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# $\mathfrak{L}^*(K)$ AND OTHER LATTICES OF RECURSIVELY ENUMERABLE SETS[1]

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ABSTRACT. We study the direct product operation on lattices which are principal filters of $\mathfrak{E}^*$, the lattice of r.e. sets modulo finite sets, to generate new isomorphism types of such filters and to characterize the one generated by the complete r.e. set $K$.

A major trend in the long term project of analyzing the lattice $\mathfrak{E}$ of recursively enumerable sets and $\mathfrak{E}^*$ its quotient modulo the finite sets has been the investigation of the class $\mathfrak{F}$ of principal filters of $\mathfrak{E}^*$, i.e. of the lattices $\mathfrak{L}^*(A) = \{B \in \mathfrak{E}^* | B^* \supseteq A\}$ for r.e. $A$. (Note that $A \subseteq^* B$ iff $A \triangle B$ is finite.) Of course the principal ideals of $\mathfrak{E}^*$ are irrelevant since $\{B \in \mathfrak{E}^* | B \subseteq^* A\} \cong \mathfrak{E}^*$ for every $A \neq {}^*\varnothing$. The first such conscious investigations began with Myhill **[1956]** who defined maximal r.e. sets, i.e. sets $M$ such that $\mathfrak{L}^*(M) \cong \{0, 1\}$ (the two element Boolean algebra). Indeed the hyperhypersimple sets of Post **[1944]**, although defined in terms of the intersection of arrays with the sets complement, also turned out to be related to this line of thought. Lachlan **[1968]** showed that they are precisely the r.e. sets $A$ such that $\mathfrak{L}^*(A)$ is a Boolean algebra. He was also able to completely characterize the members of $\mathfrak{F}$ which are Boolean algebras as exactly the $\Sigma_3$ presentable ones.

At the other extreme one finds the $r$-maximal sets. These are easily seen to be equivalent to those with $\mathfrak{L}^*(A)$ having no complemented elements. Classifying the isomorphism types of the $r$-maximal sets however seems to be a difficult open problem. The only other commonly recognized principal filter in $\mathfrak{E}^*$ is the nearly ubiquitous one $-\mathfrak{E}^*$ itself. Of course if $A$ is recursive it is immediate that $\mathfrak{L}^*(A) \cong \mathfrak{E}^*$ but Soare **[1974]**, **[1981]** has shown that this type is extremely common: If $A$ is an r.e. infinite set and $\bar{A}$ is semilow (i.e. $\{e | W_e \cap \bar{A} \neq \varnothing\} < \varnothing'$) then $\mathfrak{L}^*(A) \cong \mathfrak{E}^*$. This means that there are r.e. sets $A$ in every r.e. degree with $\mathfrak{L}^*(A) \cong \mathfrak{E}^*$ and all low r.e. sets $A$ (i.e., $A' <_T \varnothing'$) have this property.

Our goal here is simply to provide some additional examples of types of principal filters in $\mathfrak{E}^*$. We will do this by describing some simple properties of the direct product of lattices in $\mathfrak{F}$. We will then use it to generate new isomorphism types in $\mathfrak{F}$. In addition these properties will enable us to make one really new identification. We will characterize the isomorphism type of $\mathfrak{L}^*(K)$ by an absorption property

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with respect to products in $\mathfrak{L}$. Some connections between the structure of $\mathfrak{L}^*(A)$ and the degree of $A$ will also be pointed out.

Our starting point is a simple fact from lattice theory. We work with distributive lattices with 0 and 1. Basic references are Birkhoff [**1948**] for lattice theory and Rogers [**1967**] for recursion theory. An excellent current survey of r.e. sets and degrees is Soare [**1978**].

LEMMA 1. *$L_1 \otimes L_2 \cong L$ iff there are $x_1$ and $x_2$ in $L$ such that $x_1 \wedge x_2 = 0$, $x_1 \vee x_2 = 1$ and $L_i \cong L(x_i)$ where $L(x_i) = \{y \in L_i | y \geq x_i\}$.*

PROOF. The idea is just that $x_1, x_2$ are the images of $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$ respectively. See Birkhoff [**1948**, p. 26].

Our first observation is that $\mathfrak{F}$ is closed under products. Consider $\mathfrak{L}^*(A_1)$ and $\mathfrak{L}^*(A_2)$. Let $R$ be an infinite coinfinite recursive set with complement $\bar{R}$. We map $f_1$: $\mathbf{N} \to R$, $f_2$: $\mathbf{N} \to \bar{R}$ by one-one onto recursive maps. It is then immediate that $\mathfrak{L}^*(\bar{R} \cup f_1[A_1]) \cong \mathfrak{L}^*(A_1)$ and $\mathfrak{L}^*(R \cup f_2[A_2]) \cong \mathfrak{L}^*(A_2)$. Thus

$$\mathfrak{L}^*(A_1) \otimes \mathfrak{L}^*(A_2) \cong \mathfrak{L}^*\big(\bar{R} \cup f_1[A_1]\big) \otimes \mathfrak{L}^*(R \cup f_2[A_2])$$

but by the lemma this is just $\mathfrak{L}^*(f_1[A_1] \cup f_2[A_2])$. Note that if $A_1$ and $A_2$ are simple so is $f_1[A_1] \cup f_2[A_2]$. Thus the class of principal filters generated by simple sets is also closed under direct product. Of course $\mathfrak{L}^*(\mathbf{N}) = \mathbf{1}$, the trivial one-element lattice, is an identity for products in $\mathfrak{F}$.

We next consider $\mathfrak{E}^*$ and see that it is an indecomposable idempotent.

COROLLARY 2. $\mathfrak{E}^* \otimes \mathfrak{E}^* \cong \mathfrak{E}^*$.

PROOF. Let $x_1$ and $x_2$ be given by any infinite coinfinite recursive set and its complement.

COROLLARY 3. *If $L_1 \otimes L_2 \cong \mathfrak{E}^*$ then $L_i = \mathbf{1}$ or $\mathfrak{E}^*$.*

PROOF. By the lemma the $L_i$ are isomorphic to $\mathfrak{L}^*(A_i)$ for $A_1 \cap A_2 =^* \varnothing$ and $A_1 \cup A_2 =^* \mathbf{N}$. Thus the $A_i$ are recursive and $\mathfrak{L}^*(A_i) \cong \mathfrak{E}^*$ or $\mathbf{1}$ (if one is $\mathbf{N}$).

Thus products of $\mathfrak{E}^*$ give no new isomorphism types in $\mathfrak{F}$ and of course products of Boolean algebras are still Boolean algebras. We can however combine these two known types to generate new ones. We use a more general version of Lemma 1 to prove that one gets new types in this way.

THEOREM 4. *If $\mathfrak{E}^* \otimes L_1 \cong \mathfrak{E}^* \otimes L_2$ then $L_1 \cong L_2$ or $L_1 \cong \mathfrak{E}^* \otimes L_2$ or $L_2 = \mathfrak{E}^* \otimes L_1$.*

PROOF. By Theorem 7 on p. 26 of Birkhoff [**1948**] there are lattices $Z_1^1, Z_1^2, Z_2^1, Z_2^2$ such that $Z_1^1 \otimes Z_1^2 \cong \mathfrak{E}^*$, $Z_1^1 \otimes Z_2^1 \cong \mathfrak{E}^*$, $Z_2^1 \otimes Z_2^2 \cong L_1$ and $Z_1^2 \otimes Z_2^2 \cong L_2$. By Corollary 3, $Z_1^1$, $Z_1^2$ and $Z_2^1$ are $\mathbf{1}$ or $\mathfrak{E}^*$. If $Z_1^1 \cong \mathbf{1}$ then $Z_1^2 = \mathfrak{E}^* = Z_2^1$ and so $L_1 \cong \mathfrak{E}^* \otimes Z_2^2 \cong L_2$ as required. Suppose now that $Z_1^1 \cong \mathfrak{E}^*$. If $Z_2^1 \cong Z_1^2$ ($\cong \mathbf{1}$ or $\mathfrak{E}^*$) then again $L_1 \cong L_2$ ($\cong Z_2^2$ or $\mathfrak{E}^* \otimes Z_2^2$ respectively). The remaining cases are ($Z_2^1 \cong \mathbf{1} \& Z_1^2 \cong \mathfrak{E}^*$) and ($Z_2^1 \cong \mathfrak{E}^* \& Z_1^2 \cong \mathbf{1}$). In the first case $L_1 \cong Z_2^2$ and $L_2 \cong \mathfrak{E}^* \otimes Z_2^2$ as required. The second of course gives $L_2 \cong Z_2^2$ and $L_1 \cong \mathfrak{E}^* \otimes Z_2^2$. □

COROLLARY 5. *If* $\mathfrak{B}_1 \not\cong \mathfrak{B}_2$ *are Boolean algebras in* $\mathfrak{F}$ *then* $\mathfrak{E}^* \otimes \mathfrak{B}_1$ *and* $\mathfrak{E}^* \otimes \mathfrak{B}_2$ *are nonisomorphic elements of* $\mathfrak{F}$. *Neither is a Boolean algebra or* $\mathfrak{E}^*$ *but both are isomorphic to principal filters generated by simple sets.*

PROOF. As $\mathfrak{E}^*$ is not a Boolean algebra $\mathfrak{B}_i \not\cong \mathfrak{E}^* \otimes \mathfrak{B}_i$ and we apply the theorem. As there is a simple set $A$ with $\mathfrak{L}^*(A) \cong \mathfrak{E}^*$ the $\mathfrak{E}^* \otimes \mathfrak{B}_i$ are isomorphic to principal filters generated by simple sets by the closure of this class under direct product.

We next want to point out a simple relation between products of elements of $\mathfrak{F}$ and the degrees of the r.e. sets to which they correspond.

LEMMA 6. *If* $\mathfrak{L}^*(A) \cong \mathfrak{L}^*(A_1) \otimes \mathfrak{L}^*(A_2)$ *then there are* $B_i$ *with* $\mathfrak{L}^*(B_i) \cong \mathfrak{L}^*(A_i)$ *and* $B_1 \oplus B_2 \equiv_T A$.

PROOF. The elements of $\mathfrak{L}^*(A)$ guaranteed by our basic lemma are now r.e. sets $B_1$ and $B_2$ with $B_1 \cap B_2 = A$, $B_1 \cup B_2 = \mathbf{N}$ and $\mathfrak{L}^*(B_i) \cong \mathfrak{L}^*(A_i)$. To see if $x \in B_i$ ask if $x \in A$. If so, $x \in B_i$. If not, enumerate both $B_1$ and $B_2$ until $x$ appears in one of them. If it first appears in $B_i$, $x \in B_i$ and otherwise $x \notin B_i$. Of course $x \in A$ iff $x \in B_1$ and $x \in B_2$.

Thus restrictions on $\mathfrak{L}^*(A)$ that push the degree of $A$ upward are passed on by products. So we have for example

COROLLARY 7. *If* $\mathfrak{B} \neq \mathbf{1}$ *is a Boolean algebra and* $\mathfrak{L}^*(A) \cong \mathfrak{E}^* \otimes \mathfrak{B}$ *then* $A$ *is high i.e.* $\varnothing'' \equiv_T A'$.

PROOF. By Lachlan **[1968]** any $B$ with $\mathfrak{L}^*(B) = \mathfrak{B}$ is hyperhypersimple. Martin **[1966]** then shows that $B$ must be high.

Carrying this idea to an extreme one might guess that the most complicated sets $A$ should have $\mathfrak{L}^*(A)$'s with the most factors. Indeed this gives us our characterization of $\mathfrak{L}^*(K)$. ($K = \{e \mid e \in W_e\}$ is of course a 1-complete r.e. set and so in many ways the most complicated one.)

THEOREM 8. $\mathfrak{L}^*(K) \cong \mathfrak{L}^*(K) \otimes \mathfrak{L}^*(A)$ *for every r.e.* $A$.

PROOF. Let $R_1, f_1$ and $f_2$ be as in the proof that $\mathfrak{F}$ is closed under products following Lemma 1. As before we see that $\mathfrak{L}^*(K) \otimes \mathfrak{L}^*(A) \cong \mathfrak{L}^*(f_1[K] \cup f_2[A])$. As $f_1$ is a recursive one-one and onto map of $\mathbf{N}$ to a recursive set $R, f_1[K]$ is also a complete set as is $f_1[K] \cup f_2[A]$. Thus by Myhill **[1955]** $K$ and $f_1[K] \cup f_2[A]$ are recursively isomorphic and so

$$\mathfrak{L}^*(K) \cong \mathfrak{L}^*(f_1[K] \cup f_2[A]) \cong \mathfrak{L}^*(K) \otimes \mathfrak{L}^*(A)$$

as required. □

This theorem characterizes the isomorphism type of $\mathfrak{L}^*(K)$ for if $\mathfrak{L}^*(B) \cong \mathfrak{L}^*(B) \otimes \mathfrak{L}^*(A)$ for every $A$ then $\mathfrak{L}^*(B) \cong \mathfrak{L}^*(B) \otimes \mathfrak{L}^*(K) \cong \mathfrak{L}^*(K)$. Moreover our earlier results show that the type of $\mathfrak{L}^*(K)$ is not any of the ones considered before, i.e., it is not generated as a product of lattices which are Boolean algebras or $\mathfrak{E}^*$. Finally our results on degrees show that if $\mathfrak{L}^*(K) \cong \mathfrak{L}^*(A)$ then $A$ is high.

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