int_0^L (u - w)\bar{\psi}\,dx + \beta_2\int_0^L \theta_x\bar{\psi}\,dx - \alpha_1\int_0^L (v - \eta)\bar{\psi}\,dx = -\rho_2\int_0^L p\bar{\psi}\,dx \end{aligned}$$

$\forall \varphi, \psi \in H_0^1(0, L)$. It follows that $a_{11}u + a_{12}w + b_{11}v + b_{12}\eta$ and $a_{12}u + a_{22}w + b_{12}v + b_{22}\eta$ belong to $H_0^1(0, L) \cap H^2(0, L)$ and the Eqs. (12), (13) are verified. Moreover, it is easy to show that $\|U\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}$, for a positive constant C. Therefore, we conclude that $0 \in \rho(\mathcal{A})$. □

From the Proposition 1 we can establish the following result (see Pazy [16]).

Theorem 1. *For any $U_0 \in \mathcal{H}$ there exists a unique solution $U(t) = (u, w, u_t, w_t, \theta)$ of* (1)–(3) *satisfying*

$$u, w \in C([0, \infty[: H_0^1(0, L)) \cap C^1([0, \infty[: L^2(0, L))$$

and

$$\theta \in C([0, \infty[: L_*^2(0, L)) \cap L^2(]0, \infty[: H^1(0, L))$$

If $U_0 \in \mathcal{D}(\mathcal{A})$,

$$u, w \in C^1\big([0, \infty[: H_0^1(0, L)\big) \cap C^2([0, \infty[: L^2(0, L))$$
$$a_{11}u + a_{12}w + b_{11}u_t + b_{12}w_t \in C([0, \infty[: H^2(0, L))$$
$$a_{12}u + a_{22}w + b_{12}u_t + b_{22}w_t \in C([0, \infty[: H^2(0, L))$$

and

$$\theta \in C^1([0, \infty[: L^2(0, L)) \cap C([0, \infty[: V)$$

ANALYTICITY

In this section we will prove the analyticity of the semigroup $S_{\mathcal{A}}(t)$ when the matrix B is positive definite. Our main tool will be the following theorem whose proof can be seen in Liu and Zheng [13].

Theorem 2. *Let $S(t)$ be a C_0-semigroup of contractions of linear operators in a Hilbert space $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. Suppose that $i\mathbb{R} \subset \rho(\mathcal{A})$. Then, $S(t)$ is analytic if and only if*

$$\limsup_{|\lambda|\to\infty} \|\lambda(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{16}$$

where $\mathcal{L}(\mathcal{H})$ denotes the space of continuous linear functions in $\mathcal{H}$.

Lemma 1. *Let $\mathcal{A}$ be defined in* (5) *and assume that the matrix B is positive definite. Then set $i\mathbb{R} = \{i\lambda : \lambda \in \mathbb{R}\}$ is contained in $\rho(\mathcal{A})$.*

Proof. In this lemma, we will use $\|.\|$ to denote the norm in the space $\mathcal{L}(\mathcal{H})$. Following the arguments given by Liu and Zheng [13], the proof consists of the following steps:

Step 1. Since $0 \in \rho(\mathcal{A})$, for any real number λ with $\|\lambda\mathcal{A}^{-1}\| < 1$, the linear bounded operator $(i\lambda\mathcal{A}^{-1} - I)$ is invertible, therefore $i\lambda I - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - I)$ is invertible and its inverse belongs to $\mathcal{L}(\mathcal{H})$, that is, $i\lambda \in \rho(\mathcal{A})$. Moreover, $\|(i\lambda I - \mathcal{A})^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathcal{A}^{-1}\|^{-1}, \|\mathcal{A}^{-1}\|^{-1})$.

Step 2. If $\sup\{\|(i\lambda I - \mathcal{A})^{-1}\| : |\lambda| < \|\mathcal{A}^{-1}\|^{-1}\} = M < \infty$, then for $|\lambda_0| < \|\mathcal{A}^{-1}\|^{-1}$ and $\lambda \in \mathbb{R}$ such that $|\lambda - \lambda_0| < M^{-1}$, we have $\|(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1}\| < 1$, therefore the operator

$$i\lambda I - \mathcal{A} = (i\lambda_0 I - \mathcal{A})(I + i(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1})$$

is invertible with its inverse in $\mathcal{L}(\mathcal{H})$, that is, $i\lambda \in \rho(\mathcal{A})$. Since λ_0 is arbitrary we can conclude that $\{i\lambda : |\lambda| < \|\mathcal{A}^{-1}\|^{-1} + M^{-1}\} \subset \rho(\mathcal{A})$ and the function $\|(i\lambda I - \mathcal{A})^{-1}\|$ is continuous in the interval $(-\|\mathcal{A}^{-1}\|^{-1} - M^{-1}, \|\mathcal{A}^{-1}\|^{-1} + M^{-1})$.

Step 3. Thus, it follows by item (ii) that if $i\,\mathbb{R} \subset \rho(A)$ is not true, then there exists $\omega \in \mathbb{R}$ with $\|\mathcal{A}^{-1}\|^{-1} \le |\omega|$ such that $\{i\lambda : |\lambda| < |\omega|\} \subset \rho(\mathcal{A})$ and $\sup\{\|(i\lambda I - \mathcal{A})^{-1}\| : |\lambda| < |\omega|\} = \infty$. Therefore, there exists a sequence λ_n in $\mathbb{R}$ with $\lambda_n \to \omega$, $|\lambda_n| < |\omega|$ and sequences of vector functions $U_n = (u_n, w_n, v_n, \eta_n, \theta_n) \in \mathcal{D}(\mathcal{A})$, $F_n = (f_n, g_n, h_n, p_n, q_n) \in \mathcal{H}$, such that $(i\lambda_n I - \mathcal{A})U_n = F_n$ and $\|U_n\|_{\mathcal{H}} = 1$ and $F_n \to 0$ in $\mathcal{H}$ when $n \to \infty$. Since

$$Re\langle(i\lambda_n I - \mathcal{A})U_n, U_n\rangle_{\mathcal{H}} \to 0$$

using the same idea given by (7) we have

$$\kappa\|\theta_{nx}\|^2_{L^2(0,L)} + \frac{\det B}{2b_{22}}\|v_{nx}\|^2_{L^2(0,L)} + \frac{\det B}{2b_{11}}\|\eta_{nx}\|^2_{L^2(0,L)} \to 0 \quad \text{as } n \to \infty$$

Therefore $v_n \to 0$, $\eta_n \to 0$ in $H^1_0(0, L)$ and $\theta_n \to 0$ in $H^1(0, L)$. On the other hand, we have that $i\lambda_n u_n - v_n = f_n \to 0$, $i\lambda_n w_n - \eta_n = g_n \to 0$ in $H^1_0(0, L)$. Thus we obtain $u_n \to 0$, $w_n \to 0$ in $H^1_0(0, L)$, because $\omega \neq 0$. Since $\|U_n\|_{\mathcal{H}} = 1$, for all $n \in \mathbb{N}$, we have a contradiction and the proof of the lemma is complete. □

Theorem 3. *Let $\mathcal{A}$ be defined in* (5) *and assume that B is positive definite. Then the semigroup $S_{\mathcal{A}}(t)$ is analytic.*

Proof. From the Theorem 2 and Lemma 1, it suffices to show that (16) holds. Given $\lambda \in \mathbb{R}$ and $F = (f, g, h, p, q) \in \mathcal{H}$, there is a unique $U = (u, w, v, \eta, \theta)$ in $\mathcal{D}(\mathcal{A})$, such that $(i\lambda I - \mathcal{A})U = F$, that is,

$$i\lambda u - v = f \quad \text{in } H^1_0(0, L) \tag{17}$$

$$i\lambda w - \eta = g \quad \text{in } H^1_0(0, L) \tag{18}$$

$$\begin{aligned} &i\lambda\rho_1 v - (a_{11}u + a_{12}w + b_{11}v + b_{12}\eta)_{xx} + \alpha(u - w) \\ &\quad + \alpha_1(v - \eta) + \beta_1\theta_x = \rho_1 h \quad \text{in } L^2(0, L) \end{aligned} \tag{19}$$

$$i\lambda\rho_2\eta - (a_{12}u + a_{22}w + b_{12}v + b_{22}\eta)_{xx} - \alpha(u - w)$$
$$- \alpha_1(v - \eta) + \beta_2\theta_x = \rho_2 p \quad \text{in } L^2(0, L) \tag{20}$$
$$i\lambda c\theta + \beta_1 v_x + \beta_2\eta_x - \kappa\theta_{xx} = cq \quad \text{in } L^2_*(0, L) \tag{21}$$

Note that $Re\langle(i\lambda I - \mathscr{A})U, U\rangle_{\mathscr{H}} = -Re\langle\mathscr{A}U, U\rangle_{\mathscr{H}} = Re\langle F, U\rangle_{\mathscr{H}}$. Hence

$$\|v_x\|^2_{L^2(0,L)} + \|\eta_x\|^2_{L^2(0,L)} + \|\theta_x\|^2_{L^2(0,L)} \leq C\|F\|_{\mathscr{H}}\|U\|_{\mathscr{H}} \tag{22}$$

Taking the inner product in $L^2(0, L)$ of (19), (20) with u, w respectively, and using (17), (18) we obtain

$$\begin{aligned}
&a_{11}\int_0^L |u_x|^2 dx + \alpha\int_0^L |u|^2 dx + a_{12}\int_0^L w_x\bar{u}_x\,dx - \alpha\int_0^L w\bar{u}\,dx \\
&= - b_{11}\int_0^L v_x\bar{u}_x\,dx - b_{12}\int_0^L \eta_x\bar{u}_x\,dx + \rho_1\int_0^L |v|^2 dx + \rho_1\int_0^L v\bar{f}\,dx \\
&\quad + \rho_1\int_0^L h\bar{u}\,dx - \beta_1\int_0^L \theta_x\bar{u}\,dx - \alpha_1\int_0^L (v - \eta)\bar{u}\,dx
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&a_{22}\int_0^L |w_x|^2 dx + \alpha\int_0^L |w|^2 dx + a_{12}\int_0^L u_x\bar{w}_x\,dx - \alpha\int_0^L u\bar{w}\,dx \\
&= -b_{12}\int_0^L v_x\bar{w}_x\,dx - b_{22}\int_0^L \eta_x\bar{w}_x\,dx + \rho_2\int_0^L |\eta|^2 dx + \rho_2\int_0^L \eta\bar{g}\,dx \\
&\quad + \rho_2\int_0^L p\bar{w}\,dx - \beta_2\int_0^L \theta_x\bar{w}\,dx + \alpha_1\int_0^L (v - \eta)\bar{w}\,dx
\end{aligned} \tag{24}$$

Adding (23) and (24), using the inequalities of Young and Cauchy–Schwartz and performing a straightforward calculation we obtain

$$\begin{aligned}
&\frac{\det A}{2a_{22}}\|u_x\|^2_{L^2(0,L)} + \frac{\det A}{2a_{11}}\|w_x\|^2_{L^2(0,L)} + \alpha\|u - w\|^2_{L^2(0,L)} \\
&\leq |\beta_1|\int_0^L |\theta_x||u|dx + |\beta_2|\int_0^L |\theta_x||w|dx + b_{11}\int_0^L |v_x||u_x|dx + |b_{12}|\int_0^L |\eta_x||u_x|dx \\
&\quad + b_{22}\int_0^L |\eta_x||w_x|dx + |b_{12}|\int_0^L |v_x||w_x|dx + \rho_1\int_0^L |v|^2 dx + \rho_1\int_0^L |v||f|dx \\
&\quad + \rho_1\int_0^L |h||u|dx + \rho_2\int_0^L |\eta|^2 dx + \rho_2\int_0^L |\eta||g|dx + \rho_2\int_0^L |p||u|dx \\
&\quad + \alpha_1\int_0^L |v - \eta||u - w|dx
\end{aligned}$$

It results by the Poincaré inequality and (22) that

$$\|u_x\|^2_{L^2(0,L)} + \|w_x\|^2_{L^2(0,L)} \leq C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}} \tag{25}$$

Taking the imaginary part of the inner product in $L^2(0, L)$ of (21) with θ and using (22) we obtain

$$\lambda\|\theta\|^2_{L^2(0,L)} \leq C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}} \tag{26}$$

By (17), (18), (22) and (25) we get

$$\lambda\|u_x\|^2_{L^2(0,L)} + \lambda\|w_x\|^2_{L^2(0,L)} \leq C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}} \tag{27}$$

Finally, taking the imaginary part of the inner product in $L^2(0, L)$ of (19) with v and (20) with η and using (22), (25) we obtain

$$\lambda\|v\|^2_{L^2(0,L)} + \lambda\|\eta\|^2_{L^2(0,L)} \leq C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}} \tag{28}$$

Combining (26), (27) and (28) we find that

$$\lambda\|U\|^2_{\mathscr{H}} \leq C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}} \tag{29}$$

That is,

$$|\lambda|\|(i\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \leq C$$

The theorem follows. □

THE LACK OF ANALYTICITY

In this section, the goal is to prove that the semigroup $S_{\mathscr{A}}(t)$ is not analytic when the matrix B is singular and negative definite. To simplify the computations we consider, without loss of generality, $L = \pi$ and $\rho_1 = \rho_2 = 1$. Due to the Theorem 2 it is enough to show that there is a sequence λ_n in $\mathbb{R}$, $\lambda_n \to \infty$ when $n \to \infty$, and a bounded sequence F_n in $\mathscr{H}$ such that

$$\|\lambda_n(i\lambda_n I - \mathscr{A})^{-1}F_n\|_{\mathscr{H}} \to \infty \quad \text{as } n \to \infty \tag{30}$$

Remark 1. For instance, we observe that if $a_{11} = a_{22}$, $b_{11} = b_{22} = -b_{12}$ and $\beta_1 + \beta_2 = 0$, then we can obtain solutions of the form $u = w$ and $\theta = 0$. These solutions are undamped and do not decay to zero. Therefore, the semigroups associated are not analytic. On the other hand, if $\alpha_1 = 0$, $a_{11} = a_{22}$, $b_{11} = b_{22} = b_{12}$ and $\beta_1 - \beta_2 = 0$, we can also obtain solutions undamped of the form $u = -w$ and $\theta = 0$. In the proof of the next theorem we are not going to consider these cases.

Initially, we observe that there is an orthogonal matrix $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & -p_{11} \end{pmatrix}$ and a diagonal matrix $D = \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix}$, $\xi > 0$, such that $B = PDP$. Moreover, we observe that we are going to separate the proof of the next theorem in the cases $\alpha_1 = 0$ and $\alpha_1 > 0$.

Theorem 4. *Let B be a singular matrix. Then the semigroup $S_{\mathscr{A}}(t)$ is not analytic.*

Proof. For each $n \in \mathbb{N}$, we take $F_n = (0, 0, p_{12}\sin(nx), -p_{11}\sin(nx), 0) \in \mathscr{H}$ and we denote by $U_n = (u_n, w_n, v_n, \eta_n, \theta_n) \in \mathscr{D}(\mathscr{A})$ the solution of the resolvent equation

$$(i\lambda I - \mathscr{A})U_n = F_n, \quad \lambda \in \mathbb{R} \tag{31}$$

We know that $Re\langle (i\lambda I - \mathscr{A})U_n, U_n\rangle_{\mathscr{H}} = -Re\langle \mathscr{A}U_n, U_n\rangle_{\mathscr{H}} = Re\langle F_n, U_n\rangle_{\mathscr{H}}$.

If $b_{11} > 0$, it follows from (8) that

$$\kappa\|\theta_{nx}\|^2_{L^2(0,\pi)} + \frac{1}{b_{11}}\|b_{11}v_{nx} + b_{12}\eta_{nx}\|^2_{L^2(0,\pi)} \leq \|U_n\|_{\mathscr{H}}\|F_n\|_{\mathscr{H}}, \quad \forall n \in \mathbb{N}$$

Since $\|F_n\|_{\mathscr{H}} = \sqrt{\frac{\pi}{2}}$ for all $n \in \mathbb{N}$, it follows that

$$\|U_n\|_{\mathscr{H}} \geq \sqrt{\frac{2\kappa^2}{\pi}}\|\theta_{nx}\|^2_{L^2(0,\pi)} + \sqrt{\frac{2}{\pi b_{11}^2}}\|b_{11}v_{nx} + b_{12}\eta_{nx}\|^2_{L^2(0,\pi)}, \quad \forall n \in \mathbb{N} \tag{32}$$

We observe that if $b_{11} = 0$ we can obtain a similar inequality to (32). Due to the boundary conditions, the solutions are the form $u_n = A_n\sin(nx)$, $w_n = B_n\sin(nx)$ and $\theta_n = C_n\cos(nx)$ and by Eq. (31) we have $v_n = i\lambda u_n$, $\eta_n = i\lambda w_n$ and

$$\begin{aligned} &-\lambda^2 A_n + n^2(a_{11} + i\lambda b_{11})A_n + n^2(a_{12} + i\lambda b_{12})B_n + \alpha(A_n - B_n) \\ &\quad + i\alpha_1\lambda(A_n - B_n) - \beta_1 nC_n = p_{12} \end{aligned} \tag{33}$$

$$\begin{aligned} &-\lambda^2 B_n + n^2(a_{12} + i\lambda b_{12})A_n + n^2(a_{22} + i\lambda b_{22})B_n - \alpha(A_n - B_n) \\ &\quad - i\alpha_1\lambda(A_n - B_n) - \beta_2 nC_n = -p_{11} \end{aligned} \tag{34}$$

$$c\lambda C_n + \lambda\beta_1 nA_n + \lambda\beta_2 nB_n - i\kappa n^2 C_n = 0 \tag{35}$$

We can rewrite (33), (34) as

$$-\lambda^2 X_n + n^2 AX_n + i\lambda n^2 BX_n + \Phi X_n + i\lambda\Phi_1 X_n - n\Psi C_n = G \tag{36}$$

such that

$$X_n = \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix},$$

$$\Phi = \begin{pmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} \alpha_1 & -\alpha_1 \\ -\alpha_1 & \alpha_1 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad G = \begin{pmatrix} p_{12} \\ -p_{11} \end{pmatrix}$$

Therefore, for $Y_n = PX_n = \begin{pmatrix} \tilde{A}_n \\ \tilde{B}_n \end{pmatrix}$ we have in (36) that

$$-\lambda^2 Y_n + n^2 PAPY_n + i\lambda n^2 DY_n + P\Phi PY_n + i\lambda P\Phi_1 PY_n - P\Psi nC_n = PG \tag{37}$$

We introduce the notation

$$PAP = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{12} & \tilde{a}_{22} \end{pmatrix}, \quad \tilde{a}_{11} > 0, \ \tilde{a}_{22} > 0$$

$$P\Phi P = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}, \quad \alpha_{11} \geq 0, \ \ \alpha_{22} \geq 0$$

$$P\Phi_1 P = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}, \quad \gamma_{11} \geq 0, \ \ \gamma_{22} \geq 0$$

$$P\Psi = \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \end{pmatrix}$$

Replacing in (37) together with (35) we obtain

$$(-\lambda^2 + \tilde{a}_{11}n^2 + \alpha_{11} + i\lambda\xi n^2 + i\lambda\gamma_{11})\widetilde{A}_n + (\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda\gamma_{12})\widetilde{B}_n - \tilde{\beta}_1 nC_n = 0 \quad (38)$$

$$(\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda\gamma_{12})\widetilde{A}_n + (-\lambda^2 + \tilde{a}_{22}n^2 + \alpha_{22} + i\lambda\gamma_{22})\widetilde{B}_n - \tilde{\beta}_2 nC_n = 1 \quad (39)$$

$$\left(c\lambda - i\kappa n^2\right) nC_n + \lambda\tilde{\beta}_1 n^2 \widetilde{A}_n + \lambda\tilde{\beta}_2 n^2 \widetilde{B}_n = 0 \quad (40)$$

We take $\lambda = \lambda_n = \sqrt{\tilde{a}_{22}n^2 + \alpha_{22}}$, $n \in \mathbb{N}$, in (31).

Case $\alpha_1 = 0$. From (38)–(40) we get

$$(-\lambda_n^2 + \tilde{a}_{11}n^2 + i\lambda_n\xi n^2 + \alpha_{11})\widetilde{A}_n + (\tilde{a}_{12}n^2 + \alpha_{12})\widetilde{B}_n - \tilde{\beta}_1 nC_n = 0 \quad (41)$$

$$(\tilde{a}_{12}n^2 + \alpha_{12})\widetilde{A}_n + (-\lambda_n^2 + \tilde{a}_{22}n^2 + \alpha_{22})\widetilde{B}_n - \tilde{\beta}_2 nC_n = 1 \quad (42)$$

$$\left(c\frac{\lambda_n}{n^2} - i\kappa\right) nC_n + \lambda_n\tilde{\beta}_1\widetilde{A}_n + \lambda_n\tilde{\beta}_2\widetilde{B}_n = 0 \quad (43)$$

We divide the proof in three items.

Item I. Assume that (β_1, β_2) is not an eigenvector of B associated to the eigenvalue ξ, that is, $\tilde{\beta}_2 = p_{12}\beta_1 - p_{11}\beta_2 \neq 0$.

If $\tilde{a}_{12} = \alpha_{12} = 0$, from (42) we get

$$nC_n = -\frac{1}{\tilde{\beta}_2}$$

and

$$\int_0^\pi |\theta_{nx}|^2 dx = \frac{\pi}{2\tilde{\beta}_2^2}$$

Using (32) we obtain (30).

If $\tilde{a}_{12} \neq 0$ or $\alpha_{12} \neq 0$, then from (42) we have

$$\widetilde{A}_n = \frac{1}{\tilde{a}_{12}n^2 + \alpha_{12}} + \frac{\tilde{\beta}_2}{\tilde{a}_{12}n^2 + \alpha_{12}} nC_n \quad (44)$$

Substituting (44) in (41) we obtain

$$\widetilde{B}_n = \frac{(\lambda_n^2 - \tilde{a}_{11}n^2 - i\lambda_n\xi n^2 - \alpha_{11})}{(\tilde{a}_{12}n^2 + \alpha_{12})^2} + \frac{\tilde{\beta}_2(\lambda_n^2 - \tilde{a}_{11}n^2 - i\lambda_n\xi n^2 - \alpha_{11}) + \tilde{\beta}_1(\tilde{a}_{12}n^2 + \alpha_{12})}{(\tilde{a}_{12}n^2 + \alpha_{12})^2} nC_n$$

Substituting the above term in (43) we get

$$\left(\frac{c\lambda_n - i\kappa n^2}{n^2} + \frac{\tilde{\beta}_2^2\lambda_n(\lambda_n^2 - \tilde{a}_{11}n^2 - i\lambda_n\xi n^2 - \alpha_{11}) + 2\lambda_n\tilde{\beta}_1\tilde{\beta}_2(\tilde{a}_{12}n^2 + \alpha_{12})}{(\tilde{a}_{12}n^2 + \alpha_{12})^2}\right)nC_n$$
$$= -\frac{\lambda_n\tilde{\beta}_1(\tilde{a}_{12}n^2 + \alpha_{12}) + \lambda_n\tilde{\beta}_2(\lambda_n^2 - \tilde{a}_{11}n^2 - i\lambda_n\xi n^2 - \alpha_{11})}{(\tilde{a}_{12}n^2 + \alpha_{12})^2}$$

Defining

$$P_n = -\tilde{\beta}_2 n^2\lambda_n^3 + i\tilde{\beta}_2\xi n^4\lambda_n^2 + \lambda_n(\tilde{\beta}_2\tilde{a}_{11}n^4 - \tilde{\beta}_1\tilde{a}_{12}n^4 + \tilde{\beta}_2\alpha_{11}n^2 - \tilde{\beta}_1\alpha_{12}n^2)$$
$$Q_n = (c\lambda_n - i\kappa n^2)(\tilde{a}_{12}n^2 + \alpha_{12})^2 + 2\lambda_n\tilde{\beta}_1\tilde{\beta}_2(\tilde{a}_{12}n^4 + \alpha_{12}n^2)$$
$$+ \tilde{\beta}_2^2\lambda_n(\lambda_n^2 n^2 - \tilde{a}_{11}n^4 - i\lambda_n\xi n^4 - \alpha_{11}n^2)$$

we obtain

$$nC_n = \frac{P_n}{Q_n}$$

Thus

$$\lim_{n\to\infty}\frac{P_n}{n^6} = i\tilde{\beta}_2\xi\tilde{a}_{22}, \quad \lim_{n\to\infty}\frac{Q_n}{n^6} = -i(\tilde{\beta}_2^2\xi\tilde{a}_{22} + \kappa\tilde{a}_{12}^2)$$

and

$$\lim_{n\to\infty} nC_n = -\frac{\tilde{\beta}_2\xi\tilde{a}_{22}}{\tilde{\beta}_2^2\xi\tilde{a}_{22} + \kappa\tilde{a}_{12}^2}$$

Since

$$\int_0^\pi |\theta_{nx}|^2 dx = \frac{\pi}{2}n^2C_n^2$$

we have (30).

Item II. Assume that (β_1, β_2) is an eigenvector of the matrix A and an eigenvector of B associated to the eigenvalue ξ, that is, $\tilde{\beta}_2 = 0$ and $\tilde{a}_{12} = 0$.

Due to Remark 1, we can suppose $\alpha_{12} \neq 0$. From (42) and (43) we get

$$\widetilde{A}_n = \frac{1}{\alpha_{12}}, \quad C_n = -\frac{\lambda_n\tilde{\beta}_1 n}{\alpha_{12}(c\lambda_n - i\kappa n^2)}$$

It follows using (41) that

$$\widetilde{B}_n = -\frac{(\tilde{a}_{11} - \tilde{a}_{22})n^2 - \alpha_{22} + \alpha_{11} + i\lambda_n\xi n^2}{\alpha_{12}^2} - \frac{\tilde{\beta}_1^2\lambda_n n^2}{\alpha_{12}^2(c\lambda_n - i\kappa n^2)}$$

Since

$$Im\,\widetilde{B}_n = n^2\lambda_n\left(-\frac{\xi}{\alpha_{12}^2} - \frac{\tilde{\beta}_1^2\kappa n^2}{\alpha_{12}^2(c^2\lambda_n^2+\kappa^2 n^4)}\right)$$

then

$$\lim_{n\to\infty}|\widetilde{B}_n| = \infty$$

Thus

$$\int_0^\pi |u_{nx}|^2dx = \frac{\pi}{2}|nA_n|^2 = \frac{\pi n^2}{2}|p_{11}\widetilde{A}_n + p_{12}\widetilde{B}_n|^2 \tag{45}$$

$$\int_0^\pi |w_{nx}|^2dx = \frac{\pi}{2}|nB_n|^2 = \frac{\pi n^2}{2}|p_{12}\widetilde{A}_n - p_{11}\widetilde{B}_n|^2 \tag{46}$$

Hence

$$\lim_{n\to\infty}\|u_{nx}\|_{L^2(0,\pi)} = \lim_{n\to\infty}\|w_{nx}\|_{L^2(0,\pi)} = \infty$$

Therefore

$$\lim_{n\to\infty}\|U_n\|_{\mathscr{H}} = \infty$$

and we have (30).

Item III. We assume that (β_1, β_2) is not an eigenvector of the matrix A and is an eigenvector of B associated to the eigenvalue ξ, that is, $\tilde{\beta}_2 = 0$, $\tilde{a}_{12} \neq 0$. Then the Eq. (42) yields

$$\widetilde{A}_n = \frac{1}{\tilde{a}_{12}n^2 + \alpha_{12}}.$$

On the other hand, we observe that

$$\begin{aligned}
\beta_1 A_n + \beta_2 B_n &= \beta_1(p_{11}\widetilde{A}_n + p_{12}\widetilde{B}_n) + \beta_2(p_{12}\widetilde{A}_n - p_{11}\widetilde{B}_n)\\
&= (\beta_1 p_{11} + \beta_2 p_{12})\widetilde{A}_n + (\beta_1 p_{12} - \beta_2 p_{11})\widetilde{B}_n\\
&= \tilde{\beta}_1\widetilde{A}_n + \tilde{\beta}_2\widetilde{B}_n\\
&= \tilde{\beta}_1\widetilde{A}_n
\end{aligned}$$

and

$$\beta_1 v_{nx} + \beta_2\eta_{nx} = i\tilde{\beta}_1\widetilde{A}_n\lambda_n n\cos(nx) \tag{47}$$

Thus

$$\lim_{n\to\infty}\int_0^\pi |\beta_1 v_{nx} + \beta_2\eta_{nx}|^2dx = \frac{\pi\tilde{a}_{22}\tilde{\beta}_1^2}{2\tilde{a}_{12}^2}.$$

Since $\{(b_{11}, b_{12}), (\beta_1, \beta_2)\}$ and $\{(b_{12}, b_{22}), (\beta_1, \beta_2)\}$ are linearly dependents, assuming $b_{11} > 0$ we have

$$\lim_{n\to\infty} \int_0^{\pi} |b_{11} v_{nx} + b_{12} \eta_{nx}|^2 dx > 0$$

It follows from (32) that (30) holds.

Case $\alpha_1 > 0$. From (38)–(40) we get

$$(m_1 n^2 + m_2 + i\lambda_n(\xi n^2 + \gamma_{11}))\widetilde{A}_n + (\tilde{a}_{12} n^2 + \alpha_{12} + i\lambda_n \gamma_{12})\widetilde{B}_n - \tilde{\beta}_1 n C_n = 0 \quad (48)$$

$$(\tilde{a}_{12} n^2 + \alpha_{12} + i\lambda_n \gamma_{12})\widetilde{A}_n + i\lambda_n \gamma_{22} \widetilde{B}_n - \tilde{\beta}_2 n C_n = 1 \quad (49)$$

$$(c\lambda_n - i\kappa n^2) n C_n + \lambda_n \tilde{\beta}_1 n^2 \widetilde{A}_n + \lambda_n \tilde{\beta}_2 n^2 \widetilde{B}_n = 0 \quad (50)$$

with $m_1 = a_{11} - a_{22}$ and $m_2 = \alpha_{11} - \alpha_{22}$. In this case we also divide the proof in three items.

Item I. Assume that (β_1, β_2) is not an eigenvector of B associated to the eigenvalue $\xi(\tilde{\beta}_2 \neq 0)$.

Suppose that $\gamma_{22} = 0$. It results that $\alpha_{12} = \gamma_{12} = 0$ and from (49) we get

$$\tilde{a}_{12} n^2 \widetilde{A}_n - \tilde{\beta}_2 n C_n = 1$$

If $\tilde{a}_{12} = 0$, then

$$n C_n = -\frac{1}{\tilde{\beta}_2} \quad \text{and} \quad \int_0^{\pi} |\theta_{nx}|^2 dx = \frac{\pi}{2\tilde{\beta}_2^2}$$

Using (32) we conclude (30).

If $\tilde{a}_{12} \neq 0$, then

$$\widetilde{A}_n = \frac{1}{\tilde{a}_{12} n^2} + \frac{\tilde{\beta}_2}{\tilde{a}_{12} n^2} n C_n \quad (51)$$

Substituting (51) in (48) we obtain

$$\widetilde{B}_n = -\frac{m_1 n^2 + m_2 + i\lambda_n(\xi n^2 + \gamma_{11})}{\tilde{a}_{12}^2 n^4} - \frac{\tilde{\beta}_2(m_1 n^2 + m_2 + i\lambda_n(\xi n^2 + \gamma_{11})) - \tilde{\beta}_1 \tilde{a}_{12} n^2}{\tilde{a}_{12}^2 n^4} n C_n$$

Substituting the above term in (50) we get

$$\left(\frac{c\lambda_n - i\kappa n^2}{n^2} + \frac{\tilde{\beta}_2^2 \lambda_n(-m_1 n^2 - m_2 - i\lambda_n(\xi n^2 + \gamma_{11})) + 2\lambda_n \tilde{\beta}_1 \tilde{\beta}_2 \tilde{a}_{12} n^2}{\tilde{a}_{12}^2 n^2}\right) n C_n$$
$$= \frac{-\tilde{\beta}_1 \tilde{a}_{12} \lambda_n n^2 + \lambda_n \tilde{\beta}_2(m_1 n^2 + m_2 + i\lambda_n(\xi n^2 + \gamma_{11}))}{\tilde{a}_{12}^2 n^2}$$

Defining

$$Z_n = n^2\lambda_n(m_1\tilde{\beta}_2 - \tilde{\beta}_1\tilde{a}_{12}) + m_2\tilde{\beta}_2\lambda_n + i\tilde{\beta}_2\lambda_n^2(\xi n^2 + \gamma_{11})$$
$$W_n = (c\lambda_n - i\kappa n^2)\tilde{a}_{12}^2 + 2\lambda_n\tilde{\beta}_1\tilde{\beta}_2\tilde{a}_{12}n^2 - \tilde{\beta}_2^2\lambda_n(m_1n^2 + m_2) - i\tilde{\beta}_2^2\lambda_n^2(\xi n^2 + \gamma_{11})$$

we obtain

$$nC_n = \frac{Z_n}{W_n}$$

Thus

$$\lim_{n\to\infty}\frac{Z_n}{n^4} = i\tilde{\beta}_2\xi\tilde{a}_{22}, \quad \lim_{n\to\infty}\frac{W_n}{n^4} = -i\tilde{\beta}_2^2\xi\tilde{a}_{22}$$

and

$$\lim_{n\to\infty} nC_n = -\frac{1}{\tilde{\beta}_2}$$

Since

$$\lim_{n\to\infty}\|\theta_{nx}\|_{L^2(0,L)} = \frac{1}{|\tilde{\beta}_2|}\sqrt{\frac{\pi}{2}}$$

we have (30).

Now, suppose that $\gamma_{22} \neq 0$. Multiplying (49) by $\tilde{\beta}_2 n^2$ and (50) by $-i\gamma_{22}$ and adding the results we get

$$[\tilde{\beta}_2n^2(\tilde{a}_{12}n^2 + \alpha_{12}) + in^2\lambda_n(\tilde{\beta}_2\gamma_{12} - \tilde{\beta}_1\gamma_{22})]\widetilde{A}_n - [(\tilde{\beta}_2^2 + \kappa\gamma_{22})n^2 + ic\gamma_{22}\lambda_n]nC_n = \tilde{\beta}_2n^2 \tag{52}$$

Multiplying (48) by $i\gamma_{22}\lambda_n$ and (49) by $-(\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda_n\gamma_{12})$ we get

$$\begin{aligned}&[i\gamma_{22}\lambda_n(m_1n^2 + m_2) - \gamma_{22}\lambda_n^2(\xi n^2 + \gamma_{11}) - (\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda_n\gamma_{12})^2]\widetilde{A}_n\\&\quad + [i\lambda_n(\gamma_{12}\tilde{\beta}_2 - \gamma_{22}\tilde{\beta}_1) + \tilde{\beta}_2(\tilde{a}_{12}n^2 + \alpha_{12})]nC_n = -(\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda_n\gamma_{12})\end{aligned} \tag{53}$$

Defining

$$\begin{aligned}M_n = &-\tilde{\beta}_2n^2[i\gamma_{22}\lambda_n(m_1n^2 + m_2) - \gamma_{22}\lambda_n^2(\xi n^2 + \gamma_{11}) - (\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda_n\gamma_{12})^2]\\&- [\tilde{\beta}_2n^2(\tilde{a}_{12}n^2 + \alpha_{12}) + in^2\lambda_n(\tilde{\beta}_2\gamma_{12} - \tilde{\beta}_1\gamma_{22})](\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda_n\gamma_{12})\end{aligned}$$

and

$$\begin{aligned}N_n = &[(\tilde{\beta}_2^2 + \kappa\gamma_{22})n^2 + ic\gamma_{22}\lambda_n][i\gamma_{22}\lambda_n(m_1n^2 + m_2) - \gamma_{22}\lambda_n^2(\xi n^2 + \gamma_{11})\\&\quad - (\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda_n\gamma_{12})^2]\\&+ [\tilde{\beta}_2n^2(\tilde{a}_{12}n^2 + \alpha_{12}) + in^2\lambda_n(\tilde{\beta}_2\gamma_{12} - \tilde{\beta}_1\gamma_{22})][i\lambda_n(\gamma_{12}\tilde{\beta}_2 - \gamma_{22}\tilde{\beta}_1) + \tilde{\beta}_2(\tilde{a}_{12}n^2 + \alpha_{12})]\end{aligned}$$

we obtain from (52) and (53)

$$nC_n = \frac{M_n}{N_n}$$

Thus

$$\lim_{n\to\infty} \frac{M_n}{n^6} = \tilde{\beta}_2 \xi \tilde{a}_{22}\gamma_{22}, \quad \lim_{n\to\infty} \frac{N_n}{n^6} = -(\tilde{\beta}_2^2 + \kappa\gamma_{22})\xi\tilde{a}_{22}\gamma_{22} - \kappa\gamma_{22}\tilde{a}_{12}^2$$

and

$$\lim_{n\to\infty} nC_n = -\frac{\tilde{\beta}_2\xi\tilde{a}_{22}}{(\tilde{\beta}_2^2 + \kappa\gamma_{22})\xi\tilde{a}_{22} + \kappa\tilde{a}_{12}^2}$$

Since

$$\lim_{n\to\infty} \|\theta_{nx}\|_{L^2(0,L)} = \sqrt{\frac{\pi}{2}}\frac{|\tilde{\beta}_2|\xi\tilde{a}_{22}}{(\tilde{\beta}_2^2 + \kappa\gamma_{22})\xi\tilde{a}_{22} + \kappa\tilde{a}_{12}^2}$$

we have (30).

Item II. Assume that (β_1, β_2) is an eigenvector of the matrix A and an eigenvector of B associated to the eigenvalue $\xi(\tilde{\beta}_2 = \tilde{a}_{12} = 0)$.

Due to Remark 1, we can suppose $\gamma_{22} \neq 0$. From (49) and (50) we get

$$\widetilde{B}_n = \frac{1}{i\gamma_{22}\lambda_n} - \frac{\alpha_{12} + i\gamma_{12}\lambda_n}{i\gamma_{22}\lambda_n}\widetilde{A}_n, \quad C_n = -\frac{\tilde{\beta}_1\lambda_n n}{c\lambda_n - i\kappa n^2}\widetilde{A}_n$$

It follows using (48) that

$$\widetilde{A}_n = -\frac{G_n}{H_n}$$

with

$$\begin{aligned} G_n &= c\alpha_{12}\lambda_n + \kappa\gamma_{12}\lambda_n n^2 + i(c\gamma_{12}\lambda_n^2 - \kappa\alpha_{12}n^2) \\ H_n &= -c\gamma_{22}\lambda_n^3(\xi n^2 + \gamma_{11}) + \kappa\gamma_{22}\lambda_n n^2(m_1 n^2 + m_2) - c\alpha_{12}^2\lambda_n + c\gamma_{12}^2\lambda_n^3 \\ &\quad - 2\kappa\alpha_{12}\gamma_{12}\lambda_n n^2 + i[(\tilde{\beta}_1^2\gamma_{22} - \kappa\gamma_{12}^2 + \kappa\gamma_{11}\gamma_{22})\lambda_n^2 n^2 + \kappa\alpha_{12}^2 n^2 - 2c\alpha_{12}\gamma_{12}\lambda_n^2 \\ &\quad + \kappa\gamma_{22}\xi\lambda_n^2 n^4 + c\gamma_{22}\lambda_n^2(m_1 n^2 + m_2)] \end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{nG_n}{n^6} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{H_n}{n^6} = i\kappa\gamma_{22}\xi$$

then

$$\lim_{n\to\infty} n\widetilde{A}_n = 0 \quad \text{and} \quad \lim_{n\to\infty} n\widetilde{B}_n = -i\frac{1}{\gamma_{22}\sqrt{\tilde{a}_{22}}}$$

Therefore, by (45) and (46) we have (30).

Item III. We assume that (β_1, β_2) is not an eigenvector of the matrix A and is an eigenvector of B associated to the eigenvalue $\xi(\tilde{\beta}_2 = 0, \tilde{a}_{12} \neq 0)$. Then the Eq. (49) yields

$$(\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda_n\gamma_{12})\widetilde{A}_n + i\lambda_n\gamma_{22}\widetilde{B}_n = 1. \tag{54}$$

If $\gamma_{22} = 0$ then

$$\widetilde{A}_n = \frac{1}{\tilde{a}_{12}n^2 + \alpha_{12}}.$$

The limit (30) follows from (47).

If $\gamma_{22} \neq 0$, from (54) and (50) we have

$$\widetilde{B}_n = \frac{1}{i\gamma_{22}\lambda_n} - \frac{\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda_n\gamma_{12}}{i\gamma_{22}\lambda_n}\widetilde{A}_n$$
$$nC_n = -\frac{\tilde{\beta}_1\lambda_n n^2}{c\lambda_n - ikn^2}\widetilde{A}_n$$

From (51) it results that

$$\widetilde{A}_n = -\frac{R_n}{S_n}$$

with

$$R_n = c\lambda_n(\tilde{a}_{12}n^2 + \alpha_{12}) + \kappa\gamma_{12}n^2\lambda_n - i(\kappa\tilde{a}_{12}n^4 + \kappa\alpha_{12}n^2 - c\gamma_{12}\lambda_n^2)$$
$$S_n = \gamma_{22}\lambda_n(\kappa n^2 + ic\lambda_n)(m_1n^2 + m_2 + i\lambda_n(\xi n^2 + \gamma_{11}))$$
$$- (\tilde{a}_{12}n^2 + \alpha_{12} + i\lambda_n\gamma_{12})^2(c\lambda_n - i\kappa n^2) + i\tilde{\beta}_1^2\gamma_{22}n^2\lambda_n^2$$

Thus

$$\lim_{n\to\infty}\frac{n\lambda_n R_n}{n^6} = -i\kappa\sqrt{\tilde{a}_{22}}\tilde{a}_{12}, \quad \lim_{n\to\infty}\frac{S_n}{n^6} = \kappa(\tilde{a}_{12}^2 + \xi\gamma_{22}\tilde{a}_{22})i$$

and

$$\lim_{n\to\infty} n\lambda_n\widetilde{A}_n = -\frac{\tilde{a}_{12}\sqrt{\tilde{a}_{22}}}{\tilde{a}_{12}^2 + \xi\gamma_{22}\tilde{a}_{22}}$$

Therefore, from (47) we can obtain (30). The theorem follows. □

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