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Analyticity Properties Implied by the Many-Particle Structure of the n-Point Function in General Quantum Field Theory

I. Convolution of n-Point Functions Associated with a Graph

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Abstract. This is the first of a series of papers devoted to the derivation of analyticity properties in the non-linear program of general quantum field theory, following the line of the "many particle structure analysis" due to Symanzik. In this preliminary paper, "convolution products" are associated with graphs whose vertices v represent general n_v-point functions. Under convergence assumptions in Euclidean directions, it is proved that any such convolution product H^G associated with a graph G with N external lines is well defined as an analytic function of the corresponding N four-momentum variables. The analyticity domain of H^G is proved to contain the corresponding N-point "primitive domain" implied by causality and spectrum and the various real boundary values of H^G satisfy all the relevant linear relations. For appropriate boundary values, the convolution products generalize the perturbative Feynmann prescription. As a by-product of this study, it is proved that in any perturbative theory using "superpropagators" with Euclidean convergence, Feynmann amplitudes that satisfy all the requirements of the linear program can be defined without the help of a regularization.

1. Introduction

Since the axioms of general quantum field theory were proposed [1, 2], many papers have been devoted to the study of the analyticity properties of the n-point Green's functions of the fields. Interest was particularly taken in the existence of regions of analyticity lying inside the complex mass shell manifold $\mathscr{M}^c$ since in view of the reduction formulae [3, 4] this entails the analytic character of the n-particle scattering amplitudes.

It was by exploiting the so called *linear properties* of the Green's functions that the notion of a "primitive domain" of analyticity of the n-point function in momentum space was derived [5–7]. Then it was soon realized that on one hand this primitive domain did not intersect

$\mathscr{M}^c$, but that on the other hand it was not a natural domain of holomorphy, so that general techniques of analytic completion could be applied which allowed to reach the mass shell. In fact using different methods, it was proved that the four-particle scattering amplitude is the boundary value of an analytic function on the complex mass shell in a domain including Lehmann ellipses in t [8], cut planes [9] or crossing regions in s [10, 11], etc. .... More recently, it was proved that in the neighbourhood of any physical point, the general n-particle scattering amplitude can always be decomposed as the sum of a relatively small number of boundary values of functions which are analytic on $\mathscr{M}^c$ [12]. However when n increases, the analyticity properties on the mass shell are more and more difficult to derive from the linear program. For instance, the number of analytic functions obtained so far in the framework of [12] may be larger than one and the regions where this critical situation occurs are in fact wide neighbourhoods of some thresholds singularities.

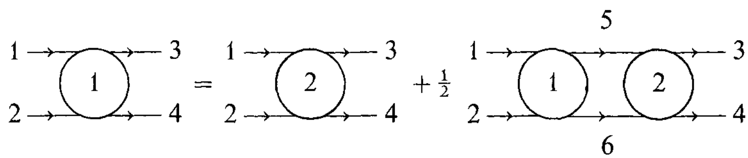
Now if we have in mind the macrocausal approach of S-matrix theory [13, 14] as well as all examples of perturbation theory which have been studied by now, we could think that local analyticity at all the points of the physical region outside the Landau surfaces should be a rather reasonable goal to be reached starting from the axioms of local field theory, provided that we make an extensive use of all the *non-linear properties* of the theory.

At present there are at least two lines of investigation to incorporate the non-linear properties of field theory in the analytic framework. The first one starts from the results of the linear program on the mass shell and applies the positivity conditions of absorptive parts and the unitarity relations of scattering amplitudes to get new analyticity properties. Up to now, it has been very successful in the case of the four-point function [15, 16] and has led to important improvements of the analyticity domain (the Martin ellipses). The use of unitarity (for instance through partial waves) has also led to continuation in a second sheet across the elastic region. Recently a generalization of these methods has been undertaken with promising results [17].

The purpose of the present series of papers is to develop an alternative and rather different way of incorporating the non-linear properties in the analytic framework. Indeed as early as 1960, Symanzik sketched a very interesting program [18] which he called "Many Particle Structure Analysis" (M.P.S.A.) of Green's functions, where he proposed to incorporate in general quantum field theory the perturbative notion of the *p-particle irreducible part* of a Green's function with respect to a certain channel. This, he conjectured, should lead to important improvements in the knowledge of the analytic structure of Green's functions, since a larger analyticity domain could be expected for these irreducible parts. However at that time, the analyticity properties of the n-point functions in the framework of the linear program had not yet been studied far enough to allow the exploitation of Symanzik's ideas from the point of view of analyticity properties. It is our purpose in the present series of papers to develop the M.P.S.A. program in a way which should make clear the analyticity properties there involved, following a plan proposed by Bros in 1967 [19, 20].

Let us first recall the main ideas of Symanzik's program. The perturbative expansion of n-point Green's functions in terms of Feynmann graphs introduces the notion of the *p-particle irreductible part of a Green's function* in the following way. Having chosen a certain channel, namely a partition of the set of all external four-momentum variables into two subsets (n_{in} incoming and n_{out} outgoing four-momenta), any term of a perturbative expansion is said to be p-particle irreducible with respect to this channel if at least $(p+1)$ internal lines of the corresponding Feynmann graph F must be untied in order to get two disjoint connected subgraphs in such a way that in this process the set of all external lines of F should be split up according to the above partition $(n_{\text{in}}, n_{\text{out}})$. The formal sum of all such p-particle irreducible contributions to the expansion of any n-point Green's function is called the p-particle irreducible part of this function with respect to the considered channel.

Now it turns out that the various p-particle irreducible n-point Green's functions thus introduced satisfy as formal series of Feynmann graphs certain integral relations, among which the complete Bethe-Salpeter equation is the simplest and best known example. This equation usually written in the graphic form



has the following algebraic meaning:

$$\tilde{t}_c^{(1)}(p_1,\dots,p_4)=\tilde{t}_c^{(2)}(p_1,\dots,p_4)+\tfrac{1}{2}\int_{\mathbb{R}^4}\tilde{t}_c^{(1)}(p_1,p_2,p_5,p_6)\,\tilde{t}_c^{(2)}(-p_5,-p_6,p_3,p_4)$$
$$\cdot\, G^{-1}(p_5)\,G^{-1}(p_6)\,\delta^{(4)}(p_1+p_2-p_5-p_6)\,d^4p_5\,d^4p_6\,.$$

Here $\sum_{i=1}^{4} p_i=0$ and $\tilde{t}_c^{(p)}(p_1,\dots,p_4)$ denotes the p-particle irreducible part $(p=1,2)$ of the connected (or truncated) four-point time-ordered product in momentum space (that is the Fourier transform of the connected vacuum expectation value of the time-ordered product of four fields, up to the factor $\delta^{(4)}\left(\sum_{i=1}^{4} p_i\right)$ with respect to the channel $[(1,2)_{\mathrm{in}}; (3,4)_{\mathrm{out}}]$. $G(p)$ denotes the complete two-point Green's function (with the Feynmann prescription).

The basic idea of Symanzik's program was then the following: once extracted from the perturbative framework, all the integral identities of this type should provide a basis for a rigourous introduction of the p-particle irreducible parts of all the n-point Green's functions in general quantum field theory. Indeed from a more thorough investigation of the algebra there involved, which will be found in a forthcoming paper [21], it results that any p-particle irreducible n-point function can be defined as the solution of a Fredholm equation, if all the $(p-1)$-particle irreducible n-point functions are supposed to be given.

Symanzik then conjectured that a double recursion over p and n, starting from the n-point Green's functions submitted to all the requirements of the linear program, should allow one to introduce p-particle irreducible n-point functions satisfying the following properties:

i) these functions exist for any p and n, and any channel;

ii) each of them satisfies all the analytic and algebraic properties of the linear program as well as the complete Green's functions do;

iii) once extracted from the perturbative graphic interpretation, the *irreducibility property* gets the following general meaning: for any p-particle irreducible n-point function (in a given channel) the absorptive part in this channel takes its threshold value at the total square mass of a system of $(p+1)$ particles (with relevant quantum numbers) travelling together in the corresponding channel. It is in the proof of this third property that the non-linear properties of general quantum field theory should play a crucial role.

However the functional formalism which was used in [18], though very powerful in many respects, does not seem to be the most suitable for the derivation of analyticity properties since it does not yield the complete analytic structure of the Steinmann-Ruelle-Araki generalized Green's functions [5–7]. Actually as early as 1967, Bros has proposed a general method of incorporating the M.P.S.A. program in the framework of analyticity properties [19, 20] which he carried through successfully in the special case $n=4$, $p=2$ which makes use of the Bethe-Salpeter equation. The aim of the present series of papers is to investigate this general non-linear program whose main steps are the following.

i) Once extracted from the perturbative framework, the integral equations which relate the various irreducible functions must be made meaningful in the sense of analytic functions. For this purpose, once we have recalled the primitive structure of the n-point functions deduced from the linear properties of the theory (namely existence of a primitive domain of analyticity, Steinmann relations, connection with the time-ordered product), we shall need to prove that this structure is preserved by "G-convolution" in the following sense.

Consider any connected graph G with N external lines. With any vertex v incident to n_v lines, associate a n_v-point function $H^{(n_v)}$.

Define a prescription of integration on the complex four-momenta associated with the internal lines. Then the object H^G thus obtained (which we shall call "convolution product associated with G") is an analytic function of the N external four-momentum variables which enjoys all the analytic and algebraic properties of the primitive structure of a N-point function.

The present paper is devoted to the proof of this conservation property which is the first step of the non-linear program.

ii) As a second step, we shall turn to the introduction of irreducible functions through a double recursion over p and n: assuming the existence of p-particle irreducible n-point functions for any channel, with $p \leqq p_0 - 1$ and n arbitrary, any p_0-particle irreducible n-point function is introduced as the solution of a certain integral relation suggested by the perturbative framework. As a result of i), this equation makes sense as a Fredholm equation in the *complex* n-point primitive domain. Then the classical Fredholm theory allows one to prove that the solution enjoys all the analytic and algebraic properties of the primitive structure of the n-point function. However as already pointed out in [19], additional regularity assumptions have to be made to ensure that the boundary values of this function are actually temperate distributions.

This second step of the program can be successfully achieved and will be reported in a forthcoming second paper in the announced series [21].

iii) The next step of the program will be the proof that the n-point function thus introduced is indeed p_0-particle irreducible in the considered channel. It is in this third step of the program that the non-linear properties of general quantum field theory will play a crucial role through the Glaser-Lehmann-Zimmermann relations [22]. These relations, obtained through an extensive use of reduction formulae and completeness of asymptotic states, are the field theoretic off-shell extrapolations of the unitarity relations.

Partial results have been obtained so far in this direction and will be reported in [21]. A generalization of the method there used is at present under study.

iv) Once it is proved, the existence of irreducible functions can certainly lead to important improvements of our knowledge of the analyticity domain of the n-point function. Indeed we notice that all the results of the linear program (Lehmann ellipses, crossing regions, etc. ...) apply to the irreducible functions with the improvements due to the raising of the thresholds. Then this *improved* primitive analytic structure allows one to enlarge the analyticity domain of the complete n-point function in a way which we shall give only a brief account of by getting back to the special case investigated in [19].

Using the Bethe-Salpeter equation as a Fredholm equation in the opposite way, it was shown there that the better analyticity properties of the two-particle irreducible four-point function entail the existence of an analytic continuation of the complete four-point function in a second sheet across the elastic part of the physical region, a result which can certainly not be obtained in the framework of the linear program.

This fact indicates the possibility that, using larger and larger values of p, the improved analytic structure of the p-particle irreducible n-point functions would yield *local analyticity* properties for the complete n-point function in higher and higher parts of the physical region. Pieces of the *Landau surfaces* would then probably come out as isolated singularities through pinching-type arguments quite similar to those already encountered in the study of the analyticity properties of Feynmann integrals.

This step will be fully investigated further in the announced series.

Let us now come back to the present paper which is devoted to the first step of the non-linear program. In Section 2, the algebraic and analytic primitive structure of the n-point functions is fully recalled. Section 3 introduces the notion of convolution product associated with a graph. The central property to be proved in this paper (the conservation of the primitive structure of the n-point functions) is exposed there as well as the general method used to get it. Sections 4 and 5 are devoted to the proof. In Section 6 it is shown that the convolution products generalize Feynmann graphs, in a sense which will be made clearer there. An application to perturbation theory is indicated in Section 7 as a by-product of our study. Finally an appendix is devoted to some technical aspects of the central proof.

2. The Primitive Structure of the n-Point Function

We start from the following facts which have been proved from the general principles of local field theory [5–7, 9, 23]:

i) The existence of a primitive domain of analyticity for the n-point function $H^{(n)}(k)$. Here the argument $k = p + iq$ denotes the set of n complex four-vectors $\{k_i = p_i + iq_i, 1 \leqq i \leqq n\}$ linked by the relation $\sum_{i=1}^{n} k_i = 0$. The first part of this section is devoted to the description of this domain.

ii) The so called "Steinmann relations" which are linear relations between the various real boundary values of the n-point function. These relations will be studied in the second part of this section.

iii) The "Ruelle prescription" which connects the various real boundary values of $H^{(n)}$ with the Fourier transform of the connected (or truncated) vacuum expectation value of the time-ordred product of n fields. It will be described in the third part of this section.

2.1 The Primitive Domain

It is composed of the union of a certain family of tubes $\{\mathcal{T}_\lambda, \lambda \in \Lambda^{(n)}\}$ in the linear manifold $\left\{k \in \mathbb{C}^{4n} : \sum_{i=1}^{n} k_i = 0\right\}$ with appropriate complex neighbourhoods of real regions which connect the various tubes together. (We recall that a tube is a domain invariant under real translations.) In order to describe the family of tubes $\{\mathcal{T}_\lambda, \lambda \in \Lambda^{(n)}\}$ some definitions are needed. Let us first specify our notations: in the following X will always denote the set $\{1, 2, \ldots, n\}$ of indices numbering the different four-vectors $\{k_1, \ldots, k_n\}$. $\mathcal{P}^*(X)$ will denote the set of proper subsets of X and $(I, X \backslash I)$ any partition of X. We shall consider the space $\mathbb{R}^{n-1}$ of the n independent variables $\{s_1, s_2, \ldots, s_n\}$ linked by the relation $\sum_{i=1}^{n} s_i = 0$; and similarly the space $\mathbb{R}^{4(n-1)}$ of the n four-vectors $\{p_1, p_2, \ldots, p_n\}$ linked by the relation $\sum_{i=1}^{n} p_i = 0$. Moreover s_I will denote $\sum_{i \in I} s_i$ and similarly $p_I = \sum_{i \in I} p_i$. Then we have the following

Definition 1. Let us consider the "triangulation" of the space $\mathbb{R}^{n-1}$ of the variable $s = \{s_1, \ldots, s_n\}$ by the family of planes $\{s_I = 0, I \in \mathcal{P}^*(X)\}$. The planes $s_I = 0$ and $s_{X \backslash I} = 0$ are of course identical. We call a *cell* any open convex cone γ_λ into which $\mathbb{R}^{n-1}$ is thus divided and $\{\gamma_\lambda, \lambda \in \Lambda^{(n)}\}$ the family of these cones.

The reason for this notation is the following: any open convex cone γ_λ of the triangulation can be written:

$$\gamma_\lambda = \{s \in \mathbb{R}^{n-1} : \lambda(I)\, s_I > 0 \ \forall\, I \in \mathcal{P}^*(X)\}$$

where λ is a sign function defined on $\mathcal{P}^*(X)$ and taking its values on $\{-1, +1\}$ which satisfies certain compatibility conditions, namely

i) $\forall\, I \in \mathcal{P}^*(X) \quad \lambda(I) = -\lambda(X \backslash I)$,

ii) $\forall\, I, J \in \mathcal{P}^*(X)$ with $I \cap J = \emptyset$ and $\lambda(I) = \lambda(J)$, then $\lambda(I \cup J) = \lambda(I) = \lambda(J)$.

We are now in a position to describe the family of tubes inside which the n-point function $H^{(n)}$ is analytic. With every cell γ_λ let us associate indeed the following cone $\mathscr{C}_\lambda$ defined in $\mathbb{R}^{4(n-1)}$:

$$\mathscr{C}_\lambda = \{q \in \mathbb{R}^{4(n-1)} : \lambda(I)\, q_I \in V^+ \ \forall\, I \in \mathscr{P}^*(X)\}\,.$$

Clearly $\mathscr{C}_\lambda$ is obtained from γ_λ by replacing all the conditions $\gtrless 0$ by the corresponding conditions $\in V^\pm$. But while the union of the closures of the cells γ_λ exhausts the whole space $\mathbb{R}^{n-1}$, nothing of this type occurs for the cones $\mathscr{C}_\lambda$ since the vectors q_I can never be space-like.

Then the n-point function $H^{(n)}$ is analytic inside the union of the family of tubes $\{\mathscr{T}_\lambda, \lambda \in \Lambda^{(n)}\}$, where $\mathscr{T}_\lambda$ denotes the tube in $\mathbb{C}^{4(n-1)}$ whose basis is the cone $\mathscr{C}_\lambda$, namely:

$$\mathscr{T}_\lambda = \{k \in \mathbb{C}^{4(n-1)} : \lambda(I)\, \mathrm{Im}\, k_I \in V^+ \ \forall\, I \in \mathscr{P}^*(X)\}\,.$$

In order to study in detail the connections between all these tubes we shall introduce the various boundary values which $H^{(n)}(k)$ can take when k tends to a real value p inside any tube:

$$H_\lambda^{(n)}(p) = \lim_{k \to p,\, q \in \mathscr{C}_\lambda} H^{(n)}(k)\,.$$

These boundary values (the so called Ruelle-Araki generalized retarded functions) have certain properties of coincidence (in the sense of distributions) that we are going to recall.

Definition 2. Two cells γ_{λ_1} and γ_{λ_2} (resp. two tubes $\mathscr{T}_{\lambda_1}$ and $\mathscr{T}_{\lambda_2}$) are called *adjacent* if there exists one partition $(I, X \backslash I)$ of X such that the indices λ_1 and λ_2 take the same value on any proper set of X but I and $X \backslash I$.

Namely

$$\lambda_1(I) = \lambda_2(X \backslash I) = -\lambda_2(I) = -\lambda_1(X \backslash I)\,.$$

We shall say that the partition $(I, X \backslash I)$ *separates* the two cells (resp. tubes). In other words, the corresponding plane $s_I = s_{X \backslash I} = 0$ is a common $(n-2)$-dimensional face of the two cells γ_{λ_1} and γ_{λ_2}.

Let us then consider the boundary values $H_{\lambda_1}^{(n)}(p)$ and $H_{\lambda_2}^{(n)}(p)$ associated with two adjacent tubes $\mathscr{T}_{\lambda_1}$ and $\mathscr{T}_{\lambda_2}$, separated by a partition $(I, X \backslash I)$. It is a consequence of the spectral condition that these two distributions coincide on the real region $\mathscr{R}_I$ thus defined:

$$\mathscr{R}_I = \{p \in \mathbb{R}^{4(n-1)} : p_I^2 \neq m_I^2,\, p_I^2 < M_I^2\}$$

where m_I is the discrete mass and M_I the threshold mass associated with the channel $(I, X \backslash I)$ by the spectral condition. That is

$$\forall\, p \in \mathscr{R}_I \qquad H_{\lambda_1}^{(n)}(p) = H_{\lambda_2}^{(n)}(p)\,.$$

This fact ensures through the general edge of the wedge theorem [24] that $H^{(n)}$ is certainly analytic inside a small complex region $\mathcal{N}(\mathcal{R}_I)$ which connects the two tubes $\mathcal{T}_{\lambda_1}$ and $\mathcal{T}_{\lambda_2}$, and turns out to be the intersection of a complex neighbourhood of $\mathcal{R}_I$ with the convex hull of the union of the two given tubes. In the following, $\mathcal{R}_I$ will be called the "edge of the wedge region" of the two tubes.

A special role is played in this context by the real region $\mathcal{R} = \bigcap_{I \in \mathcal{P}^*(X)} \mathcal{R}_I$. Actually $\mathcal{R}$ is the region where all the distributions $H_\lambda^{(n)}(p)$ coincide. Therefore by applying the edge of the wedge theorem to any couple of opposite tubes, it is proved that $\mathcal{R}$ is a real region of analyticity.

2.2 The Steinmann Relations

Let us consider the set of the various boundary values $\{H_\lambda^{(n)}(p), \lambda \in \Lambda^{(n)}\}$. These distributions are linked by certain linear relations known under the general name of "Steinmann relations". In this section we intend to describe the main feature of this linear system, which will be used extensively in the following, namely the existence of a special class of Steinmann relations, the so called "quartet relations" which generate all the others[1].

Definition 3. Two partitions $(I, X\backslash I)$ and $(J, X\backslash J)$ are called *transverse* if $I \cap J \neq \emptyset$ and $I \cap (X\backslash J) \neq \emptyset$. If not they are called *incident* and we have either $I \subset J$ or $I \subset X\backslash J$.

Definition 4. We say that four cells $\{\gamma_{\lambda_i}, 1 \leqq i \leqq 4\}$ compose a *quartet* if there exist two *transverse* partitions $(I, X\backslash I)$ and $(J, X\backslash J)$ such that:

i) γ_{λ_1} and γ_{λ_2} (resp. γ_{λ_3} and γ_{λ_4}) are adjacent cells separated by the partition $(J, X\backslash J)$;

ii) γ_{λ_1} and γ_{λ_3} (resp. γ_{λ_2} and γ_{λ_4}) are adjacent cells separated by the partition $(I, X\backslash I)$.

Then we also use the denomination of quartet for the set of the four corresponding tubes $\{\mathcal{T}_{\lambda_i}, 1 \leqq i \leqq 4\}$.

In the following we shall always represent the four tubes which compose a quartet with the notation $\{\mathcal{T}_{\lambda_{++}}, \mathcal{T}_{\lambda_{-+}}, \mathcal{T}_{\lambda_{--}}, \mathcal{T}_{\lambda_{+-}}\}$ where the first index refers to the sign of the partial sum s_I, the second to the sign of s_J, and where λ stands for the common value of the various $\{\lambda_i, 1 \leqq i \leqq 4\}$ for other proper subsets of X.

Now the four boundary values of the n-point function occuring in each quartet $\{\mathcal{T}_{\lambda_{++}}, \mathcal{T}_{\lambda_{-+}}, \mathcal{T}_{\lambda_{--}}, \mathcal{T}_{\lambda_{+-}}\}$ satisfy the following relation, for all real arguments, in the sense of distributions:

$$H_{\lambda_{++}}^{(n)}(p) + H_{\lambda_{--}}^{(n)}(p) = H_{\lambda_{+-}}^{(n)}(p) + H_{\lambda_{-+}}^{(n)}(p)$$

[1] The proof of this property is a consequence of the analysis made by Ruelle [6] but has remained still unpublished [26]. A general proof in the framework of Bros [23] rather different from Ruelle's will be reported in a coming paper [27].

and the set of these *quartet relations* generates the whole set of Steinmann relations satisfied by the various boundary values $\{H_\lambda^{(n)}(p), \lambda \in \Lambda^{(n)}\}$.

2.3 The Ruelle Prescription

This prescription refers to the fact proved in Ref. [6] that the Fourier transform of the connected (or truncated) vacuum expectation value of the time-ordered product of n fields $\tilde{t}_c(p)$ coincides with the various distributions $H_\lambda^{(n)}(p)$ in relevant regions. More precisely let us consider the following family of open sets $\{\Omega_\lambda, \lambda \in \Lambda^{(n)}\}$ with

$$\Omega_\lambda = \{p \in \mathbb{R}^{4(n-1)} : \lambda(I)\, p_I \in \complement\, \bar{V}_I^- \ \forall\, I \in \mathscr{P}^*(X)\}\,.$$

Here $\bar{V}_I^\pm$ denotes the following closed set in $\mathbb{R}^4$:

$$\bar{V}_I^+ = -\bar{V}_I^- = \{p \in \mathbb{R}^4 : p^0 \geqq \sqrt{\mathbf{p}^2 + M_I^2}\} \cup \{p \in \mathbb{R}^4 : p^0 = \sqrt{\mathbf{p}^2 + m_I^2}\}\,.$$

It is easy to see that the collection of all Ω_λ forms an open covering of the whole space $\mathbb{R}^{4(n-1)}$. Ruelle has proved:

$$\forall\, p \in \Omega_\lambda \qquad \tilde{t}_c(p) = H_\lambda^{(n)}(p)$$

that is, the connected time-ordered product is the boundary value of the n-point function according to a prescription (the so called "Ruelle prescription") which connects the tube inside which the boundary value is taken with the location of the real limit point.

More precisely:

$$\tilde{t}_c(p) = \lim_{\substack{k \to p \\ p \in \Omega_\lambda,\, q \in \mathscr{C}_\lambda}} H^{(n)}(k)\,.$$

This is of course very important since in the L.S.Z. theory [3] the truncated time-ordered product is the fundamental object which allows to express the connected scattering amplitudes in terms of the fields. More precisely, we have the well known "reduction formulae" [3, 4]:

$$\langle p^I |S| - p^{X\setminus I}\rangle_c = \delta^{(4)}(p_X)\, \hat{\tilde{t}}_c(p)|_{\mathscr{M}_I}\,.$$

Here $\langle p^I |S| - p^{X\setminus I}\rangle_c$ denotes the connected scattering amplitude describing the process $I \to X\setminus I$ and $\hat{\tilde{t}}_c(p)$ the connected *amputated* time-ordered product

$$\hat{\tilde{t}}_c(p) = \prod_{i \in X} (p_i^2 - m_i^2)\, \tilde{t}_c(p)$$

whose restriction to the relevant physical region

$$\mathscr{M}_I = \{p \in \mathbb{R}^{4(n-1)} : p_i^2 = m_i^2 \ \forall\, i \in X,\ p_i \in V^- \ \forall\, i \in I,\ p_i \in V^+ \ \forall\, i \in X\setminus I\}$$

is meaningful in the sense of distributions on the mass shell [4, 12].

2.4 General n-Point Functions

In the following we shall meet functions enjoying all the analytic and algebraic properties of the n-point function stated above except the physical connection with products of n fields.

We shall call these functions *general n-point functions* in order to distinguish then from the "physical" one. More precisely we call general n-point function any function f defined on $\mathbb{C}^{4(n-1)}$ and enjoying the following properties:

i) analyticity and slow increase near the real inside the tube $\{\mathcal{T}_\lambda, \lambda \in \Lambda^{(n)}\}$. The real boundary values $\{f_\lambda(p), \lambda \in \Lambda^{(n)}\}$ are then distributions;

ii) Steinmann relations between the various real bondary values $\{f_\lambda(p), \lambda \in \Lambda^{(n)}\}$;

iii) coincidence relations between two adjacent boundary values $f_{\lambda_1}(p)$ and $f_{\lambda_2}(p)$ on a real region $\mathcal{R}_I$ thus defined:

$$\mathcal{R}_I = \{p \in \mathbb{R}^{4(n-1)} : p_I^2 < M_I^2,\ p_I^2 \neq m_I^2\}$$

where m_I and M_I are now general positive numbers.

Moreover the Ruelle prescription allows to introduce the *t-boundary value* of the function f, denoted $f_t(p)$, by the following prescription:

$$f_t(p) = \lim_{\substack{k \to p \\ p \in \Omega_\lambda,\, q \in \mathscr{C}_\lambda}} f(k)\,.$$

Of course the masses m_I and M_I which occur in the definition of Ω_λ are now those of Conditions iii). As in Section 2.3, such a definition makes sense: the collection of all Ω_λ forms an open covering of $\mathbb{R}^{4(n-1)}$ and it is easy to check that the intersection of any two open sets Ω_λ and Ω_λ, $\Omega_{\lambda'}$ lies in the coincidence region of $f_\lambda(p)$ and $f_{\lambda'}(p)$. Actually as seen in Section 2.3 the t-boundary value of the physical n-point function is nothing but the connected time-ordered product $\tilde{t}_c(p)$.

In this paper we shall always deal with *general* n-point functions. Moreover these functions will always be assumed to enjoy integrability properties at infinity in the set of all the Euclidean directions of the tubes $\{\mathcal{T}_\lambda, \lambda \in \Lambda^{(n)}\}$ [2]. This will be stated more precisely in the course of Section 4.6. Let us only notice that we shall not assume any regularity properties in real directions at infinity. In particular no use will be made of the possible temperate character of the boundary values.

[2] For applications to the non linear program [21], this will necessitate the use of an analytic regularization process, as already pointed out in [19].

3. Convolution Product Associated with a Graph: the Method

In this section we shall briefly describe the general method which will be used further. We first sketch the relevant terminology. Let us consider a *graph* G. Being given a vertex v and a line l, we say that v and l are *incident* if v is an endpoint of l. We call *external* those lines which are incident to one vertex only. The set of lines incident to a given vertex v is called the *star* of v and denoted $\mathscr{S}_v$. We avoid disconnected graphs because they reduce to the discussion of their connected components. Similarly we shall suppose that every vertex is incident to at least three lines. Simply connected graphs (or "trees") will play a special role in the following. We introduce the numbers

n: number of external lines of G,

l: number of independent loops of G.

The successive steps of the method are then the following:

i) With every line of G associate a complex four-vector so as to satisfy the energy-momentum conservation law at every vertex. Let $\{k_i, 1 \leqq i \leqq n\}$ be the n complex four-vectors thus associated with the *external* lines of G. It should be understood that, in order to satisfy the global conservation law $\sum_{i=1}^{n} k_i = 0$, it is necessary to associate opposite vectors with each of the two endpoints of every *internal* line of G.

ii) With every vertex v of G incident to n_v lines, associate a *general* n_v-point function of the relevant complex arguments.

iii) The external variables being held fixed in the n-point domain, define a prescription of integration on the four-vectors associated with the internal lines.

iv) Then study the algebraic and analytic properties of the object there obtained which we shall call *convolution product associated with the graph* G (shortly *G-product*) and denote $H^G(k_1, k_2, \dots, k_n)$.

The aim of this paper is to prove that H^G is a general n-point function (in the sense of Section 2.4). In other words to prove that the primitive structure of the n-point functions is preserved by G-convolution. We shall use the following ideas:

i) Among all others, the case of simply connected graphs appears to be the easiest. This is because conservation laws at every internal vertex are sufficient to determine internal momenta once known the external ones so that the prescription of integration is trivial. Section 4 is devoted to this simple case.

ii) One is then led to define and study the G-product by recursion over the number l of independent loops of G. The case of trees is indeed the case $l=0$ which initiates the recursion and it remains to prove the following property:

"Assume that for any graph G with $l \leqq l_0 - 1$ independent loops and any number n of external lines, it is possible to define a G-product which is a general n-point function.

Then, the same can be done for any graph with $l = l_0$ independent loops and any number of external lines."

This will be proved in Section 5.

4. The Case of Trees

4.1 Preliminary Remarks

Let us consider a tree T and denote $\{k_i, 1 \leqq i \leqq n\}$ the n four-vectors associated with its external lines. We shall first prove some simple properties.

Proposition 1. *With any internal line i of T it is possible to associate a partition $(I, X \backslash I)$ of the set $X = \{1, 2, \dots, n\}$.*

Proof. By "cutting" i, we get two subtrees T_1 and T_2. The complex four-vectors associated with the external lines of T_1 and T_2 are then $\{k_i, i \in I; -k_I\}$ for T_1 and $\{k_i, i \in X \backslash I; -k_{X \backslash I}\}$ for T_2.

Proposition 2. *Let us consider a vertex v of T and denote $\mathscr{S}_v$ the set of lines l incident to v. Then it is possible to associate with v a partition $\{I(l, v), l \in \mathscr{S}_v\}$ of X.*

Proof. With every line $l \in \mathscr{S}_v$, we associate a subset $I(l, v)$ of X by the following rule:

— if l is an internal line of T, $I(l, v)$ is the subset of the associated partition (see above) $(I(l, v); X \backslash I(l, v))$ corresponding to the subtree which does not contain v.

— if l is an external line of T associated with the four-vector k_i, $I(l, v)$ is reduced to the set $\{i\}$.

Then it is easy to check that the collection $\{I(l, v), l \in \mathscr{S}_v\}$ forms a partition of X and that we have for any vertex v:

$$\sum_{l \in \mathscr{S}_v} k_{I(l,v)} = \sum_{i=1}^{n} k_i = 0 .$$

In other words, it is possible to associate with any line l of T incident to v a complex four-vector $k_{I(l,v)} = \sum_{j \in I(l,v)} k_j$ and with any vertex v a general n_v-point function $H^{(n_v)}$ of the n_v arguments $\{k_{I(l,v)}, l \in \mathscr{S}_v\}$ linked by the relation $\sum_{l \in \mathscr{S}_v} k_{I(l,v)} = 0$.

Proposition 3. *Let us consider the family of tubes $\{\mathscr{T}_\lambda, \lambda \in \Lambda^{(n)}\}$ defined in the space of the complex four-vectors $\{k_i, 1 \leqq i \leqq n\}$ linked by the* *relation* $\sum_{i=1}^{n} k_i = 0$. *With any tube of this family and any vertex v of T, it is possible to associate a tube* $\mathcal{T}_{\lambda_v}$ *defined in the space of the complex four-vectors* $\{k_{I(l,v)}, l \in \mathcal{S}_v\}$ *linked by the relation* $\sum_{l \in \mathcal{S}_v} k_{I(l,v)} = 0$. *Moreover* $\mathcal{T}_\lambda$ *is contained in the intersection of the various* $\mathcal{T}_{\lambda_v}$.

Proof. $\mathcal{T}_{\lambda_v}$ is defined in $\mathbb{C}^{4(n_v-1)}$ by the restriction $\lambda_v \in \Lambda^{(n_v)}$ of the sign function λ to subsets of X which are unions of sets of the family $\{I(l, v), l \in \mathcal{S}_v\}$. Clearly $\mathcal{T}_\lambda \subset \bigcap_v \mathcal{T}_{\lambda_v}$.

We are now in a position to introduce the convolution product.

4.2 The Convolution Product Associated with T

Let us consider the family of functions $\{H_\lambda^T(k), \lambda \in \Lambda^{(n)}\}$ thus defined:

$$H_\lambda^T(k) = \prod_v H_{\lambda_v}^{(n_v)}(\{k_{I(l,v)}, l \in \mathcal{S}_v\}) \prod_i [H^{(2)}(k_I)]^{-1}. \tag{1}$$

In this formula, the first product is taken over the *vertices* of T; $H_{\lambda_v}^{(n_v)}$ denotes the restriction of the general n_v-point function $H^{(n_v)}$ associated with v to the tube $\mathcal{T}_{\lambda_v}$ defined in $\mathbb{C}^{4(n_v-1)}$; its arguments are the n_v complex four-vectors $\{k_{I(l,v)}, l \in \mathcal{S}_v\}$. The second product is taken over the *internal lines i* of T; $(I, X \setminus I)$ denotes the partition of X associated with the line i; and $[H^{(2)}(k_I)]^{-1}$ the inverse of a general two-point function $H^{(2)}(k_I)$ associated with the line i. Had this factor been omitted, H_λ^T would have presented a double pole in the channel $(I, X \setminus I)$. The reason for its introduction is to recover a simple pole.

Remark. In fact for this purpose the choice $[H^{(2)}(k_I)]^{-1} \equiv (k_I^2 - m_I^2)$ would have been sufficient. However in the context of the non-linear program [21], the use of a *complete* propagator appears necessary. Then the possible occurence of C.D.D. poles [18] (i.e. zeros of $H^{(2)}$) leads us to assume that each completely "amputated" n_v-point function

$$\hat{H}^{(n_v)}(k_1, \ldots, k_{n_v}) = \prod_{j=1}^{n_v} [H^{(2)}(k_j)]^{-1} H^{(n_v)}(k_1, \ldots, k_{n_v})$$

is analytic in the n_v-point primitive domain. In other words we assume that the possible C.D.D. poles are cancelled by corresponding zeros of $H^{(n_v)}$.

Proposition 4. H_λ^T *is analytic inside the tube* $\mathcal{T}_\lambda$.

Proof. Since each $H_{\lambda_v}^{(n_v)}$ is analytic inside the tube $\mathcal{T}_{\lambda_v}$, their product H_λ^T is analytic inside the intersection $\bigcap_v \mathcal{T}_{\lambda_v}$, which contains $\mathcal{T}_\lambda$ as shown in Proposition 3.

4.3 *Proof of Steinmann Relations*

Let us consider the set of the various real boundary values $\{H_\lambda^T(p), \lambda \in \Lambda^{(n)}\}$. We recall that in order to check they satisfy the Steinmann relations, it is enough to concentrate on the so called "quartet relations" introduced in Section 2.2:

$$H^T_{\lambda++}(p) + H^T_{\lambda--}(p) = H^T_{\lambda+-}(p) + H^T_{\lambda-+}(p)$$

which must be satisfied in the sense of distributions by the four boundary values occuring in any quartet of tubes $\{\mathscr{T}_{\lambda++}, \mathscr{T}_{\lambda-+}, \mathscr{T}_{\lambda--}, \mathscr{T}_{\lambda-+}\}$.

We shall need the following

Definition 5. Let us consider a partition $(I, X\backslash I)$. We shall say that it is a *vertex partition* for T if there exist a vertex v_0 of T and a partition $(\mathscr{I}, \mathscr{S}_{v_0}\backslash\mathscr{I})$ of its star $\mathscr{S}_{v_0}$ such that

$$I = \bigcup_{l \in \mathscr{I}} I(l, v_0)$$

where $I(l, v_0)$ is associated with the line l as indicated in Proposition 2. If so, we shall say that $(I, X\backslash I)$ is *connected* with v_0.

The reasons for which this definition is needed are the following. Let us consider two adjacent cells γ_{λ_1} and γ_{λ_2} separated by a partition $(I, X\backslash I)$. At any vertex v of T the two sign functions λ_1 and λ_2 define relevant restrictions $\lambda_{1,v}$ and $\lambda_{2,v}$. It is easily checked that if $(I, X\backslash I)$ *is not* a vertex partition for T, then λ_1 and λ_2 define the same restriction at any vertex v of T, that is: $\lambda_{1,v} = \lambda_{2,v}$. Similarly, if $(I, X\backslash I)$ *is* a vertex partition for T connected with the vertex v_0, then it is easily checked that λ_1 and λ_2 define the same restriction at any vertex v *but* v_0. This point is the basis of much that follows in this section. It should be thoroughly understood. Now we have the following algebraic lemma:

Lemma 1. *If two partitions $(I, X\backslash I)$ and $(J, X\backslash J)$ are transverse vertex partitions, they are connected with the same vertex v_0. Moreover there exist two partitions $(\mathscr{I}, \mathscr{S}_{v_0}\backslash\mathscr{I})$ and $(\mathscr{J}, \mathscr{S}_{v_0}\backslash\mathscr{J})$ of its star $\mathscr{S}_{v_0}$ which satisfy the following conditions:*

i) $I = \bigcup_{l \in \mathscr{I}} I(l, v_0)$ *and* $J = \bigcup_{l \in \mathscr{J}} I(l, v_o)$.

ii) $(\mathscr{I}, \mathscr{S}_{v_0}\backslash\mathscr{I})$ *and* $(\mathscr{J}, \mathscr{S}_{v_0}\backslash\mathscr{J})$ *are transverse in* $\mathscr{S}_{v_0}$.

Proof. We remark that if $(I, X\backslash I)$ and $(J, X\backslash J)$ were connected with two different vertices of T, they would be necessarily *incident* (in the sense of Definition 3). This point is sufficient to conclude.

Now we are in a position to prove

Proposition 5. *The four boundary values occurring in any quartet of tubes $\{\mathscr{T}_{\lambda++}, \mathscr{T}_{\lambda+-}, \mathscr{T}_{\lambda--}, \mathscr{T}_{\lambda-+}\}$ satisfy the following relation for all real* *arguments, in the sense of distributions:*

$$H^T_{\lambda_{++}}(p) + H^T_{\lambda_{--}}(p) = H^T_{\lambda_{+-}}(p) + H^T_{\lambda_{-+}}(p)\,. \tag{2}$$

Proof. In view of the analysis made above, we have to distinguish two different cases:

i) At least one of the two partitions which define the quartet is not a vertex partition for T, for example the one that corresponds to the second index.

As already noticed, λ_{++} and λ_{+-} (resp. λ_{-+} and λ_{--}) then define the same restriction at any vertex v of T and we have for all real arguments:

$$H^T_{\lambda_{++}}(p) = H^T_{\lambda_{+-}}(p) \quad \text{and} \quad H^T_{\lambda_{-+}}(p) = H^T_{\lambda_{--}}(p)$$

which proves relation (2) to be trivially satisfied.

ii) Both partitions are vertex partitions for T.

Since they are transverse partitions, we apply Lemma 1 and conclude that they are connected with the same vertex v_0. It is then easily checked that the four sign functions $\{\lambda_{++}, \lambda_{+-}, \lambda_{--}, \lambda_{-+}\}$ define the same restriction at any vertex v of T *but* v_0. We denote $\{\lambda_{++,v_0}, \lambda_{+-,v_0}, \lambda_{--,v_0}, \lambda_{-+,v_0}\}$ the four different restrictions at this vertex. Then we can apply the second part of Lemma 1, which asserts that the four tubes $\{\mathscr{T}_{\lambda_{++,v_0}}, \mathscr{T}_{\lambda_{+-,v_0}}, \mathscr{T}_{\lambda_{--,v_0}}, \mathscr{T}_{\lambda_{-+,v_0}}\}$ constitute a quartet in the space of the complex four-vectors $\{k_{I(l,v_0)}, l \in \mathscr{S}_{v_0}\}$. As the function $H^{(n_{v_0})}$ associated with the vertex v_0 is a general n_{v_0}-point function, its four boundary values satisfy a quartet relation. Then it is enough to multiply all the terms of this relation by the common factors coming from the other vertices of T, to get the announced result.

4.4 Proof of Coincidence Relations

Proposition 6. *Let us consider the boundary values $H^T_{\lambda_1}(p)$ and $H^T_{\lambda_2}(p)$ associated with two adjacent tubes $\mathscr{T}_{\lambda_1}$ and $\mathscr{T}_{\lambda_2}$ separated by a partition $(I, X\backslash I)$. Then these two distributions coincide on a real region:*

$$\mathscr{R}_I = \{p \in \mathbb{R}^{4(n-1)} : p_I^2 \neq \tilde{m}_I^2, p_I^2 < \tilde{M}_I^2\}$$

where $\tilde{m}_I$ and $\tilde{M}_I$ are certain masses associated with the channel $(I, X\backslash I)$.

Proof. We have to distinguish two different cases:

i) The partition $(I, X\backslash I)$ is not a vertex partition for T. Then we have already noticed that λ_1 and λ_2 define the same restriction at any vertex v, and therefore the coincidence relation $H^T_{\lambda_1}(p) = H^T_{\lambda_2}(p)$ is satisfied for *all* real arguments.

ii) The partition $(I, X\backslash I)$ is a vertex partition for T connected with one vertex v_0. Then, as already mentioned, λ_1 and λ_2 define the same restriction at any vertex v *but* v_0. We denote λ_{1,v_0} and λ_{2,v_0} the two different restrictions at this vertex. As the function $H^{(n_{v_0})}$ associated with the vertex v_0 is a general n_{v_0}-point function, its two boundary values $H^{(n_{v_0})}_{\lambda_1, v_0}$ and $H^{(n_{v_0})}_{\lambda_2, v_0}$ coincide on $\mathscr{R}_I$. Multiplying this coincidence relation by the common factors coming from the other vertices of T is then sufficient to conclude.

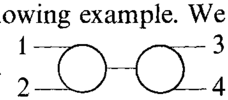
Remark. A special case may occur when the partition $(I, X\backslash I)$ is connected with *two* vertices: $(I, X\backslash I)$ is then associated with an internal line i of T (see Proposition 1) and connected with the two endpoints of i. In that case, the proof follows closely the previous one and will be left to the reader.

Now putting together Propositions 4–6 and applying the general edge of the wedge theorem, we finally obtain:

Theorem 1. *The convolution product associated with any simply connected graph with n external lines is a general n-point function.*

4.5 Additional Remark

As a final remark let us point out that the convolution product associated with a tree T is analytic in a domain larger than the "primitive domain" of the physical n-point function. Indeed it is a direct consequence of Proposition 4 that its "natural" faces are only those *corresponding to the vertex partitions.*

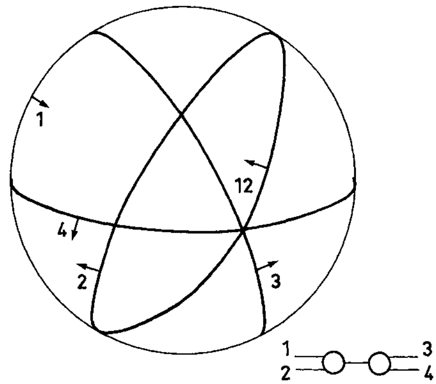
This point will be made clearer through the following example. We consider the convolution product associated with 

and we represent on the "Steinmann sphere" [5, 10] the traces of the various faces for this general four-point function. Fig. 1 shows the situation on the hemisphere $s_1 > 0$; the case of the physical four-point function is also drawn for comparison.

4.6 Bounds in the Tubes

Let us now state in a more precise way than in Section 2.5 the bounds which we shall assume for any vertex function $H^{(n_v)}$ in its primitive n_v-point domain of analyticity $\mathscr{D}^{(n_v)}$. We introduce the completely "amputated" n_v-point function

$$\hat{H}^{(n_v)}(k_1, \ldots, k_{n_v}) = \prod_{i=1}^{n_v} [H^{(2)}(k_i)]^{-1} H^{(n_v)}(k_1, \ldots, k_{n_v}).$$



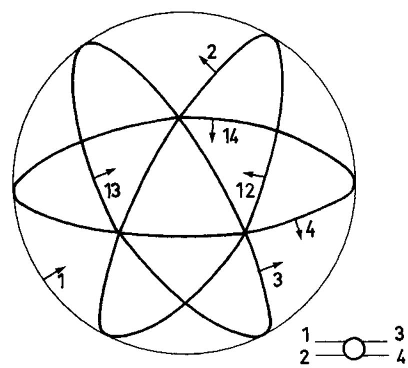


FIG. 1

Then we assume the following bounds inside the tubes:

$$|\hat{H}^{(n_v)}(k_v)| \leqq C_K \left(1 + \frac{1}{d_{\hat{e}}(k_v, \partial \mathscr{D}^{(n_v)})}\right)^{M_v} \quad \text{for} \quad n_v > 2 \tag{3a}$$

$$|H^{(2)}(k)| \leqq \frac{C_L}{1 + [(q^0)^2 + \boldsymbol{p}^2]^{2+\eta}} \quad \text{with} \quad \eta > 0\,. \tag{3b}$$

Here $k_v = p_v + iq_v$ stands for the n_v complex four-vectors $\{k_j, 1 \leqq j \leqq n_v\}$; a time direction $\hat{e}$ has been chosen as a coordinate axis, which allows to write $k_v = (\boldsymbol{k}_v, k_v^0) = (\boldsymbol{p}_v + i\boldsymbol{q}_v, p_v^0 + iq_v^0)$; $d_{\hat{e}}(k_v, \partial \mathscr{D}^{(n_v)})$ denotes the Euclidean distance of the point k_v to the boundary of $\mathscr{D}^{(n_v)}$; C_K (resp. C_L) is a positive constant which depends of a compact set K (resp. L) in p_v^0-space (resp. p^0-space); the integer M_v indicates the order of the distribution which is the boundary value of $\hat{H}^{(n_v)}$.

As a direct consequence of (3a) and (3b), we have

$$|H^{(n_v)}(k_v)| \leqq C'_K \prod_{i=1}^{n_v} \frac{1}{1+[(q_i^0)^2+\boldsymbol{p}_i^2]^{2+\eta}} \left(1+\frac{1}{d_{\hat{e}}(k_v, \partial\mathscr{D}^{(n_v)})}\right)^{M_v}. \tag{3c}$$

Now under these assumptions it is easily checked that the convolution product H^T satisfies similar bounds, namely:

$$|H^T(k)| \leqq L_K \prod_{i=1}^{n} \frac{1}{1+[(q_i^0)^2+\boldsymbol{p}_i^2]^{2+\eta}} \left(1+\frac{1}{d_{\hat{e}}(k, \partial\mathscr{D}^{(n)})}\right)^{M}.$$

In the following such bounds will be needed to ensure integrability at infinity on the set of all *Euclidean directions* defined as follows

$$k=(\varrho\boldsymbol{p}, p^0+i\varrho q^0)$$

with $\varrho \to +\infty$ and q^0 chosen in any cell of the family $\{\gamma_\lambda, \lambda \in \Lambda^{(n)}\}$.

5. The General Case

In this section we intend to prove the following property: "Assume that for any graph $\tilde{G}$ with $l \leqq l_0 - 1$ and n arbitrary, it is possible to define a $\tilde{G}$-product which is a general n-point function. Then the same can be done for any graph with $l = l_0$ and n arbitrary". We shall use the following ideas:

i) With every graph G with l_0 independent loops and n external lines, we can associate a graph with (l_0-1) independent loops and $(n+2)$ external lines which we call an *antecedent* of G. This is easily done by "cutting" any internal line of G. Of course such a process is not unique and G admits a finite number of antecedents. In the following we choose one of these which we denote $\tilde{G}$.

ii) Now it is possible to apply the recursion hypothesis to $\tilde{G}$ and to define a $\tilde{G}$-product $H^{\tilde{G}}$ which is a general $(n+2)$-point function, defined on the space $\mathbb{C}^{4(n+1)}$ of the $(n+2)$ complex four-vectors $\{k_i, 1 \leqq i \leqq n+2\}$ linked by the relation $\sum_{i=1}^{n+2} k_i = 0$.

iii) Then we define the convolution product associated with G by the following rule:

$$H^G(k) = \int_{\Gamma_k} H^{\tilde{G}}(k, k_{n+1}, k_{n+2}) \Big|_{k_{n+1}=-k_{n+2}=t} [H^{(2)}(t)]^{-1}\, dt. \tag{4}$$

Here k stands for the n complex four-vectors $\{k_i, 1 \leqq i \leqq n\}$ linked by the relation $\sum_{i=1}^{n} k_i = 0$; $t = u + iv$ varies in the space $\mathbb{C}^4$ of one complex four-vector; Γ_k is a four real dimensional integration region in this space, whose infinite part belongs to the set of the Euclidean directions and which will be defined further. The integrability assumptions made at the end of Section 2.5 are then sufficient to ensure convergence at infinity. $[H^{(2)}(t)]^{-1}$ denotes the inverse of a general two-point function $H^{(2)}(t)$ which is introduced under the same assumptions that in Section 4.2.

Finally let us remark that in the integral (4) $H^{\tilde{G}}$ is restricted to a region of the analytic hyperplane $k_{n+1}+k_{n+2}=0$ which lies on the edge of the $(n+2)$-point configuration. Analyticity of $H^{\tilde{G}}$ in this region will then result from a local procedure of analytic completion which will be made in the second part of this section.

Then the proof will go in three steps:

1) We shall first introduce the convolution product at points k lying on a certain submanifold $V_{\hat{e}}$ in $\mathbb{C}^{4(n-1)}$ associated with a chosen time direction $\hat{e}$. We shall define a contour $\Gamma_{\hat{e},k}$ such as the integral (4) $H^G_{\hat{e}}(k)$ will be shown to have certain analytic and algebraic properties.

2) In a second step, we shall move $\hat{e}$ in the light cone and the various $H^G_{\hat{e}}$ will appear as pieces of a unique analytic function H^G. Putting together the analytic and algebraic properties of every $H^G_{\hat{e}}$ will yield for H^G the relevant structure of a general n-point function.

3) Finally it will be necessary to check that H^G is independent from the original choice of an antecedent $\tilde{G}$ and this will achieve the proof of the recursion property.

5.1 Choice of a Time Direction

Let us make choice of a time direction $\hat{e}\in V^+$ and therefore of a coordinate system in $\mathbb{R}^4$. We write:

$$\forall\, 1\leqq i\leqq n \qquad k_i=p_i+iq_i=(\boldsymbol{k}_i,k_i^0)=(\boldsymbol{p}_i+i\boldsymbol{q}_i,p_i^0+iq_i^0)$$

and

$$t=u+iv=(\boldsymbol{t},t^0)=(\boldsymbol{u}+i\boldsymbol{v},u^0+iv^0)\,.$$

We introduce the following submanifold in $\mathbb{C}^{4(n-1)}$:

$$V_{\hat{e}}=\{k\in\mathbb{C}^{4(n-1)}:\boldsymbol{q}_i=0,\ 1\leqq i\leqq n\}$$

and consider the traces of the family of tubes $\{\mathscr{T}_\lambda,\lambda\in\Lambda^{(n)}\}$ on this manifold, namely the family of "flat" tubes $\{\mathscr{T}_\lambda\cap V_{\hat{e}},\lambda\in\Lambda^{(n)}\}$. Whereas the union of the closures of the tubes $\mathscr{T}_\lambda$ is indeed very far from exhausting the whole of $\mathbb{C}^{4(n-1)}$, it is not difficult to check that the union of the closures of the various $\mathscr{T}_\lambda\cap V_{\hat{e}}$ exhaust the whole of $V_{\hat{e}}$. This comes from the fact that for any point k on $V_{\hat{e}}$, any of the partial sums $\{q_I,I\in\mathscr{P}^*(X)\}$ lies along the time direction $\hat{e}$. The various conditions $q_I\in V^{\pm}$ then become $q_I^0\gtrless 0$ and we can write

$$\mathscr{T}_\lambda\cap V_{\hat{e}}=\{k\in\mathbb{C}^{4(n-1)}:\operatorname{Im}\boldsymbol{k}=0,\ \operatorname{Im}k^0\in\gamma_\lambda\}\,.$$

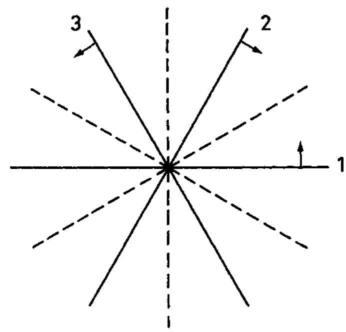


FIG. 2

Now the property already stated $\bigcup_{\lambda\in\Lambda^{(n)}} \bar{\gamma}_\lambda = \mathbb{R}^{(n-1)}$ allows to conclude.

In the following we shall first try to define the G-product H^G at points which lie on $V_{\hat{e}}$, more precisely at points inside any flat tube $\mathscr{T}_\lambda \cap V_{\hat{e}}$. We shall need some definitions.

Definition 6. Let us consider the "triangulation" of the space $\mathbb{R}^{n-1}$ of the variable $s=(s_1, s_2, \ldots, s_n)$ with $\sum_{i=1}^{n} s_i = 0$ by the following planes $\{s_I = 0, I \in \mathscr{P}^*(X)\}$ *and* $\{s_I = s_J; I, J \in \mathscr{P}^*(X)\}$. Clearly a lot of these planes are identical. We call a *pseudocell* any open convex cone σ_μ into which $\mathbb{R}^{n-1}$ is thus divided and $\{\sigma_\mu, \mu \in \Theta^{(n)}\}$ the complete family of these cones.

Of course a cell γ_λ is in general the union of several pseudocells since we have introduced new planes in the configuration. This is quite clear in the case $n=3$ showed in Fig. 2. New planes there involved are drawn in dotted lines.

With every pseudocell σ_μ we associate the following flat tube in $V_{\hat{e}}$:

$$S_\mu = \{k \in \mathbb{C}^{4(n-1)} : \operatorname{Im} \boldsymbol{k} = 0, \operatorname{Im} k^0 \in \sigma_\mu\}.$$

Clearly the union of the closures $\{\bar{S}_\mu, \mu \in \Theta^{(n)}\}$ exhausts the whole of $V_{\hat{e}}$.

Now in order to define the G-product at points which lie on $V_{\hat{e}}$, two steps are needed:

i) First for any flat tube S_μ of the family $\{S_\mu, \mu \in \Theta^{(n)}\}$ define a function $H^G_{\hat{e},\mu}$ analytic inside S_μ.

ii) Then consider the various S_μ included in a given $\mathscr{T}_\lambda \cap V_{\hat{e}}$ and prove that the corresponding functions $H^G_{\hat{e},\mu}$ are pieces of a unique function $H^G_{\hat{e},\lambda}$ analytic inside the flat tube $\mathscr{T}_\lambda \cap V_{\hat{e}}$.

Actually, as it will appear further, these two steps are made necessary by the analytic structure of the restriction of $H^{\tilde{G}}$ to the hyperplane $k_{n+1}+k_{n+2}=0$, which we shall study now.

5.2 Analyticity Properties and Bounds for the Integrand

In this section we shall study the analyticity properties of the restriction of $H^{\tilde{G}}$ to the analytic hyperplane

$$\pi=\{(k,k_{n+1},k_{n+2})\in\mathbb{C}^{4(n+1)}:k_{n+1}+k_{n+2}=0\}\,.$$

Clearly any point of π can be uniquely represented by the point (k,t) chosen in $\mathbb{C}^{4n}$ such as $t=k_{n+1}=-k_{n+2}$.

In the following we shall thus denote $H_\pi^{\tilde{G}}(k,t)$ the restriction to π of the $\tilde{G}$-product $H^{\tilde{G}}$. We shall need the following lemma of analytic completion.

Lemma 2. *A general n-point function f is analytic at all the boundary points of the tubes $\{\mathscr{T}_\lambda,\lambda\in\Lambda^{(n)}\}$ which do not belong to the union of the family of sets $\{\Delta_I, I\in\mathscr{P}^*(X)\}$ with*

$$\Delta_I=\Delta_{X\setminus I}=\{k\in\mathbb{C}^{4(n-1)}:k_I^2=m_I^2\}\cup\{k\in\mathbb{C}^{4(n-1)}:k_I^2=M_I^2+\varrho,\varrho\geqq 0\}$$

where m_I and M_I are the masses associated with the partition $(I,X\setminus I)$ by coincidence relations

The proof of this basic lemma uses the special continuity theorem of Bremermann [10, 28] and will be fully given in another forthcoming paper [27]. Then we are in a position to prove

Proposition 7. *$H_\pi^{\tilde{G}}$ is analytic at all the points (k,t) of π which are boundary points of the tubes $\{\mathscr{T}_\beta,\beta\in\Lambda^{(n+2)}\}$ and do not belong to the union Σ of the following sets:*

$$\Xi_I=\{(k,t)\in\pi:k_I^2=m_I^2\}\cup\{(k,t)\in\pi:k_I^2=M_I^2+\varrho,\varrho\geqq 0\}\quad\forall I\in\mathscr{P}^*(X)$$

$$\Sigma_I=\{(k,t)\in\pi:(k_I-t)^2=m_I'^2\}\cup\{(k,t)\in\pi:(k_I-t)^2=M_I'^2+\varrho,\varrho\geqq 0\}$$
$$\forall I\in\mathscr{P}^*(X)$$

$$\Sigma_0=\{(k,t)\in\pi:t^2=m^2\}\cup\{(k,t)\in\pi:t^2=M^2+\varrho,\varrho\geqq 0\}$$

where X stands for the set $\{1,2,\dots,n\}$ of indices numbering the external lines of G.

Proof. This is a straightforward consequence of Lemma 2 applied to the $(n+2)$-point function $H^{\tilde{G}}$ on the face $q_{n+1}+q_{n+2}=0$, with an appropriate specialization of the notations for the masses.

Now we introduce the following submanifold in π:

$$W_{\hat{e}} = \{(k,t) \in \pi : \boldsymbol{q}_i = 0,\ 1 \leqq i \leqq n;\ \boldsymbol{v} = 0\}$$

and we study the domain of analyticity of $H_\pi^{\tilde{G}}$ inside this manifold.

We shall prove:

Proposition 8. *$H_\pi^{\tilde{G}}$ is analytic in the variables $\{t^0; k_i^0, 1 \leqq i \leqq n\}$ at all the points of $W_{\hat{e}}$ which do not belong to Σ.*

Proof. This is a trivial consequence of Proposition 7 since any point lying inside $W_{\hat{e}}$ is a boundary point of the tubes $\{\mathscr{T}_\beta, \beta \in \Lambda^{(n+2)}\}$.

Let us now state the bounds which are assumed to hold for $H^{\tilde{G}}$ (in view of the recursion hypothesis) in its primitive $(n+2)$-point domain of analyticity $\mathscr{D}^{(n+2)}$, namely:

$$|H^{\tilde{G}}(\underline{k})| \leqq C_K \prod_{j=1}^{n+2} \frac{1}{1 + [(q_j^0)^2 + \boldsymbol{p}_j^2]^{2+\eta}} \left(1 + \frac{1}{d_{\hat{e}}(\underline{k}, \partial \mathscr{D}^{(n+2)})}\right)^M. \tag{5}$$

Here $\underline{k} = \underline{p} + i\underline{q}$ stands for the $(n+2)$ complex four-vectors $\{k_i, 1 \leqq i \leqq n+2\}$ and the other notations are similar to those already used in Section 4.6. This bound ensures the integrability of $H^{\tilde{G}}$ at infinity in the set of all Euclidean directions $\underline{k} = (\varrho \underline{\boldsymbol{p}}, \underline{p}^0 + i\varrho \underline{q}^0)$, with $\varrho \to +\infty$ and $\underline{q}^0$ chosen in any $(n+2)$-cell of the family $\{\gamma_\beta, \beta \in \Lambda^{(n+2)}\}$.

The following proposition will show that $H_\pi^{\tilde{G}}$ satisfies similar bounds.

Proposition 9. *Under the above assumptions we have*

$$\begin{aligned} |H_\pi^{\tilde{G}}(k,t)| \leqq C'_{\tilde{K}} \prod_{j=1}^{n} &\frac{1}{1 + [(q_j^0)^2 + \boldsymbol{p}_j^2]^{2+\eta}} \frac{1}{1 + [(v^0)^2 + \boldsymbol{u}^2]^{4+2\eta}} \\ &\cdot \left(1 + \frac{1}{d[(k,t); \Sigma]}\right)^M. \end{aligned} \tag{6}$$

Proof. We represent $\underline{k}^0$ by the variables (k^0, t, ξ) with $k^0 = \{k_i^0, 1 \leqq i \leqq n\}$, $t = k_{n+1}^0$, $\xi = k_{n+1}^0 + k_{n+2}^0$. For fixed (k^0, t) we make the following argument in the complex ξ-plane: in view of Lemma 2, $H^{\tilde{G}}(\xi)$ is analytic inside a cut neighbourhood of the real. Moreover $H_\pi^{\tilde{G}}$ is nothing but the restriction $H^{\tilde{G}}(\xi = 0)$. Then let us choose a contour γ inside the analyticity domain of $H^{\tilde{G}}(\xi)$, enclosing the origin and containing each of the two endpoints of the cuts. In view of (5) one can show that the following bound holds for $H^{\tilde{G}}$ at any point chosen on γ:

$$|H^{\tilde{G}}(\xi)| \leqq \frac{D}{(\xi^2 - \mu^2)^N}$$

with D the right-hand side of (6).

Now inside γ, $\log|H^{\tilde{G}}(\xi)|$ is certainly bounded by the harmonic function $\log\frac{D}{|\xi^2-\mu^2|^N}$ and the bound (6) follows for the restriction $H_\pi^{\tilde{G}} = H^{\tilde{G}}(\xi=0)$.

Now let us describe the analyticity domain of $H_\pi^{\tilde{G}}$ by its sections in the complex variable t^0 when k is held fixed outside the union of the family of sets $\{\Xi_I,\ I\in\mathscr{P}^*(X)\}$ introduced by Proposition 7 and $\boldsymbol{t}$ is fixed real and arbitrary.

Proposition 10. *$H_\pi^{\tilde{G}}$ is analytic at all the points of the complex plane t^0 which do not belong to the union $\hat{\Sigma}$ of the following "cuts":*

$$\hat{\Sigma}_0=\{t^0\in\mathbb{C}: t^{0^2}=\boldsymbol{t}^2+m^2\}\cup\{t^0\in\mathbb{C}: t^{0^2}=\boldsymbol{t}^2+M^2+\varrho,\varrho\geqq 0\}$$

$$\hat{\Sigma}_I=\{t^0\in\mathbb{C}:(t^0-k_I^0)^2=(\boldsymbol{t}-\boldsymbol{p}_I)^2+m_I'^2\}\cup\{t^0\in\mathbb{C}:(t^0-k_I^0)^2$$
$$=(\boldsymbol{t}-\boldsymbol{p}_I)^2+M_I'^2+\varrho,\varrho\geqq 0\}\quad\forall I\in\mathscr{P}^*(X).$$

Proof. Clearly $\hat{\Sigma}_0$ (resp. $\hat{\Sigma}_I$) is the trace on the complex plane t^0 of the set Σ_0 (resp. Σ_I) introduced by Proposition 7. Then, since k does not belong to the union of the sets $\{\Xi_I, I\in\mathscr{P}^*(X)\}$, it is sufficient to apply Proposition 8 to conclude.

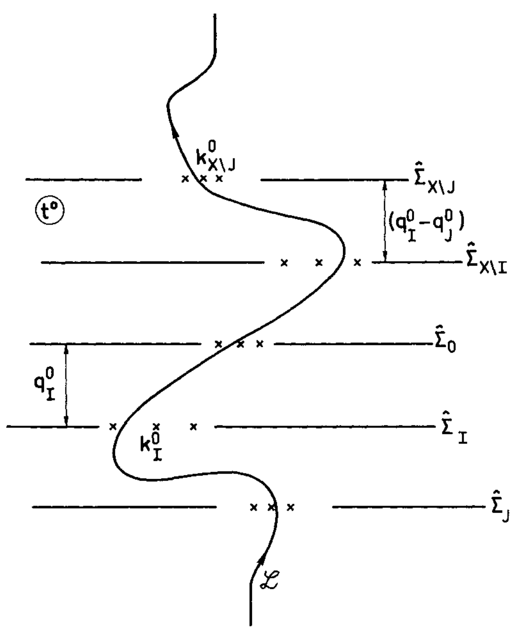


FIG. 3

Such a situation is partially pictured on Fig. 3 which shows the cut $\hat{\Sigma}_0$ and two typical couples $(\hat{\Sigma}_I, \hat{\Sigma}_{X\setminus I})$. We remark that $\hat{\Sigma}_0$ (resp. $\hat{\Sigma}_I$) is symmetrical with respect to the origin (resp. the point $t^0 = k_I^0$) and that $\hat{\Sigma}_I$ and $\hat{\Sigma}_{X\setminus I}$ are symmetrical sets with respect to the origin.

Then it is easy to check that any line $\mathscr{L}$ passing through the origin and each of the points $\{k_I^0, I \in \mathscr{P}^*(X)\}$ with a slope always different from zero lies inside the t^0-section of the domain of analyticity of $H_\pi^{\tilde{G}}$ if the following conditions are fulfilled:

i) None of the cuts $\hat{\Sigma}_I$ should be confused with $\hat{\Sigma}_0$, that is

$$\forall I \in \mathscr{P}^*(X) \qquad q_I^0 \neq 0 .$$

ii) None of the cuts $\hat{\Sigma}_I$ should be confused with one another, that is

$$\forall I, J \in \mathscr{P}^*(X) \qquad q_I^0 \neq q_J^0 .$$

Should these conditions on the external variable be unsatisfied, the domain of $H_\pi^{\tilde{G}}$ in the t^0 plane would become disconnected and the line $\mathscr{L}$ would get pinched. This analysis is the basis of much that follows and should be thoroughly understood.

5.3 The G-Product

Proposition 11. *With any point k inside a flat tube S_μ of the family $\{S_\mu,\ \mu \in \Theta^{(n)}\}$ it is possible to associate a four real dimensional region $\Gamma_{\hat{e},k}$ in $\mathbb{C}^4$ such that the function*

$$H^G_{\hat{e},\mu}(k) = \int_{\Gamma_{\hat{e},k}} H_\pi^{\tilde{G}}(k,t)\,[H^{(2)}(t)]^{-1}\,dt \tag{7}$$

is analytic inside a neighbourhood of S_μ.

Proof. Since k belongs to S_μ all the conditions stated at the end of the previous section are fulfilled. Then any line $\mathscr{L}_k$ passing through the origin and each of the points $\{k_I^0, I \in \mathscr{P}^*(X)\}$ lies inside the analyticity domain of $H_\pi^{\tilde{G}}$ in t^0. So we choose the contour $\Gamma_{\hat{e},k}$ as $\mathbb{R}^3 \times \mathscr{L}_k$. Moreover we choose $\mathscr{L}_k$ with its infinite parts *parallel* with the imaginary axis so as to make (7) convergent at infinity (see Proposition 9).

Now we can release the constraint that $\mathscr{L}_k$ should contain the origin and the various points $\{k_I^0, I \in \mathscr{P}^*(X)\}$ and only require that these points be sufficiently close to $\mathscr{L}_k$ so that $\mathscr{L}_k$ does not intersect the cuts.

Since we work with strictly positive masses, this is allowed independently of the real part $\boldsymbol{p}$ and this moderate freedom which we have to shift $\mathscr{L}_k$ is sufficient to get analyticity inside a neighbourhood of S_μ.

Proposition 12. *Inside* S_μ, $H^G_{\hat{e},\mu}$ *satisfies the bound:*

$$|H^G_{\hat{e},\mu}(k)| \leqq L_K \prod_{i=1}^{n} \frac{1}{1+[(q_i^0)^2+\boldsymbol{p}_i^2]^{2+\eta}} \left(1+\frac{1}{d(q^0,\partial\sigma_\mu)}\right)^M \tag{8}$$

with L_K *some positive constant which depends of a compact set* K *in* p^0-*space and* $d(q^0,\partial\sigma_\mu)$ *the Euclidean distance in* $\mathbb{R}^{n-1}$ *of the point* q^0 *to the boundary of the pseudocell* σ_μ. *This bound ensures integrability of* $H^G_{\hat{e},\mu}$ *in Euclidean directions. Moreover the order of increase* M *is the same that for* $H^{\tilde{G}}$.

Proof. We start from the bound (6) which can be rewritten:

$$|H^{\tilde{G}}_\pi(k,t)\,[H^{(2)}(t)]^{-1}| \leqq L_{\tilde{K}} \prod_{j=1}^{n} \frac{1}{1+[(q_j^0)^2+\boldsymbol{p}_j^2]^{2+\eta}} \; \frac{1}{1+[(v^0)^2+\boldsymbol{u}^2]^{2+\eta}} \cdot\left(1+\frac{1}{d[(k,t);\Sigma]}\right)^M .$$

Now the integral on $\Gamma_{\hat{e},k}$ of the term $[(v^0)^2+\boldsymbol{u}^2]^{-(2+\eta)}$ is certainly convergent. Moreover it is possible to choose the compact set $\tilde{K}$ as a product $K\times[-a,+a]$ in (p^0,u^0)-space. Then in order to prove (8) it is sufficient to check that for any t on $\Gamma_{\hat{e},k}$ we have the following inequality:

$$d[(k,t);\Sigma] \geqq C_1\, d(q^0,\partial\sigma_\mu) \tag{9}$$

with C_1 some positive constant.

This will be seen as follows. First we notice that in view of the equivalence of distances in $W_{\hat{e}}$, we have

$$d[(k,t);\Sigma] \geqq C_2 \inf[d(q^0,\partial\sigma_\mu), d(t^0,\hat{\Sigma})]$$

with C_2 some positive constant and $d(t^0,\hat{\Sigma})$ the Euclidean distance in $\mathbb{C}$ of the point t^0 to the union of the cuts $\hat{\Sigma}$.

Now since the complex line $\mathscr{L}_k$ can always be chosen to lie at a finite distance of the cuts (and this is easily checked), there exists a positive constant C_3 such as

$$d(t^0,\hat{\Sigma}) \geqq C_3 \inf_{I,J\in\mathscr{P}^*(X)} \{|q_I^0|, |q_I^0-q_J^0|\} .$$

But this last term is exactly $d(q^0,\partial\sigma_\mu)$ and the inequality (9) is proved.

Now let us consider a given cell γ_λ and the family $\{\sigma_\mu, \mu\in M_\lambda\}$ of the pseudocells included in γ_λ. In any flat tube S_μ of the family $\{S_\mu, \mu\in M_\lambda\}$ a function $H^G_{\hat{e},\mu}$ analytic inside a neighbourhood of S_μ has been introduced. We have the following

Proposition 13. *The functions* $\{H^G_{\hat{e},\mu}, \mu\in M_\lambda\}$ *are pieces of a unique function* $H^G_{\hat{e},\lambda}$ *which is analytic inside a neighbourhood* $\mathscr{D}_{\hat{e},\lambda}$ *of the flat tube* $\mathscr{T}_\lambda\cap V_{\hat{e}}$ *and satisfies integrability conditions of the type* (8) *in Euclidean* *directions.* $H^G_{\hat{e},\lambda}$ *has a slow increase near the real in* $\mathscr{D}_{\hat{e},\lambda}$ *and has therefore a boundary value in the sense of distributions.*

The derivation of this property is rather intricate and will be found in the appendix. Let us only stress that it is crucial for the proof that the various boundary values of $H^{\tilde{G}}$ satisfy Steinmann relations.

5.4 *Moving the Time Direction*

In the previous section we have seen that for any time direction $\hat{e} \in V^+$ and any tube $\mathscr{T}_\lambda$, there exists a function $H^G_{\hat{e},\lambda}$ which is analytic inside a neighbourhood $\mathscr{D}_{\hat{e},\lambda}$ of the flat tube $\mathscr{T}_\lambda \cap V_{\hat{e}}$. As a matter of fact $\mathscr{D}_{\hat{e},\lambda}$ itself can be chosen as a convex tube. In this section we intend to move the time direction $\hat{e}$ everywhere in the light cone and prove that the various functions $H^G_{\hat{e},\lambda}$ are pieces of the same analytic function H^G_λ.

For this purpose we shall need the following well known result. Assume that a domain $\mathscr{D}$ is covered by a family of domains $\{\mathscr{D}_i, i \in \mathscr{I}\}$ and that in each $\mathscr{D}_i$ an analytic function φ_i is given. Assume that in each non empty $\mathscr{D}_i \cap \mathscr{D}_j$ we have the compatibility condition $\varphi_i = \varphi_j$. Then it is possible to define a single valued function φ which is analytic inside $\mathscr{D}$ and in any $\mathscr{D}_i$ is identical with φ_i. Now we are in a position to prove:

Proposition 14. *For any* $\lambda \in \Lambda^{(n)}$, *the functions* $\{H^G_{\hat{e},\lambda}, \hat{e} \in V^+\}$ *introduced in the previous section are pieces of a unique function* H^G_λ *which is analytic inside the tube* T_λ *with convex conical basis* $\gamma_\lambda \otimes V^+$:

$$T_\lambda = \{k \in \mathbb{C}^{4(n-1)} : \operatorname{Im} k \in \gamma_\lambda \otimes V^+\}$$

where $\gamma_\lambda \otimes V^+$ *denotes the tensor product of the cones* γ_λ *and* V^+.

Proof. Consider the direction $\hat{e}_i$ $(i=1,2)$ and the corresponding function $H^G_{\hat{e}_i,\lambda}$ which is analytic inside the tube $\mathscr{D}_{\hat{e}_i,\lambda}$. If non empty, the intersection $\mathscr{D}_{\hat{e}_1,\lambda} \cap \mathscr{D}_{\hat{e}_2,\lambda}$ is then a connected convex tube. Moreover for any point k inside $\mathscr{D}_{\hat{e}_1,\lambda} \cap \mathscr{D}_{\hat{e}_2,\lambda}$, using the convergence at infinity and Stoke's theorem, we check easily that $\Gamma_{\hat{e}_1,k}$ can be distorted into $\Gamma_{\hat{e}_2,k}$ without modifying the value of the integral (7), which yields:

$$\forall\, k \in \mathscr{D}_{\hat{e}_1,\lambda} \cap \mathscr{D}_{\hat{e}_2,\lambda} \qquad H^G_{\hat{e}_1,\lambda}(k) = H^G_{\hat{e}_2,\lambda}(k)\,.$$

From the remark made above, we conclude to the existence of a unique function H^G_λ which is analytic inside $\bigcup_{\hat{e} \in V^+} \mathscr{D}_{\hat{e},\lambda}$ and is a common analytic continuation of the various $H^G_{\hat{e},\lambda}$.

Now $\bigcup_{\hat{e} \in V^+} \mathscr{D}_{\hat{e},\lambda}$ is not a natural domain of holomorphy since its basis is not convex. We claim that its holomorphy envelope contains the tube T_λ with convex conical basis $\gamma_\lambda \otimes V^+$. Indeed the basis of the flat tube

$\mathscr{T}_\lambda \cap V_{\hat{e}}$ can be written:

$$R_{\hat{e},\lambda} = \{q \in \mathbb{R}^{4(n-1)} : \boldsymbol{q} = 0, q^0 \in \gamma_\lambda\}$$

or similarly:

$$R_{\hat{e},\lambda} = \{q \in \mathbb{R}^{(n-1)} \otimes \mathbb{R}^4 : q = s \otimes \hat{e}, s \in \gamma_\lambda\} .$$

And it follows directly from this representation and from the definition of the tensor product of two cones that the convex hull of $\bigcup_{\hat{e} \in V^+} R_{\hat{e},\lambda}$ is precisely $\gamma_\lambda \otimes V^+$. Then it is sufficient to apply the tube theorem [10] to conclude.

5.5 *Getting the Tubes* $\mathscr{T}_\lambda$

In the previous section, we have shown that for any $\lambda \in \Lambda^{(n)}$ there exists a function H_λ^G which is analytic inside the tube T_λ with convex basis $\gamma_\lambda \otimes V^+$. However it is not yet the relevant tube $\mathscr{T}_\lambda$ since we have the following lemma:

Lemma 3. [7]. *The tube T_λ with convex basis $\gamma_\lambda \otimes V^+$ is included in the tube $\mathscr{T}_\lambda$. The equality $T_\lambda = \mathscr{T}_\lambda$ occurs if and only if the cell γ_λ is simplicial.* (We recall that a cell is *simplicial* if limited in $\mathbb{R}^{n-1}$ by $(n-1)$ planes. Non-simplicial cells appear when $n \geqq 5$.)

Thus in order to recover the analytic structure of a general n-point function as defined above in Section 2.5, we need to enlarge the analyticity domain of the function H_λ^G when the cell γ_λ is non-simplicial. However since T_λ is a natural domain of holomorphy, this cannot be made by purely geometrical techniques of analytic completion. Actually we shall need the following basic lemma:

Lemma 4. *Let us consider a family $\{f_\lambda, \lambda \in \Lambda^{(n)}\}$ of functions which satisfy the following conditions:*

i) *For any $\lambda \in \Lambda^{(n)}$, f_λ is analytic inside the tube T_λ.*

ii) *The various real boundary values $\{f_\lambda(p), \lambda \in \Lambda^{(n)}\}$ satisfy the Steinmann relations.*

Then for any $\lambda \in \Lambda^{(n)}$, f_λ is analytic inside the tube $\mathscr{T}_\lambda$.

The proof of this lemma is out of the scope of the present work and will be reported in [27]. It uses a basic property of the Steinmann relations, namely the possibility to "solve" them by introducing auxiliary functions which enjoy better analyticity properties than the original f_λ [23].

Thus in order to apply Lemma 4 to the family $\{H_\lambda^G, \lambda \in \Lambda^{(n)}\}$ we need the following

Proposition 15. *The four real boundary values* $\{H^G_{\lambda_{+\,+}}(p), H^G_{\lambda_{+\,-}}(p), H^G_{\lambda_{-\,-}}(p), H^G_{\lambda_{-\,+}}(p)\}$ *which occur in any quartet of tubes satisfy the quartet relation*

$$H^G_{\lambda_{+\,+}}(p)+H^G_{\lambda_{-\,-}}(p)=H^G_{\lambda_{+\,-}}(p)+H^G_{\lambda_{+\,-}}(p)$$

in the sense of distributions for all real arguments.

The proof of this property is rather cumbersome and lengthy and will be shown in [27]. In view of Lemma 4, it allows to state

Proposition 16. *For any* $\lambda\in\Lambda^{(n)}$, H^G_λ *is analytic inside the tube* $\mathscr{T}_\lambda$.

5.6 Proof of Coincidence Relations

In this section we intend to prove that the functions $\{H^G_\lambda, \lambda\in\Lambda^{(n)}\}$ are restrictions to the tubes $\{\mathscr{T}_\lambda, \lambda\in\Lambda^{(n)}\}$ of a unique function H^G which is a general n-point function. We shall need the following

Proposition 17. *The two boundary values* $H^G_{\lambda_1}(p)$ *and* $H^G_{\lambda_2}(p)$ *associated with two adjacent tubes* $\mathscr{T}_{\lambda_1}$ *and* $\mathscr{T}_{\lambda_2}$ *separated by a partition* $(I, X\backslash I)$ *coincide (in the sense of distributions) on a real region* $\mathscr{R}_I$

$$\mathscr{R}_I=\{p\in\mathbb{R}^{4(n-1)}: p_I^2 \neq m_I^2, p_I^2<\tilde{M}_I^2\}$$

where m_I *and* $\tilde{M}_I$ *are certain numbers associated with the partition* $(I, X\backslash I)$.

Proof. In order to prove the coincidence of $H^G_{\lambda_1}(p)$ and $H^G_{\lambda_2}(p)$ on $\mathscr{R}_I$, we remark that it is sufficient to stick to the situation where a given time direction $\hat{e}\in V^+$ has been chosen and prove the coincidence of $H^G_{\hat{e},\lambda_1}(p)$ and $H^G_{\hat{e},\lambda_2}(p)$ on the real open set:

$$\mathscr{R}_{I,\hat{e}}=\{p\in\mathbb{R}^{4(n-1)}: |p_I^0|\neq m_I,\ |p_I^0|<\tilde{M}_I\}\,.$$

Indeed any function $H^G_{\hat{e},\lambda}$ is the restriction to $V_{\hat{e}}$ of the function H^G_λ and we have for all real arguments:

$$H^G_{\hat{e},\lambda}(p)=H^G_\lambda(p)\,.$$

To conclude it is then sufficient to notice that $\mathscr{R}_I=\bigcup_{\hat{e}\in V^+}\mathscr{R}_{I,\hat{e}}$.

Now let us consider the two adjacent tubes $\mathscr{T}_{\lambda_1}\cap V_{\hat{e}}$ and $\mathscr{T}_{\lambda_2}\cap V_{\hat{e}}$, and a complex point k lying on their common face, inside the submanifold $q_I^0=0$. First, in order to ensure in (7) the analyticity of the integrand $H^{\tilde{G}}_\pi$ in the external variable k, we remark that, in view of Proposition 10, the following condition must be satisfied:

$$|p_I^0|\neq m_I \qquad |p_I^0|<M_I \tag{10}$$

where m_I and M_I are the masses which occur in the definition of Ξ_I in Proposition 7.

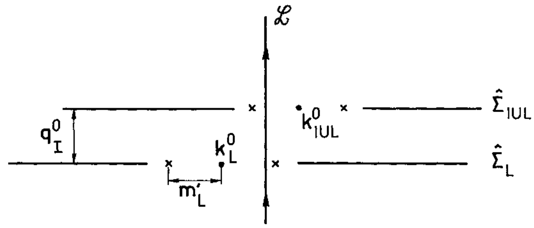


FIG. 4

Then we notice that in the complex plane t^0, when q_I^0 tends to zero, the following couples of cuts become confused: $(\hat{\Sigma}_I, \hat{\Sigma}_0)$, $(\hat{\Sigma}_{X\setminus I}, \hat{\Sigma}_0)$ and the two families $\{(\hat{\Sigma}_L, \hat{\Sigma}_{I\cup L}); L \in \mathscr{P}^*(X\setminus I)\}$ and $\{(\hat{\Sigma}_M, \hat{\Sigma}_{M\cup(X\setminus I)}); M \in \mathscr{P}^*(I)\}$.

The situation in the neighbourhood of a given couple $(\hat{\Sigma}_L, \hat{\Sigma}_{I\cup L})$ is shown in Fig. 4. Clearly at the limiting case $q_I^0 = 0$, no contour will get pinched if the two analyticity "gaps" $|u^0 - p_L^0| < m'_L$ and $|u^0 - p_{I\cup L}^0| < m'_{I\cup L}$ have a non-empty intersection: indeed the line $\mathscr{L}$ shown in Fig. 4 will provide a common distortion for each of them. This necessitates the condition $|p_I^0| < (m'_L + m'_{I\cup L})$ to be satisfied. Now this analysis goes similarly with all couples of confused cuts and there exists a certain mass M''_I such as the condition

$$|p_I^0| \neq m_I \qquad |p_I^0| < \tilde{M}_I = \inf(M_I, M''_I) \tag{11}$$

allows the two functions $H^G_{\hat{e},\lambda_1}$ and $H^G_{\hat{e},\lambda_2}$ to have a common analytic continuation on the submanifold $q_I^0 = 0$. Since for a function which is analytic and with slow increase near the real in a tube the distribution boundary value is the same for all directions inside this tube, it is enough to tend to the real inside $q_I^0 = 0$ to get the desired result.

Putting together Propositions 15–17, we finally obtain that the functions $\{H^G_\lambda, \lambda \in \Lambda^{(n)}\}$ are pieces of a unique function H^G which is a general n-point function.

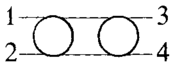
5.7 Additional Remark

As in the case of trees, we must point out that H^G is analytic in a domain larger than the "primitive domain" of the physical n-point function. Indeed let us introduce the tree T obtained by "cutting" successively l internal lines of G and denote $\{k_1, k_2, \ldots, k_n, k_{n+1}, \ldots, k_{n+2l}\}$ its $(n+2l)$ external lines. We consider the sets $I \in \mathscr{P}^*(X)$ such as the partition $[I; (X\setminus I) \cup \{n+1, n+2, \ldots, n+2l\}]$ *is not* a vertex partition for T. H^T is analytic on such faces $q_I = 0$ and consequently, in view of the analysis made above, we claim that the coincidence region $\mathscr{R}_I$ of H^G

associated with the channel $(I, X\backslash I)$ is of the form:

$$\mathscr{R}_I = \{p \in \mathbb{R}^{4(n-1)} : p_I^2 < \tilde{M}_I^2\} .$$

This comes from the fact that H^T is analytic on the face $q_I = 0$ and that the pinching of the complex line $\mathscr{L}_k$ is by nature unable to produce in $\mathscr{R}_I$ any condition of the type $p_I^2 \neq m_I^2$.

We shall say that H^G is *one-particle irreducible* in such channels (for a detailed study of the notion of p-particle irreducible functions in axiomatic field theory, see [21]). For instance the convolution product 

is one-particle irreducible in the channels t and u.

5.8 Independence of the Antecedent

In this section we shall prove that the G-product does not depend of the original choice of an antecedent $\tilde{G}$ of G.

Let $\tilde{G}_1$ (resp. $\tilde{G}_2$) be an antecedent of G obtained by cutting an internal line i_1 (resp. i_2) of G, and $\tilde{\tilde{G}}_{12}$ the *common* antecedent of $\tilde{G}_1$ and $\tilde{G}_2$ obtained from G by cutting simultaneous i_1 and i_2. For a given time direction $\hat{e}$, with $\tilde{G}_j$ $(j = 1, 2)$ we associate the G-product $H_{\hat{e}}^{G,j}$. Then we introduce the following $4(n-1)$ real dimensional submanifold $\mathscr{E}_0^{(n)}$ in $\mathbb{C}^{4(n-1)}$:

$$\mathscr{E}_0^{(n)} = \{k \in \mathbb{C}^{4(n-1)} : \boldsymbol{q}_i = 0,\ p_i^0 = 0,\ 1 \leqq i \leqq n\} .$$

$\mathscr{E}_0^{(n)}$ is the *Euclidean region* associated with the time direction $\hat{e}$. In order to check that $H_{\hat{e}}^{G,1}$ and $H_{\hat{e}}^{G,2}$ define the same function at any point of their common analyticity domain, it is enough by analytic continuation to check that they coincide on $\mathscr{E}_0^{(n)}$. As for the latter point, it will appear as a straightforward consequence of the following result, once taken into account the Fubini's theorem for multiple integrals:

Proposition 18. *At any point k inside a neighbourhood of $\mathscr{E}_0^{(n)}$, $H_{\hat{e}}^{G,j}$ $(j = 1, 2)$ satisfies the following representation:*

$$H_{\hat{e}}^{G,j}(k) = \int_{\Gamma_{\hat{e},0}} \int_{\Gamma_{\hat{e},0}} H^{\tilde{\tilde{G}}_{12}}(k, k_{n+1}, \ldots, k_{n+4}) \Big|_{\substack{k_{n+1} = -k_{n+2} = t \\ k_{n+3} = -k_{n+4} = t'}} [H^{(2)}(t)]^{-1} [H^{(2)}(t')]^{-1}\, dt\, dt'$$

with $\Gamma_{\hat{e},0} = \mathbb{R}^3 \times L_0$ and $L_0 = i\mathbb{R}$ the imaginary axis in the complex plane t^0 (resp. t'^0).

Proof. Once k is held fixed inside a sufficient small neighbourhood of $\mathscr{E}_0^{(n)}$, the situation in the t^0-plane is as pictured on Fig. 5: all the cuts are centered on the imaginary axis L_0. Then we can choose L_0 as the integration line and write:

$$H_{\hat{e}}^{G,j}(k) = \int_{\Gamma_{\hat{e},0} = \mathbb{R}^3 \times L_0} H_\pi^{\tilde{G}_j}(k, t)\, [H^{(2)}(t)]^{-1}\, dt .$$

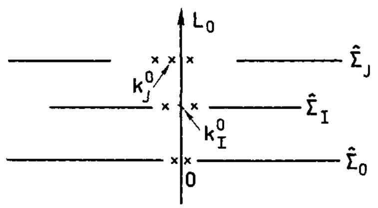


FIG. 5

Now the following basic argument is easily checked: when k lies inside $\mathscr{E}_0^{(n)}$ and t varies on $\Gamma_{\hat{e},0}$, the point $(k, t, -t)$ lies inside the corresponding Euclidean region in $\mathbb{C}^{4(n+1)}$, namely

$$\{(k, k_{n+1}, k_{n+2}) \in \mathbb{C}^{4(n+1)} : \boldsymbol{q}_i = 0,\ p_i^0 = 0,\ 1 \leqq i \leqq n+2\}.$$

But at such points in $\mathbb{C}^{4(n+1)}$, the same argument still allows to write, with the *same* contour $\Gamma_{\hat{e},0}$:

$$H_{\hat{e}}^{\tilde{G}_j}(k, k_{n+1}, k_{n+2}) = \int_{\Gamma_{\hat{e},0}} H^{\tilde{\tilde{G}}_{12}}(k, k_{n+1}, \ldots, k_{n+4}) \Big|_{k_{n+3} = -k_{n+4} = t'} [H^{(2)}(t')]^{-1}\, dt'$$

which allows to conclude.

Thus we have shown that H^G does not depend of the original choice of an antecedent $\tilde{G}$ of G and this achieves the proof of the recursion property[3]. Taking into account the initial case of trees, we finally obtain:

Theorem 2. *The convolution product associated with any graph G with n external lines is a general n-point function.*

5.9 A Class of Global Representations

In a neighbourhood of $\mathscr{E}_0^{(n)}$ the method used in the proof of Proposition 18 allows to write a *global representation* of H^G, which brings out the individual contributions of the vertex functions as well as the multiple integral prescription associated with the internal lines. Indeed the following fact has been put into evidence: if k belongs to a neighbourhood of $\mathscr{E}_0^{(n)}$ the axis L_0 is a suitable integration line and then the point $(k, t, -t)$ belongs to a neighbourhood of $\mathscr{E}_0^{(n+2)}$. Iterating this argument we get the following representation of $H_{\hat{e}}^G$, at any point k inside a sufficiently small neighbourhood of $\mathscr{E}_0^{(n)}$:

$$H_{\hat{e}}^G(k) = \int_{(\mathbb{R}^3 \times L_0)^l} \prod_i [H^{(2)}(k_i)]^{-1} \prod_v H^{(n_v)}(\{k_j, j \in \mathscr{S}_v\})\, d\mu \qquad (12)$$

with l the number of independent loops of G, and $d\mu$ the usual Euclidean measure in the $4l$ real dimensional linear space $(\mathbb{R}^3 \times L_0)^l$. The first

[3] The argument applies for $l \geqq 2$. In the case $l = 1$ it is straightforward to check that the tree-products $H^{\tilde{G}_1}$ and $H^{\tilde{G}_2}$ define the same relevant restriction.

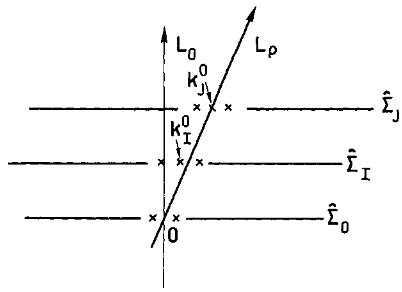


FIG. 6

product extends to the *internal lines* of G; k_i denotes the internal four vector associated with the line i. The second product extends to the *vertices* of G; $H^{(n_v)}$ denotes the general n_v-point function associated with the n_v lines $\{j \in \mathscr{S}_v\}$ of G incident to v. As a result of Sections 4.6 and 5.3, in the course of this iteration the convergence of the multiple integral (12) is ensured by the bounds (3) which we have assumed for any vertex function $H^{(n_v)}$.

Actually a natural extension of this representation is provided by the classical argument of the *Wick rotation* of the contours. Let us consider the family $\{\mathscr{E}_\varrho^{(n)}, \varrho \in \mathbb{R}\}$ of $4(n-1)$ real dimensional submanifolds:

$$\mathscr{E}_\varrho^{(n)} = \{k \in \mathbb{C}^{4(n-1)} : \boldsymbol{q}_i = 0,\ p_i^0 = \varrho q_i^0,\ 1 \leqq i \leqq n\}\,.$$

The following argument is easily checked on Fig. 6: if k belongs to a sufficiently small neighbourhood of $\mathscr{E}_\varrho^{(n)}$, all the cuts are centered along the line L_ϱ (with equation $u^0 = \varrho v^0$) in the t^0-plane. L_ϱ is a suitable integration line and then the point $(k, t, -t)$ belongs to a neighbourhood of $\mathscr{E}_\varrho^{(n+2)}$. Iterating the argument we get the following representation of $H_{\hat{e}}^G$ in a neighbourhood of $\mathscr{E}_\varrho^{(n)}$:

$$H_{\hat{e}}^G(k) = \int\limits_{(\mathbb{R}^3 \times L_\varrho)^l} \prod_i [H^{(2)}(k_i)]^{-1} \prod_v H^{(n_v)}(\{k_j, j \in \mathscr{S}_v\})\, d\mu \tag{13}$$

with $d\mu$ the usual Euclidean measure in $(\mathbb{R}^3 \times L_\varrho)^l$.

However this representation converges only if any vertex function $H^{(n_v)}$ is assumed to enjoy bounds of the type (3) at infinity inside the manifold $\mathscr{E}_\varrho^{(n_v)}$ in $\mathbb{C}^{4(n_v-1)}$.

In this context a special role is played by the limiting case $\varrho \to +\infty$. Then the submanifold $\mathscr{E}_\varrho^{(n)}$ tends to the real space $\mathbb{R}^{4(n-1)}$ and of course the previous representation (13) is nothing but a formal recipe which we must give a precise meaning. This will be done in the next section: under appropriate bounds at all infinite directions *near the real*, we shall prove that (13) still holds *in the sense of the t-boundary values* of the various n-point functions there involved.

6. T-Boundary Value of the Convolution Product

In order to give sense to (13) when the external arguments lie inside the real space $\mathbb{R}^{4(n-1)}$, let us first recall that the t-boundary value of a general n-point function is defined by the following prescription:

$$f_t(p) = \lim_{\substack{k \to p \\ q \in \mathscr{C}_\lambda, p \in \Omega_\lambda}} f(k)$$

where the open covering $\{\Omega_\lambda, \lambda \in \Lambda^{(n)}\}$ has been introduced in Section 1.5. That is, f_t is defined by the collection of boundary values $\{f_\lambda(p), \lambda \in \Lambda^{(n)}\}$ with the consistency conditions $f_\lambda(p) = f_{\lambda'}(p)$ in any $\Omega_\lambda \cap \Omega_{\lambda'}$. Now we intend to prove:

Proposition 19. *If a sufficiently strong decrease is assumed near the real at infinity, the following representation holds for the t-boundary value H_t^G (in the sense of distributions):*

$$H_t^G(p) = \int_{\mathbb{R}^{4l}} \prod_i [H_t^{(2)}(p_i)]^{-1} \prod_v H_t^{(n_v)}(\{p_j, j \in \mathscr{S}_v\})\, d\mu \tag{14}$$

with $d\mu$ the Euclidean measure in the space $\mathbb{R}^{4l}$ of internal four-momenta and $H_t^{(n_v)}$ (resp. $H_t^{(2)}$) the t-boundary value of the corresponding general n_v-point (resp. two-point) function.

As in the previous section, the proof will go by recursion over the number l of independent loops of G.

6.1 The Case of Trees

In order to prove that the $l = 0$ form of (14):

$$H_t^T(p) = \prod_v H_t^{(n_v)}(\{p_{I(l,v)}, l \in \mathscr{S}_v\}) \prod_i [H_t^{(2)}(p_I)]^{-1}$$

holds in the sense of distributions, it is enough to check that on any open set of the covering $\{\Omega_\lambda, \lambda \in \Lambda^{(n)}\}$, all factors are boundary values of functions (with slow increase) analytic is the same domain. We start from the following representation of H^T inside the tube $\mathscr{T}_\lambda$:

$$H_\lambda^T(k) = \prod_v H_{\lambda_v}^{(n_v)}(\{k_{I(l,v)}, l \in \mathscr{S}_v\}) \prod_i [H^{(2)}(k_I)]^{-1}$$

which yields in the sense of distributions for all real arguments:

$$H_\lambda^T(p) = \prod_v H_{\lambda_v}^{(n_v)}(\{p_{I(l,v)}, l \in \mathscr{S}_v\}) \prod_i [H_{\lambda(I)}^{(2)}(p_I)]^{-1}$$

with $\lambda(I)$ the sign of s_I in the cell γ_λ. If we restrict the above relation in the relevant open set Ω_λ, it is easy to check that the n_v real four-vectors $\{p_{I(l,v)}, l \in \mathscr{S}_v\}$ associated with the vertex v belong to the open set Ω_{λ_v}

defined in the relevant space by λ_v. Thus we can apply the Ruelle prescription at each vertex v and replace the factor $H_{\lambda_v}^{(n_v)}$ by $H_t^{(n_v)}$ (and similarly each factor $H_{\lambda(I)}^{(2)}$ by $H_t^{(2)}$), which achieves the proof.

6.2 The General Case

We shall need the following:

Lemma 5. *In order to define the t-boundary value H_t^G it is sufficient to stick to the case of a chosen time direction $\hat{e}$ and apply the following prescription:*

$$\forall\, p \in \Omega_{\hat{e},\lambda}\; H_t^G(p) = H_{\hat{e},\lambda}^G(p)$$

with $\Omega_{\hat{e},\lambda} = \{p \in \mathbb{R}^{4(n-1)} : \lambda(I)p_I^0 > -\tilde{m}_I \;\forall\, I \in \mathscr{P}^*(X)\}$ and $\tilde{m}_I < m_I$, with m_I the discrete mass associated with the partition $(I, X\backslash I)$.

Proof. Since the sets of the family $\{\Omega_{\hat{e},\lambda}, \lambda \in \Lambda^{(n)}\}$ form an open covering of $\mathbb{R}^{4(n-1)}$ such that $\Omega_{\hat{e},\lambda} \subset \Omega_\lambda$ for any $\lambda \in \Lambda^{(n)}$, it is a direct consequence of the consistency relations:

$$\forall\, p \in \Omega_\lambda \cap \Omega_{\lambda'} \qquad H_{\hat{e},\lambda}^G(p) = H_{\hat{e},\lambda'}^G(p)\,.$$

Now the proof of Proposition 19 will be a direct consequence of

Proposition 20. *If a sufficiently strong decrease is assumed near the real at infinity, the t-boundary value of the G-product satisfies (in the sense of distributions):*

$$H_t^G(p) = \int_{\mathbb{R}^4} H_t^{\tilde{G}}(p, p_{n+1}, p_{n+2})\Big|_{p_{n+1} = -p_{n+2} = u} [H_t^{(2)}(u)]^{-1}\, du$$

Proof. In the following we use the Ruelle prescription such as provided by Lemma 5 with a covering $\{\Omega_{\hat{e},\lambda}, \lambda \in \Lambda^{(n)}\}$ defined by the following choice of masses:

$$\forall\, I \in \mathscr{P}^*(X) \qquad \tilde{m}_I = \inf(m_I, m_I')\,.$$

(We recall that m_I is the discrete mass in the channel $(I, X\backslash I)$ and m_I' half the minimum of the distance between the two poles of the cut $\hat{\Sigma}_I$.) The reasons for such a choice will appear clearer in the following. Now let us consider a given point $p = (\boldsymbol{p}, p^0)$ in a chosen open set $\Omega_{\hat{e},\lambda}$. We have:

$$\forall\, I \in \mathscr{P}^*(X) \qquad \lambda(I)p_I^0 > -\tilde{m}_I\,.$$

In the following we denote $\mathscr{I}$ the family of proper subsets of X such that

$$\forall\, I \in \mathscr{I} \quad \lambda(I)p_I^0 > 0 \qquad \forall\, I \notin \mathscr{I} \quad -\tilde{m}_I < \lambda(I)p_I^0 \leqq 0\,.$$

We introduce the auxiliary point $\hat{p}^0$ inside the cell γ_λ as follows: if $I \in \mathscr{I}$, we choose $\hat{p}_I^0 = p_I^0$; if $I \notin \mathscr{I}$ we choose $\lambda(I)\hat{p}_I^0$ arbitrary inside the interval $]0, \lambda(I)p_I^0 + \tilde{m}_I[$ in such a way that $\hat{p}^0$ belongs to γ_λ. If we consider the

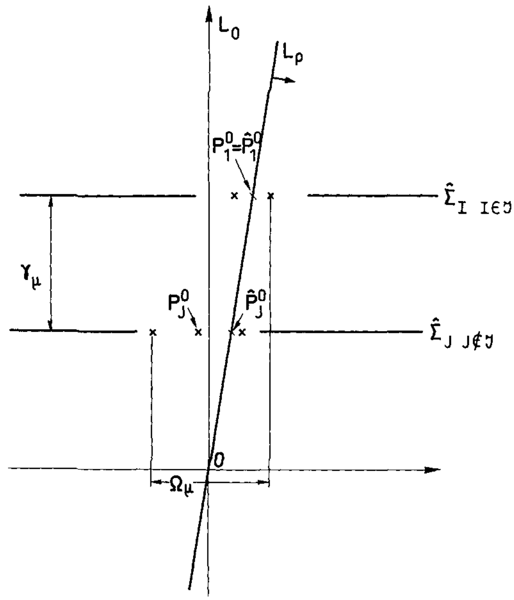


FIG. 7

point $\hat{k}=(\boldsymbol{p},\hat{p}^0+iq^0)$ with $\hat{p}^0=\varrho q^0$ and ϱ positive, it is straightforward that $\hat{k}$ belongs to the submanifold $\mathscr{E}_\varrho^{(n)}$ and q^0 to γ_λ. Since the line L_ϱ is a suitable contour for $\hat{k}$, we can write:

$$H^G_{\hat{e},\lambda}(\hat{k})=\int\limits_{\mathbb{R}^3\times L_\varrho} H^{\tilde{G}}_\pi(\hat{k},t)\,[H^{(2)}(t)]^{-1}\,dt\,.$$

Now if we consider the point $k=(\boldsymbol{p},p^0+iq^0)$ it is clear that k lies in a neighbourhood of $\mathscr{E}_\varrho^{(n)}$ in the following sense: if we tend from $\hat{k}$ to k by keeping q^0 fixed and letting $\hat{p}^0$ tend towards p^0, none of the moving cuts $\{\hat{\Sigma}_I, I\notin\mathscr{I}\}$ will intersect the line L_ϱ. This is indeed checked easily on Fig. 7 where the two situations for a partial sum have been pictured. Thus the line L_ϱ is still a suitable contour for k and we write:

$$H^G_{\hat{e},\lambda}(k)=\int\limits_{\mathbb{R}^3\times L_\varrho} H^{\tilde{G}}_\pi(k,t)\,[H^{(2)}(t)]^{-1}\,dt\,. \tag{15}$$

Now we have the Ruelle prescription:

$$\forall\,(\boldsymbol{p},p^0)\in\Omega_{\hat{e},\lambda}\qquad H^G_t(\boldsymbol{p},p^0)=\lim_{\substack{q^0\to 0\\ q^0\in\gamma_\lambda}} H^G_{\hat{e},\lambda}(\boldsymbol{p},p^0+iq^0)$$

which we choose to write under the form:

$$H^G_t(\boldsymbol{p},p^0)=\lim_{\varrho\to+\infty} H^G_{\hat{e},\lambda}\left(\boldsymbol{p},p^0+\frac{i}{\varrho}\hat{p}^0\right).$$

Taking into account (15) this yields:

$$H_t^G(\boldsymbol{p}, p^0) = \lim_{\varrho \to +\infty} \int_{\mathbb{R}^3 \times L_\varrho} H_\pi^{\tilde{G}}\left(\left(\boldsymbol{p}, p^0 + \frac{i}{\varrho}\hat{p}^0\right), t\right) \quad [H^{(2)}(t)]^{-1}\, dt \tag{16}$$

provided that sufficiently strong decrease properties at infinity near the real have been assumed.

It is easy to check that when k is fixed as above in a neighbourhood of $\mathscr{E}_\varrho^{(n)}$ and t varies on $\mathbb{R}^3 \times L_\varrho$, the point (k, t) is in the situation when the Ruelle prescription in $\mathbb{C}^{4n}$ can be applied (in the limit $\varrho \to \infty$). This is straightforward on Fig. 7 which shows the traces on the plane t^0 of a $(n+2)$-cell γ_μ and of its associated open set Ω_μ. Thus we have:

$$\lim_{\varrho \to +\infty} H_\pi^{\tilde{G}}\left(\left(\boldsymbol{p}, p^0 + \frac{i}{\varrho}\hat{p}^0\right), t\right) [H^{(2)}(t)]^{-1} = H_t^{\tilde{G}}(p, p_{n+1}, p_{n+2})\Big|_{p_{n+1} = -p_{n+2} = u} [H_t^{(2)}(u)]^{-1}.$$

Here, in view of the analyticity properties of $H_\pi^{\tilde{G}}$, the right-hand side is meaningful in the sense of distributions. Then (16) can be rewritten (in the sense of distributions):

$$H_t^G(p) = \int_{\mathbb{R}^4} H_t^{\tilde{G}}(p, p_{n+1}, p_{n+2})\Big|_{p_{n+1} = -p_{n+2} = u} [H_t^{(2)}(u)]^{-1}\, du$$

which is the desired result.

7. An Application to Perturbation Theory

Since the idea of G-convolution originates from the consideration of Feynmann perturbative series, it is natural to set the question of the applications of our study to perturbation theory. In this paper we do not consider the case of a polynomial interaction lagrangian since a special study would have there to be made in view of renormalization. However we can deal with all theories in which convergence in Euclidean directions is assumed to hold [29].

In fact as a consequence of Theorem 2, we shall see that the Feynmann amplitudes associated with a certain class of "superpropagators" which is defined below satisfy all the requirements of the linear program.

The superpropagators $w(k)$ which we consider are general two-point functions which satisfy a bound of the following type in the Euclidean directions:

$$|w(k)| < \frac{C_K}{[\boldsymbol{p}^2 + (q^0)^2]^{2+\eta}} \tag{17}$$

with η strictly positive and $p^0 \in K$.

Then, with any function w and any graph G with n external lines, we associate the amplitude:

$$F^G(k) = \int_{(\mathbb{R}^3 \times L_0)^l} \prod_i w(k_i)\, d\mu\,. \tag{18}$$

Here the external variables k are taken in a neighbourhood of the Euclidean region $\mathscr{E}_0^{(n)}$; l is the number of independent loops of G; the product extends to the *internal lines* i of G; k_i denotes the internal four-vector associated with the line i; $d\mu$ is the usual Euclidean measure in the Euclidean region of the internal variables $(\mathbb{R}^3 \times L_0)^l$. In view of (17), this integral is absolutely convergent. As a straightforward consequence of Theorem 2, we shall prove:

Proposition 21. *The integral* (18) *defines an analytic function* F^G *whose analytic continuation satisfies all the requirements of the linear program (i.e. is a general n-point function).*

Proof. With each vertex v of G, we associate the function

$$H^{(n_v)}(\{k_j, j \in \mathscr{S}_v\}) = \prod_{j \in \mathscr{S}_v} w(k_j)$$

(the product extends to all the lines of G incident to the vertex v). It is clear that each function $H^{(n_v)}$ is a general n_v-point function: indeed it is analytic in a domain which is much larger than the n_v-point primitive domain, namely the whole space $\mathbb{C}^{4(n_v-1)}$ minus all the cuts $\{k_j^2 = m_j^2, k_j^2 = M_j^2 + \varrho\}$. With this definition of the vertex functions all the factors w^{-1} cancel out in (12) which is reduced to (18). Then we apply Theorem 2.

In particular this result shows that in the Efimov "non local" field theory (which assumes Euclidean convergence) the amplitudes associated with Feynmann graphs satisfy all the requirements of the linear program. In contrast with [29] no regularization is needed here. Arbitrary singular behaviour of the superpropagator $w(k)$ for $k^2 \to +\infty$ are moreover allowed.

8. Final Remark

An alternative method has been recently proposed by Glaser [17] for a new derivation of the central proof of this paper. This method is an extension of the one already used by Epstein and Glaser in their basic work on renormalization theory [30]. In fact these authors have shown that the linear properties of general quantum field theory, expressed in the present paper in terms of the Steinmann-Ruelle-Araki boundary values of the n-point function in momentum space, can be equivalently formulated in terms of the vacuum expectation values of chronological products submitted to the requirements of "local factorization".

In this context, a general notion of G-convolution can also be defined and a corresponding version of Theorem 2 can be given, in which one is led to prove that the local factorization of the chronological products is preserved by G-convolution. There most of the proof has to be worked out in real position space.

This method presents real advantages of algebraic simplicity, in particular it avoids the technical problems encountered with Steinmann relations. On the other hand the method used in the present paper deals more directly with analyticity properties in momentum space. It shows that most of the facts involved in G-convolution are direct consequences of the geometry of the n-point primitive domain and that in order to define the G-product integrability conditions have only to be postulated in the Euclidean region, thus allowing to include the case of arbitrary singular behaviour on the real.

Appendix

This appendix is devoted to the proof of Proposition 13.

Among all the planes $s_I = s_J$ introduced by Definition 6 and which divide $\mathbb{R}^{n-1}$ into the family of pseudocells $\{\sigma_\mu, \mu \in \Theta^{(n)}\}$, the only ones which do not coincide with a face $s_K = 0$ of the original cell configuration are those corresponding to two non empty *disjoint* subsets.

Thus let us consider a given cell γ_λ and, in the family $\{\sigma_\mu, \mu \in M_\lambda\}$ of the pseudocells contained in it, two adjacent cones σ_{μ_+} and σ_{μ_-} separated by such a plane $s_I = s_J$ with $I \cap J = \emptyset \cdot \sigma_{\mu_+}$ is chosen to lie on the side $s_I > s_J$. Now Proposition 11 ensures that it is possible to define a function $H^G_{\hat{e},\mu_+}$ (resp. $H^G_{\hat{e},\mu_-}$) which is analytic inside a neighbourhood of the flat tube S_{μ_+} (resp. S_{μ_-}) and thus defined:

$$\forall\, k \in S_{\mu_\pm} \qquad H^G_{\hat{e},\mu_\pm}(k) = \int_{\mathbb{R}^3 \times \mathscr{L}_k} H^{\tilde{G}}_\pi(k,t)\,[H^{(2)}(t)]^{-1}\,dt\,.$$

i) In a first step, let us prove that these two functions are pieces of a unique function, analytic in a neighbourhood of the convex hull of $S_{\mu_+} \cup S_{\mu_-}$.

For this purpose let us consider a point k_+ (resp. k_-) inside S_{μ_+} (resp. S_{μ_-}). We choose k_+ and k_- symmetrical with respect to the common face $q^0_I = q^0_J$ and denote k their common *fixed* projection onto this face.

In view of Proposition 12, we can define the discontinuity:

$$[H^G_{\hat{e},\mu_+} - H^G_{\hat{e},\mu_-}]\,(k) = \lim_{q^0_I - q^0_J \to 0} [H^G_{\hat{e},\mu_+}(k_+) - H^G_{\hat{e},\mu_-}(k_-)]$$

in the sense of distributions in the variable $(p_I - p_J)$.

Now we shall prove that this discontinuity is equal to zero at any point k inside the common face. Since $q^0_I = q^0_J$ the domain of analyticity of $H^{\tilde{G}}_\pi$ in the t^0-plane becomes disconnected. Actually the following couples of cuts are confused: for any $L \in \mathscr{P}(X \backslash (I \cup J))$, $(\hat{\Sigma}_{I \cup L}, \hat{\Sigma}_{J \cup L})$ with $\mathscr{P}(A)$ the set of subsets of A. The situation is as shown on Fig. 8. It is convenient to describe the two limiting contours $\mathscr{L}_{k_+}$ (resp. $\mathscr{L}_{k_-}$) by giving their projection l_+ (resp. l_-) onto the space of the variables

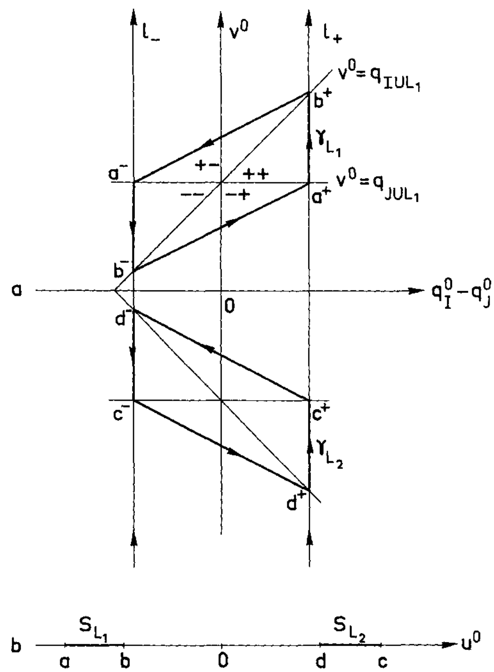


FIG. 8

$(q_I^0 - q_J^0, v^0)$, as in Fig. 8a. Fig. 8b represents their common projection onto the space of the variable u^0. (We recall that $t^0 = u^0 + iv^0$.) The couples (a, a^+), (b, b^+) (resp. $(a, a^-), (b, b^-)$) define points on $\mathscr{L}_{k_+}$ (resp. $\mathscr{L}_{k_-}$). We choose $\mathscr{L}_{k_+}$ (resp. $\mathscr{L}_{k_-}$) as a broken line and it is defined by associating with every point of l_+ (resp. l_-) inside the interval $a^+ b^+$ (resp. $a^- b^-$) the corresponding barycenter on ab.

Now $H_\pi^{\tilde{G}}$ is analytic on the submanifold $q_I^0 = q_J^0$ and using the Stoke's theorem, it is not difficult to get:

$$H^G_{\hat{e},\mu_+}(k_+) - H^G_{\hat{e},\mu_-}(k_-) = \sum_{L \in \mathscr{P}(X \setminus (I \cup J))} \int_{\mathbb{R}^3 \times \gamma_L} H_\pi^{\tilde{G}}(k, t)\,[H^{(2)}(t)]^{-1}\,dt + \varepsilon(q_I^0 - q_J^0)\,.$$

Here the vanishing contribution $\varepsilon(q_I^0 - q_J^0)$ comes from the difference of the infinite parts of the cycles $\mathscr{L}_{k_+}$ and $\mathscr{L}_{k_-} \cdot \gamma_L$ is the cycle with projection onto the space $(q_I^0 - q_J^0, v^0)$ as shown on Fig. 8a: it is limited by the straight lines $v^0 = q^0_{I \cup L}$ and $v^0 = q^0_{J \cup L}$ and corresponds to the pinching of the contours between the cuts $\hat{\Sigma}_{I \cup L}$ and $\hat{\Sigma}_{J \cup L}$. γ_{L_1} and γ_{L_2} are symmetrical with respect to the $(q_I^0 - q_J^0)$ axis if $L_1 \cup L_2 = X \setminus (I \cup J)$.

Now let us concentrate on the edge $v^0 = q^0_{I \cup L} = q^0_{J \cup L}$: it is the trace in the face $q^0_{n+1} + q^0_{n+2} = 0$ of the common edge of a *quartet* of tubes associated with the two following *transverse* partitions:

$$[I \cup L \cup \{n+2\}, (X \backslash (I \cup L)) \cup \{n+1\}]$$

and

$$[J \cup L \cup \{n+2\}, (X \backslash (J \cup L)) \cup \{n+1\}].$$

If we denote $\{H^{\tilde{G}}_{++}, H^{\tilde{G}}_{+-}, H^{\tilde{G}}_{--}, H^{\tilde{G}}_{-+}\}$ the corresponding branches of $H^{\tilde{G}}$, we can write the following quartet relation on $\mathbb{R}^{4(n+1)}$:

$$H^{\tilde{G}}_{++}(\underline{p}) + H^{\tilde{G}}_{--}(\underline{p}) = H^{\tilde{G}}_{+-}(\underline{p}) + H^{\tilde{G}}_{-+}(\underline{p}).$$

Now it is a consequence of the edge of the wedge theorem that such a relation is still valid for the restriction $H^{\tilde{G}}_{\pi}$ taken at any *complex* point (k, t) [27]:

$$H^{\tilde{G}}_{\pi,++}(k,t) + H^{\tilde{G}}_{\pi,--}(k,t) = H^{\tilde{G}}_{\pi,+-}(k,t) + H^{\tilde{G}}_{\pi,-+}(k,t). \qquad (19)$$

When $(q^0_I - q^0_J)$ tends to zero the cycle γ_L shrinks to nothing, so that in the limit we are led to integrate the combination $[H^{\tilde{G}}_{\pi,++} - H^{\tilde{G}}_{\pi,+-} + H^{\tilde{G}}_{\pi,--} - H^{\tilde{G}}_{\pi,-+}]$ on the segment s_L as shown on Fig. 8b. More precisely:

$$[H^G_{\hat{e},\mu+} - H^G_{\hat{e},\mu-}](k) = \sum_{L \in \mathscr{P}(X \backslash (I \cup J))} \int_{\mathbb{R}^3 \times s_L} [H^{\tilde{G}}_{\pi,++} - H^{\tilde{G}}_{\pi,+-} + H^{\tilde{G}}_{\pi,--} - H^{\tilde{G}}_{\pi,-+}](k,u) \cdot [H^{(2)}(u)]^{-1} \, du$$

where the summation is easily checked to be meaningful in the sense of distributions. Then in view of (19), we have proved $H^G_{\hat{e},\mu+}(k) = H^G_{\hat{e},\mu-}(k)$ at any point k inside the common face of $S_{\mu+}$ and $S_{\mu-}$ and the edge of the wedge theorem ensures analyticity inside a neighbourhood of the convex hull of $S_{\mu+} \cup S_{\mu-}$.

ii) In a second step, we remark that any two pseudocells of the family $\{\sigma_\mu, \mu \in M_\lambda\}$ can always be connected by a finite chain of adjacent pseudocells of the same family. In view of i), this entails the existence of a function $H^G_{\hat{e},\lambda}$ analytic at all points lying in a neighbourhood of $\mathscr{T}_\lambda \cap V_{\hat{e}}$, except at those exceptional points which belong to several submanifolds $q^0_I = q^0_J$. But these "edges" belong to the convex hull of the analyticity domain of $H^G_{\hat{e},\lambda}$ and the tube theorem [10] allows to conclude.

This ends the proof of Proposition 13.

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