Analyzing Expected Outcomes and (Positive) Almost-Sure Termination of Probabilistic Programs is Hard

Benjamin Kaminski Joost-Pieter Katoen

Software Modeling and Verification Chair

RWTH AACHEN UNIVERSITY

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How hard is it to solve these analysis problems?

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[Esparza *et al.* 2012]

"[Ordinary] termination is a purely topological property [. . .], but almost–sure termination is not.
[. . .] proving almost–sure termination requires arithmetic reasoning not offered by termination provers."

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- Class Σ_n^0 is defined as

$$\Sigma_n^0 \;=\; \Big\{ A \;\Big|\; A = \{\vec{x} \mid \exists y_1\, \forall y_2\, \exists y_3 \;\cdots\; \exists/\forall y_n : (\vec{x},\, y_1,\, y_2,\, y_3,\, \ldots,\, y_n) \in R\},\; R \text{ is a decidable relation} \Big\}$$

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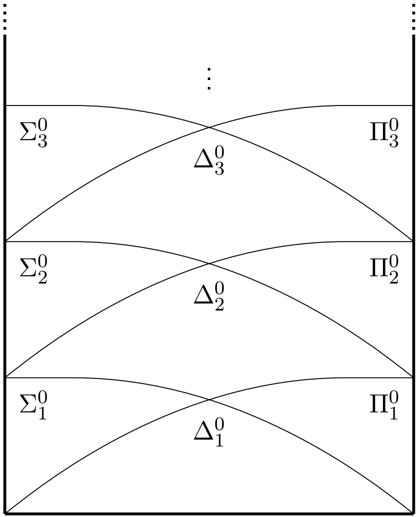
- Class Δ_n^0 is defined as $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

The Arithmetical Hierarchy — The Bigger Picture

The following inclusion diagram holds (all inclusions are strict):



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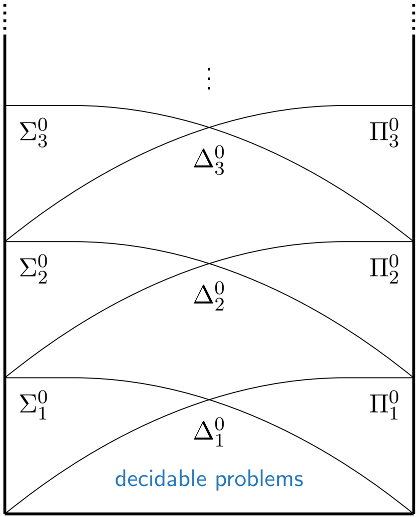

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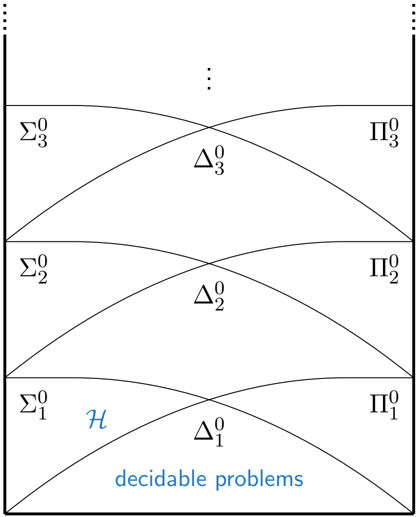

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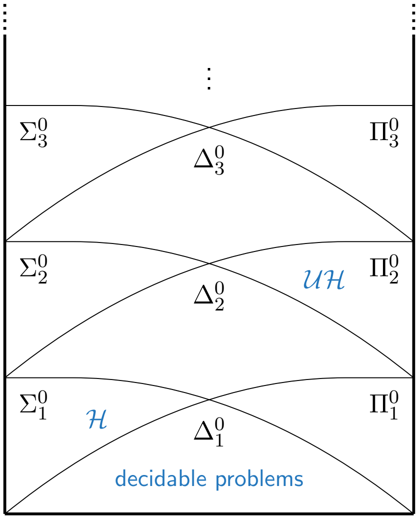

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- The expected number of steps until P terminates on input η:

$$\mathsf{E}_{P,\eta}(\downarrow)$$

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Lower and Upper Bounds, and Exact Expected Outcomes

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\begin{aligned}
(P,\, v,\, q) \in \mathcal{LEXP} \quad &:\Longleftrightarrow \quad q < \mathsf{E}_P(v) \\
(P,\, v,\, q) \in \mathcal{UEXP} \quad &:\Longleftrightarrow \quad q > \mathsf{E}_P(v) \\
(P,\, v,\, q) \in \mathcal{EXP} \quad &:\Longleftrightarrow \quad q = \mathsf{E}_P(v)
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Almost–Sure Termination $\mathcal{AST}$

$$(P,\, \eta) \in \mathcal{AST} \;:\Longleftrightarrow\; \mathsf{Pr}_{P,\eta}(\downarrow) = 1$$

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Positive Almost–Sure Termination $\mathcal{PAST}$

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Universal Versions of $\mathcal{AST}$ and $\mathcal{PAST}$

$$P \in \mathcal{UAST} \;:\Longleftrightarrow \forall \eta\colon (P,\, \eta) \in \mathcal{AST}$$
$$P \in \mathcal{UPAST} \;:\Longleftrightarrow \forall \eta\colon (P,\, \eta) \in \mathcal{PAST}$$

A (very) Simple Example Program

Consider the program P_{geo}:

x := 0;
$\{$continue := 0$\}$ [0.5] $\{$continue := 1$\}$;
while (continue $\neq$ 0)$\{$
    x := x + 1;
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- $\mathsf{E}_{P_{geo}}(x) = 2$
- $\mathsf{E}_{P_{geo}}(\texttt{continue}) = 0$
- P_{geo} terminates almost–surely on all inputs
- Expected runtime of P_{geo} is $\mathcal{O}\left(\mathsf{E}_{P_{geo}}(x)\right)$ on all inputs

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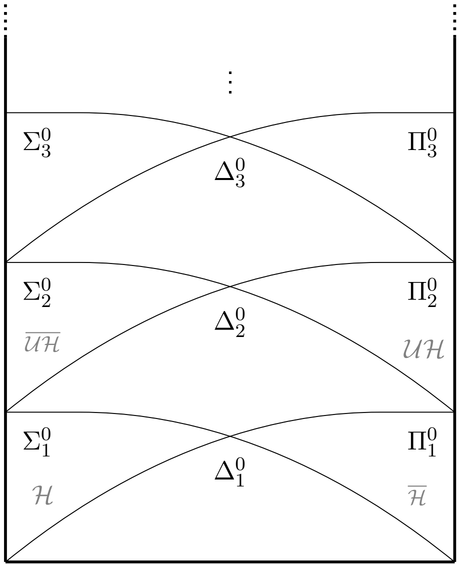

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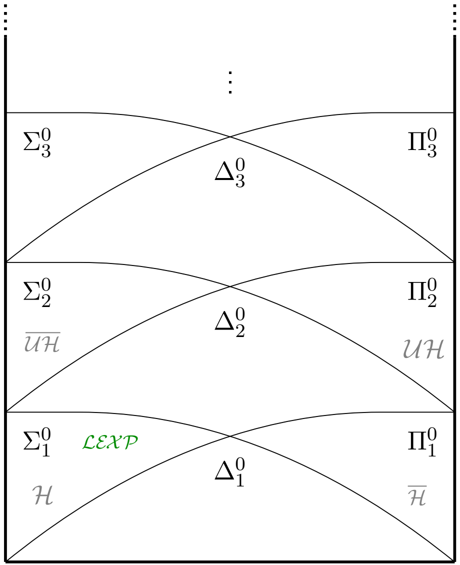

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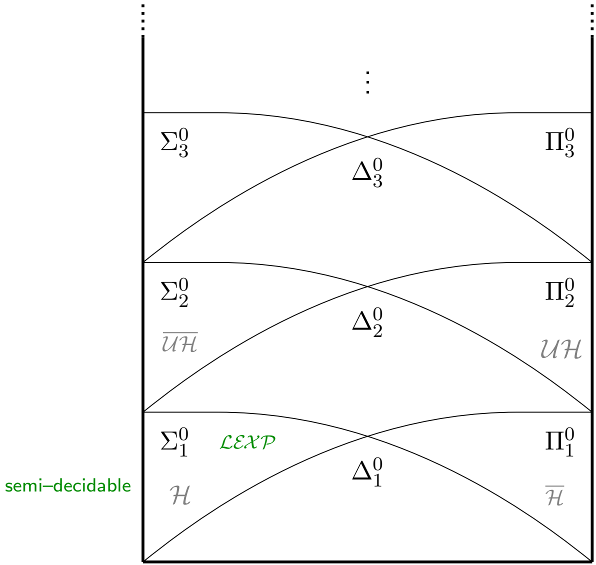

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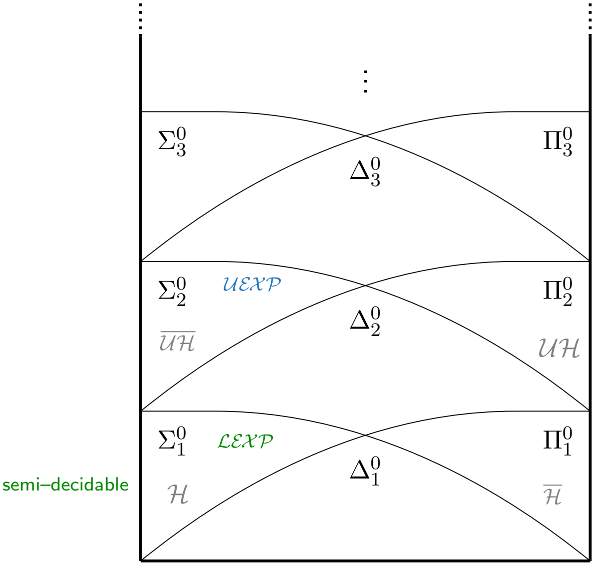

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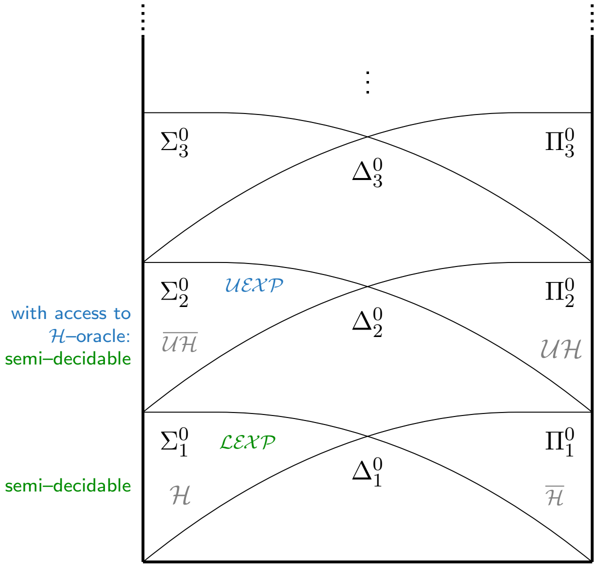

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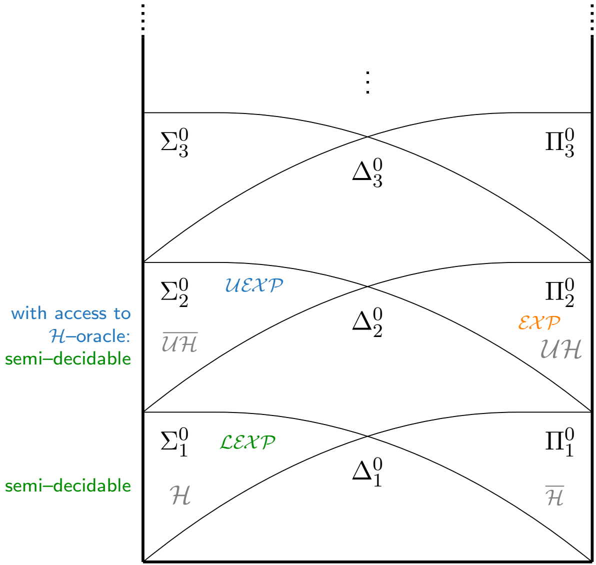

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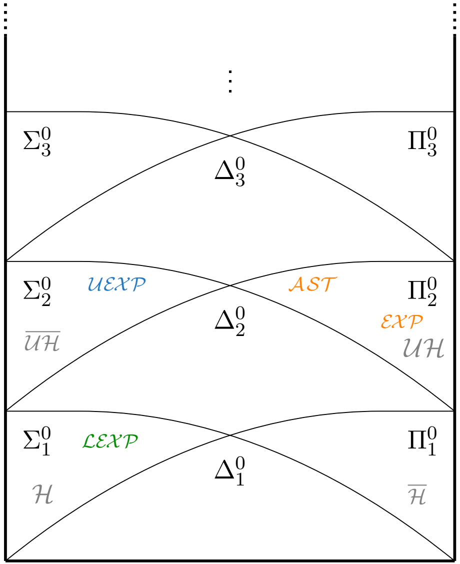

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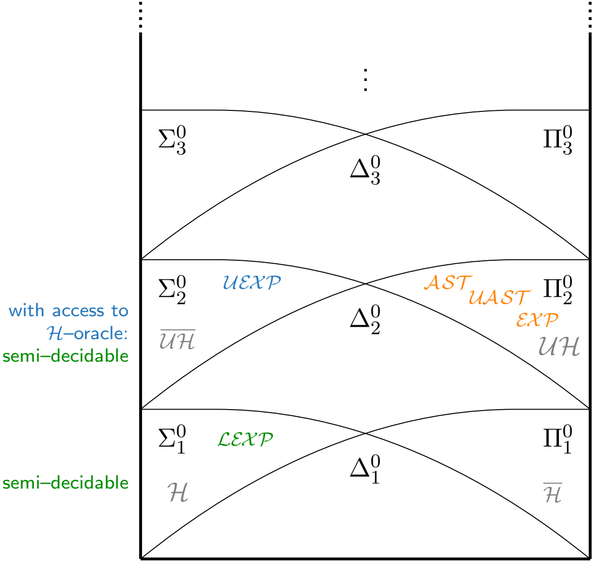

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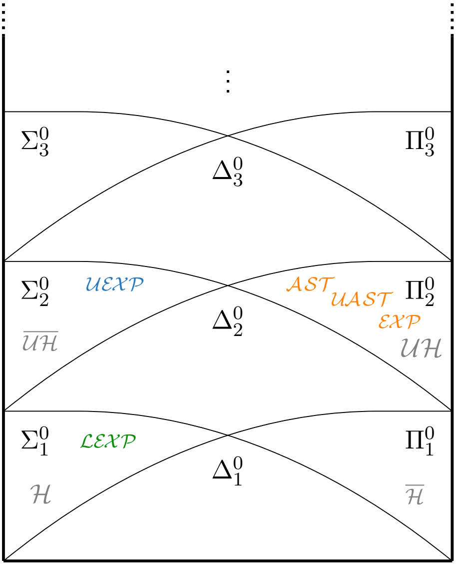

not semi–decidable; even with access to $\mathcal{H}$–oracle

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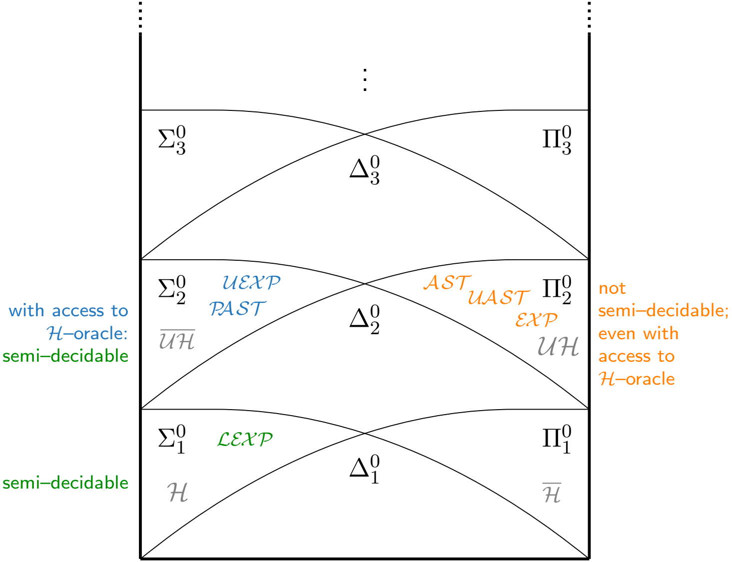

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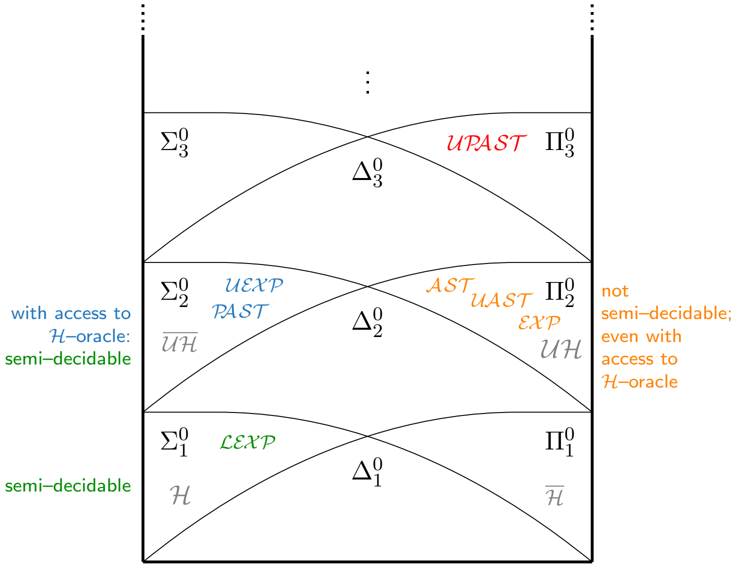

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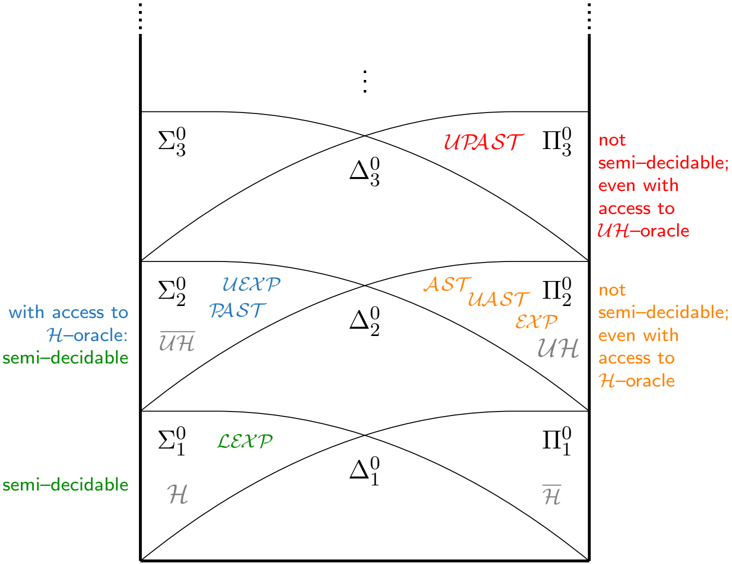

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