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Anderson's orthogonality catastrophe

Peter Müller (LMU München)

joint with Martin Gebert, Heinrich Küttler, Peter Otte

arXiv:1302.6124 and in preparation

1 Historic background

Motivation: explanation of anomalies in X-ray absorption in metals

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INFRARED CATASTROPHE IN FERMI GASES WITH LOCAL SCATTERING POTENTIALS

P. W. Anderson
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(Received 27 March 1967)

We prove that the ground state of a system of N fermions is orthogonal to the ground state in the presence of a finite range scattering potential, as $N\to\infty$. This implies that the response to application of such a potential involves only emission of excitations into the continuum, and that certain processes in Fermi gases may be blocked by orthogonality in a low-T, low-energy limit.

- ground-state overlap $\lesssim N^{\text{-const.}}$ in the macroscopic limit
- controverse discussion in the Physics literature in the 1970ies
- still of interest in Physics today
- no mathematical explanation

Model and result

- Schrödinger operators on $L^2(\mathbb{R}^d)$

$$H := -\Delta + V_0 \qquad \text{and} \qquad H' := H + V$$

V_0 Kato decomposable perturbation $0 \le V \in L_c^\infty(\mathbb{R}^d)$

- Finite-volume restrictions to box $\Lambda_L := [-L, L]^d$ with Dirichlet b.c.

$$H_L = \sum_{j \in \mathbb{N}} \lambda_j^L \, |\varphi_j^L\rangle\langle\varphi_j^L| \qquad \text{and} \qquad H_L' = \sum_{j \in \mathbb{N}} \mu_j^L \, |\psi_j^L\rangle\langle\psi_j^L|$$

- Non-interacting system of N spinless fermions on $\bigwedge_{j=1}^N L^2(\Lambda_L)$

$$\mathbf{H}_L^{(\prime)} := \sum_{j=1}^N \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes H_L^{(\prime)} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

- Ground states

$$\Phi_N^L := \frac{1}{\sqrt{N!}} \varphi_1^L \wedge \cdots \wedge \varphi_N^L \quad \text{and} \quad \Psi_N^L := \frac{1}{\sqrt{N!}} \psi_1^L \wedge \cdots \wedge \psi_N^L$$

- Particle number to yield given Fermi energy $E \in \mathbb{R}$ in the mac. limit

$$N \equiv N_L(E) := \#\{j \in \mathbb{N} : \lambda_j^L \le E\}$$

- Ground-state overlap

$$\mathcal{S}_L(E) := \left\langle \Phi^L_{N_L(E)}, \Psi^L_{N_L(E)} \right\rangle = \det \begin{pmatrix} \langle \varphi_1^L, \psi_1^L \rangle & \cdots & \langle \varphi_1^L, \psi^L_{N_L(E)} \rangle \\ \vdots & & \vdots \\ \langle \varphi^L_{N_L(E)}, \psi_1^L \rangle & \cdots & \langle \varphi^L_{N_L(E)}, \psi^L_{N_L(E)} \rangle \end{pmatrix}$$

Theorem A. [Gebert, Küttler, M. – to appear in CMP]

$\forall$ sequence of lengths $(L_n)_{n \in \mathbb{N}}$, $L_n \uparrow \infty$, $\exists$ subsequence $(L_{n_k})_{k \in \mathbb{N}}$ such that for Leb.-a.e. $E \in \mathbb{R}$

$$\limsup_{k \to \infty} \frac{\ln \left| \mathcal{S}_{L_{n_k}}(E) \right|}{\ln L_{n_k}} \le -\frac{\gamma(E)}{2}$$

with

$$\gamma(E) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \operatorname{tr} \left(\sqrt{V} \, 1_{]E-\varepsilon, E]}(H_{ac}) \, V \, 1_{[E, E+\varepsilon[}(H'_{ac}) \sqrt{V} \right).$$

Theorem A. [Gebert, Küttler, M. – to appear in CMP]

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with

$$\gamma(E):=\lim_{\varepsilon\to0}\frac{1}{\varepsilon^2}\,\mathrm{tr}\left(\sqrt{V}\,1_{]E-\varepsilon,E]}(H_{ac})\,V\,1_{[E,E+\varepsilon[}(H'_{ac})\sqrt{V}\right).$$

- $\gamma(E)$ well defined for Leb.-a.e. $E\in\mathbb{R}$
- $|\mathcal{S}_{L_{n_k}}(E)|\leq\exp\left\{-\frac{a\gamma(E)}{2}\ln L_{n_k}+o_a(\ln L_{n_k})\right\}\quad\forall\,0<a<1$
- if $d=1$ or $\ell^2(\mathbb{Z}^d)$ no subsequence necessary
- $\gamma(E)=0\quad\forall\,E\notin spec_{ac}(H)\ \ [=spec_{ac}(H')]$
- connection with Frank, Lewin, Lieb, Seiringer (2011).

Relation to scattering theory

$$\gamma(E) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \operatorname{tr}\Big(\sqrt{V}\, 1_{]E-\varepsilon,E]}(H_{ac})\, V\, 1_{[E,E+\varepsilon[}(H'_{ac})\sqrt{V}\Big).$$

Proposition.

For Leb.-a.e. $E \in spec_{ac}(H)$

$$\gamma(E) = (2\pi)^{-2}\, \|S(E) - \mathbb{1}\|_{HS}^2 = (2\pi)^{-2}\, \|T(E)\|_{HS}^2$$

with fixed-energy scattering matrix $S(E) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$.

Corollary.

Assume $d = 3$, $V_0 = 0$, and V radially symmetric. Then for Leb.-a.e. $E \geq 0$

$$\gamma(E) = \frac{1}{\pi^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \big(\sin\delta_\ell(E)\big)^2$$

with scattering phases $\delta_\ell(E)$, $\ell \in \mathbb{N}_0$.

Coincides with Anderson's decay exponent (only point interaction there)!

3 Sketch of the proof

$$A := \begin{pmatrix} \langle \varphi_1^L, \psi_1^L \rangle & \cdots & \langle \varphi_1^L, \psi_N^L \rangle \\ \vdots & & \vdots \\ \langle \varphi_N^L, \psi_1^L \rangle & \cdots & \langle \varphi_N^L, \psi_N^L \rangle \end{pmatrix}$$

For simplicity: $H = H_{ac}$ and $H' = H'_{ac}$

Lemma 1. Let $E \in \mathbb{R}$ and recall $N \equiv N_L(E)$. Then

$$|\mathcal{S}_L(E)| = |\det A| = \exp\left\{-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\operatorname{tr}\left((P\,\Pi\,P)^k\right)\right\} \le \exp\left\{-\frac{1}{2}\operatorname{tr}(P\,\Pi)\right\}$$

with $P := 1_{]-\infty,\lambda_N^L]}(H_L)$ and $\Pi := 1_{[\mu_{N+1}^L,\infty[}(H'_L)$.

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with $P := 1_{]-\infty, \lambda_N^L]}(H_L)$ and $\Pi := 1_{[\mu_{N+1}^L, \infty[}(H'_L)$.

Definition. Anderson integral

$$\mathcal{I}_L(E) := \operatorname{tr}(P\, \Pi)$$

Lower bound on $\mathcal{I}_L(E)$ needed!

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Lower bound on $\mathcal{I}_L(E)$ needed!

[Küttler, Otte, Spitzer – AHP to appear]

$$\mathcal{I}_L(E) = \gamma(E) \ln L + o(\ln L)$$

($d = 1$, $V_0 = 0$, V more general)

③ Sketch of the proof

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Lemma 2. $\forall$ sequence of lengths $(L_n)_{n\in\mathbb{N}}$, $L_n \uparrow \infty$, $\exists$ subsequence $(L_{n_k})_{k\in\mathbb{N}}$ such that for Leb.-a.e. $E \in \mathbb{R}$

$$\left| \mathcal{I}_{L_{n_k}}(E) - \operatorname{tr}\left(1_{]-\infty,E]}(H_{L_{n_k}})\, 1_{]E,\infty[}(H'_{L_{n_k}}) \right) \right| = o(\ln L_{n_k})$$

Lemma 3. Let $E \in \mathbb{R}$. Then

$$\operatorname{tr}\left(1_{]-\infty,E]}(H_L)\, 1_{]E,\infty[}(H_L')\right) = \int_{]-\infty,E]\times]E,\infty[} \frac{d\mu_L(x,y)}{(y-x)^2}$$

with $\mu_L(B \times B') := \operatorname{tr}\left(\sqrt{V} 1_B(H_L)\, V\, 1_{B'}(H_L')\sqrt{V}\right)$

Lemma 3. Let $E \in \mathbb{R}$. Then $\forall\ a > 0$

$$\operatorname{tr}\Big(1_{]-\infty,E]}(H_L)\,1_{]E,\infty[}(H_L')\Big) \geq \int_{]-\infty,E]\times]E,\infty[} \frac{d\mu_L(x,y)}{(y-x+L^{-a})^2}$$

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Lemma 4. $\forall\ 0 < a < 1$ for Leb.-a.e. $E \in \mathbb{R}$

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with $\mu(B\times B') := \operatorname{tr}\Big(\sqrt{V}1_B(H)\,V\,1_{B'}(H')\sqrt{V}\Big)$

Idea of the proof: regularise, Helffer-Sjöstrand, remove regularisation!

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Lemma 5. $\forall\ a > 0$ for Leb.-a.e. $E \in \mathbb{R}$

$$\int_{]-\infty,E]\times]E,\infty[} \frac{d\mu(x,y)}{(y-x+L^{-a})^2} = a\,\gamma(E) \ln L + o_a(\ln L)$$

4 Towards the exact asymptotics ...

PHYSICAL REVIEW VOLUME 164, NUMBER 2 10 DECEMBER 1967

Ground State of a Magnetic Impurity in a Metal

PHILIP W. ANDERSON
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Theorem B. [Gebert, Küttler, M., Otte – in prep.]

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$$\limsup_{k\to\infty} \frac{\ln|\mathcal{S}_{L_{n_k}}(E)|}{\ln L_{n_k}} \leq -\frac{\tilde{\gamma}(E)}{2}$$

with

$$\tilde{\gamma}(E) := \frac{\| \arcsin|T(E)/2| \|_{HS}^2}{\pi^2}.$$

Compare Theorem A:

$$\gamma(E) := \frac{\| T(E)/2 \|_{HS}^2}{\pi^2}.$$