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RESEARCH MEMORANDUM

ANGULAR ACCURACY OF A PHASED ARRAY RADAR

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SUMMARY

One type of phased-array radar of current interest employs an array of separate receiving elements, each followed by an individual amplifier. These individual signals are combined coherently to form one or more receiving beams for searching, tracking, or performing both functions simultaneously. This memorandum presents an approach to the theory of angle measurement with a phased array of this type.

In the one-dimensional problem considered here the receiving antenna consists of a linear array of individual antenna-amplifier elements. The receiver-noise-limited case is considered, in which accuracy is limited by the additive normally distributed noise present in each channel. An expression is derived for the limiting accuracy of angular measurement when a single set of samples is available. This set of samples is obtained simultaneously, one sample from each channel. Next, two methods of implementing the angular measurements are discussed. These are amplitude comparison monopulse and a coherent or phase comparison technique. For large signal-to-noise ratios and for either a square law or a linear envelope detector, the accuracy of amplitude comparison monopulse approaches the theoretical limit. The same accuracy can be achieved with the coherent technique by proper weighting of the individual signals.

CONTENTS

SYMBOLS

A_1, B_1 amplitudes of the two quadrature components of E_1

A_2, B_2 amplitudes of the two quadrature components of E_2

a_1 $\sum_{k=1}^{N} \cos(\phi + k\delta + k\xi)$

a_2 $\sum_{k=1}^{N} \cos(\phi + k\delta - k\xi)$

b_1 $\sum_{k=1}^{N} \sin(\phi + k\delta + k\xi)$

b_2 $\sum_{k=1}^{N} \sin(\phi + k\delta - k\xi)$

c an arbitrary constant

D_3 difference signal; phase comparison, equal weights case

D_4 difference signal; phase comparison, optimum weights case

d spacing between elements of the array

E_1, E_2 signals in the two squinted beams formed for amplitude comparison monopulse

e_k signal (voltage or current) in the k^{th} channel; includes noise component

$F(\psi)$ a function defined in Eq. 37

k number of the channel (an index)

$L(x_1, y_1, \ldots, x_n, y_n \mid \delta, \phi)$

likelihood function for the sample of x_k, y_k, given δ and ϕ

N number of channels

n_k noise component in k^{th} channel

S	the sum signal (Eq. 47), used to determine sense of the error signal in phase comparison monopulse
$\frac{S}{N}$	signal-to-noise power ratio in one channel
t	time
U_1	$\sum_{k=1}^{N} u_k \cos k\xi$
U_2	$\sum_{k=1}^{N} u_k \sin k\xi$
u_k, v_k	amplitudes of the two quadrature components of n_k
V_1	$\sum_{k=1}^{N} v_k \cos k\xi$
V_2	$\sum_{k=1}^{N} v_k \sin k\xi$
w_k	set of weights used in forming difference signal for phase comparison monopulse
x_k, y_k	amplitudes of the two quadrature components of e_k

γ	reference phase at center of array
Δ_1	difference signal; amplitude comparison monopulse, square law detector
Δ_2	difference signal; amplitude comparison monopulse, linear detector
Δ_3	difference signal; coherent signal processing, constant weights
Δ_4	difference signal; coherent signal processing, optimum weights
δ	incremental phase shift between adjacent elements of array due to target displacement from crossover axis
δ^*	the estimated value of δ

θ angle between axis of array and line of sight to target

λ wavelength

ξ incremental phase shift between adjacent elements of array to provide monopulse squint angles

ρ_1, ρ_2 envelopes of signals in the two channels corresponding to squinted beams for amplitude comparison monopulse

σ rms noise (voltage or current) in the individual channels

$\sigma_{\Delta 1}$ rms error in Δ_1 due to noise

$\sigma_{\Delta 2}$ rms error in Δ_2 due to noise

$\sigma_{\Delta 3}$ rms error in Δ_3 due to noise

$\sigma_{\Delta 4}$ rms error in Δ_4 due to noise

$\sigma_{\delta 1}$ rms error in δ; amplitude comparison monopulse, square law detector

$\sigma_{\delta 2}$ rms error in δ; amplitude comparison monopulse, linear law detector

$\sigma_{\delta 3}$ rms error in δ; coherent signal processing, constant weights

$\sigma_{\delta 4}$ rms error in δ; coherent signal processing, optimum weights

σ_{δ^*} lower bound for the rms error in δ

ϕ reference phase of signal

ψ $N\xi$

ω $2\pi \cdot$ signal frequency

I. INTRODUCTION

There is increasing interest today in phased array radars which use a set of separate antenna elements, each followed by an individual amplifier, in place of a more conventional receiving antenna. These individual signals can be combined coherently to form several receiving beams simultaneously. The theory of angle measurement with this type of array is discussed here. For simplicity, only the one-dimensional problem is considered and the antenna elements are assumed to be equally spaced in a linear array.

In practice, the angular accuracy of this type of radar can be limited by any of several effects, including receiver noise, external noise, component phase or amplitude errors, and atmospheric refraction. The following analysis considers the errors due to receiver noise, i.e., noise originating in the individual antenna-amplifier channels.

II. STATEMENT OF THE PROBLEM

A linear array of equally spaced antenna elements is assumed, each followed by an individual amplifier, as illustrated in Fig. 1. When a plane wave is incident at an angle θ from the axis of the linear array, the individual amplifier outputs (voltage or current) can be expressed in the form:

$$e_k = \cos(\omega t + \phi + k\delta) + n_k \tag{1}$$

where e_k = signal in k^{th} channel

ω = $2\pi \cdot$ signal frequency

ϕ = reference phase of signal

δ = incremental phase shift between channels

n_k = noise in k^{th} channel

The incremental phase shift, δ, is related to the angle of arrival by the equation:

$$\delta = \frac{2\pi d}{\lambda} \cos\theta \tag{2}$$

where λ = wave length

d = spacing between elements

θ = angle of arrival measured from axis of array

For convenience, the following analysis considers errors in the incremental phase shift, δ. These can be converted readily to errors in angle of arrival, θ, using Eq. 2.

It is assumed that the reference phase of the signal, ϕ, is unknown, as is generally the case in radar systems. The effects of equipment errors,

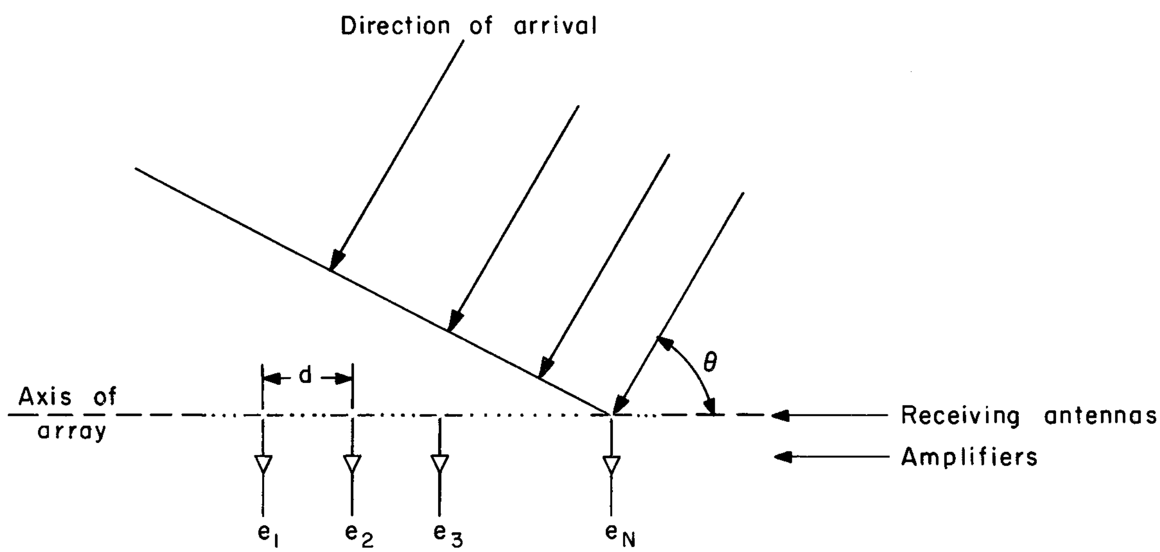

Fig. 1 — Linear phased array geometry

either random or systematic, are not considered here, nor are errors in normalizing the signals in the individual channels.

The noise components in the individual channels are assumed to be independent and normally distributed. It will be convenient to express them in the form:

$$n_k = u_k \cos \omega t + v_k \sin \omega t \tag{3}$$

where: u_k, v_k = quadrature components of noise in the k^{th} channel

and:

$$\overline{u_k} = \overline{v_k} = 0$$

$$\overline{u_k^2} = \overline{v_k^2} = \overline{n_k^2} = \sigma^2$$

The bar above a quantity is used to denote average value. The signal-to-noise power ratio in each individual channel is then:

$$\frac{S}{N} = \frac{1}{2\sigma^2} \tag{5}$$

In terms of the above definitions, the specific problem considered here is the following: Given a set of signals, e_k, obtained by sampling the outputs of the N channels of a linear array simultaneously, how accurately can the incremental phase shift, δ, be estimated? Since a single set of e_k samples is assumed, this corresponds to the single-pulse accuracy of a pulsed radar.

In Section III, a lower bound is obtained for the rms error in estimating δ. Sections IV and V consider two specific methods of estimating δ. These are amplitude comparison monopulse and a coherent phase comparison technique.

III. LOWER BOUND ON ERROR IN δ

The accuracy with which the incremental phase shift, δ, can be estimated from one sample of the set of signals, e_k , is limited by noise. In this section, a theorem of statistics is used to obtain a lower bound on the rms error in the estimate of δ. This theorem applies specifically to all regular unbiased estimates. Denoting the estimated value of δ by δ^*, an unbiased estimate is one for which the expectation of δ^* is equal to δ, i.e.,

(6) $$E(\delta^*) = \delta$$

The conditions for regularity are more subtle and are discussed in Ref. 1.

It is convenient to express the signals, e_k, in the form:

(7) $$e_k = x_k \cos \omega t - y_k \sin \omega t$$

where x_k and y_k are the two quadrature components of one sample from the k^{th} channel. The likelihood function for the sample $\left\{x_k, y_k\right\}$ is then:

$$L\left(x_1, y_1, \ldots, x_N, y_N \middle| \delta, \emptyset\right) = \frac{1}{(2\pi\sigma^2)^N} \prod_1^N \exp\left\{-\frac{\left[x_k - \cos(k\delta+\emptyset)\right]^2 + \left[y_k - \sin(k\delta+\emptyset)\right]^2}{2\sigma^2}\right\}$$

(8)

This expression follows directly from the assumption that the noise components in the individual channels are independent and normally distributed.

Although we are concerned only with estimating δ, there are two unknown parameters in this case, δ and $\emptyset$, so the expression for a joint estimate must be used. From Ref. 1, the mean square error for all regular unbiased estimates of δ has the lower bound:

$$\sigma^2_{\delta^*} \geq \frac{E\left\{\left(\frac{\partial \log L}{\partial \emptyset}\right)^2\right\}}{E\left\{\left(\frac{\partial \log L}{\partial \emptyset}\right)^2\right\} \cdot E\left\{\left(\frac{\partial \log L}{\partial \delta}\right)^2\right\} - \left[E\left\{\frac{\partial \log L}{\partial \emptyset} \cdot \frac{\partial \log L}{\partial \delta}\right\}\right]^2}$$

(9)

From Eq. 8:

$$(10) \qquad \frac{\partial(\log L)}{\partial \phi} = -\frac{1}{\sigma^2}\sum_{1}^{N}\left\{\left[x_k - \cos(k\delta+\phi)\right]\sin(k\delta+\phi) - \left[y_k - \sin(k\delta+\phi)\right]\cos(k\delta+\phi)\right\}$$

and:

$$(11) \qquad \frac{\partial(\log L)}{\partial \delta} = -\frac{1}{\sigma^2}\sum_{1}^{N}k\left\{\left[x_k - \cos(k\delta+\phi)\right]\sin(k\delta+\phi) - \left[y_k - \sin(k\delta+\phi)\right]\cos(k\delta+\phi)\right\}$$

Since the noise components in the individual channels are independent of each other, as are the two quadrature components in each channel, we have:

$$(12) \qquad E\left\{\left[x_k - \cos(k\delta+\phi)\right]\left[y_\ell - \sin(\ell\delta+\phi)\right]\right\} = 0 \qquad \text{all } k, \ell$$

and:

$$(13) \qquad E\left\{\left[x_k - \cos(k\delta+\phi)\right]\left[x_\ell - \cos(\ell\delta+\phi)\right]\right\} = E\left\{\left[y_k - \sin(k\delta+\phi)\right]\left[y_\ell - \sin(\ell\delta+\phi)\right]\right\} = \begin{cases} 0 & k \neq \ell \\ \sigma^2 & k = \ell \end{cases}$$

Combining Eqs. 9 through 13 gives the following expression for the lower bound on mean square error in δ:

$$(14) \qquad \sigma^2_{\delta^*} \geq \frac{12\sigma^2}{N^3 - N}$$

IV. AMPLITUDE COMPARISON MONOPULSE

In amplitude comparison monopulse, two beams are formed, as illustrated in Fig. 2. Each beam is formed by summing all N outputs with the appropriate relative phases. The two resulting signals are envelope-detected and the difference of the two envelopes is calibrated in terms of angle from crossover, θ, as shown in Fig. 2. The approximate incremental phase shift (or angle of incidence) must be known a priori to permit formation of two beams which are pointed roughly in the direction of incidence. A square law envelope detector is assumed in Subsections IV-A through IV-C, and the linear detector case is discussed in Subsection IV-D. In both cases a large signal-to-noise ratio is assumed so that noise cross-modulation terms can be neglected.

In terms of the individual channel voltages, e_k, the sum voltages for the two beams are:

$$E_1 = \sum_1^N e_k \underline{/k\xi}$$

$$(15) \qquad = \sum_1^N \left[\cos(\omega t+\phi+k\delta+k\xi) + u_k \cos(\omega t+k\xi) + v_k \sin(\omega t+k\xi)\right]$$

$$E_2 = \sum_1^N e_k \underline{/-k\xi}$$

$$= \sum_1^N \left[\cos(\omega t+\phi+k\delta-k\xi) + u_k \cos(\omega t-k\xi) + v_k \sin(\omega t-k\xi)\right]$$

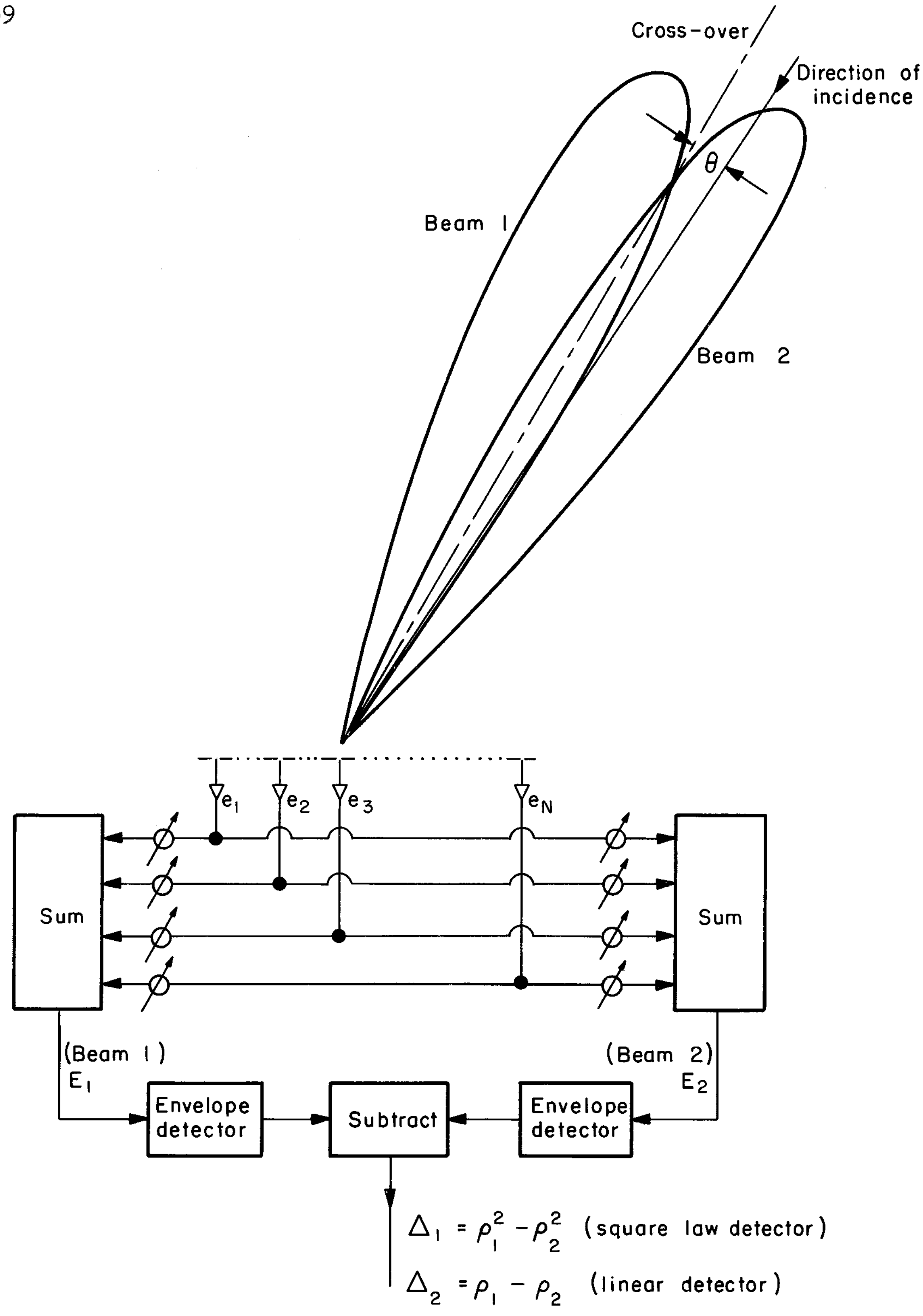

Fig. 2 —— Amplitude comparison monopulse

where: δ = incremental phase shift due to displacement of cross-over from incidence angle (θ in Fig. 2)

ξ = incremental phase shift for beam squinting

Since the noise voltages in the two channels are correlated and will partially cancel in the difference signal Δ, it is important to retain information on the relative noise phases.

The sum signals can also be expressed in the following form:

(16) $$E_1 = A_1 \cos \omega t + B_1 \sin \omega t$$

$$E_2 = A_2 \cos \omega t + B_2 \sin \omega t$$

where:

$$A_1 = a_1 + U_1 + V_2$$

$$B_1 = -b_1 - U_2 + V_1$$

(17) $$A_2 = a_2 + U_1 - V_2$$

$$B_2 = -b_2 + U_2 + V_1$$

and:

$$a_1 = \sum_1^N \cos (\phi + k\delta + k\xi)$$

$$a_2 = \sum_1^N \cos (\phi + k\delta - k\xi)$$

$$b_1 = \sum_1^N \sin (\phi + k\delta + k\xi)$$

$$b_2 = \sum_1^N \sin (\phi + k\delta - k\xi)$$

(18)
$$U_1 = \sum_1^N u_k \cos k\xi$$

$$U_2 = \sum_1^N u_k \sin k\xi$$

$$V_1 = \sum_1^N v_k \cos k\xi$$

$$V_2 = \sum_1^N v_k \sin k\xi$$

In terms of these quantities, the outputs of the two square-law envelope detectors are:

(19)
$$\rho_1^2 = A_1^2 + B_1^2$$

$$\rho_2^2 = A_2^2 + B_2^2$$

Finally, the error signal, which is denoted by Δ_1 in this case, is:

(20)
$$\Delta_1 = \rho_1^2 - \rho_2^2$$

To obtain the error in the measurement of incremental phase shift δ, two quantities must be calculated: the rms value of Δ_1 due to noise, σ_{Δ_1}; and the error slope or derivative of Δ_1 with respect to δ. The rms error in estimating δ is then:

(21)
$$\sigma_{\delta 1} = \frac{\sigma_{\Delta_1}}{\left(\frac{\partial \Delta_1}{\partial \delta}\right)}$$

A. THE ERROR SLOPE -- SQUARE LAW DETECTOR

First, we will obtain an expression for the error slope in the absence of noise. In this case, the difference signal is:

$$(22) \qquad \Delta_1 = \rho_1^2 - \rho_2^2 = \left(a_1^2 + b_1^2\right) - \left(a_2^2 + b_2^2\right)$$

Substituting the sums of Eq. 18 for a_1, a_2, b_1, and b_2 into this expression gives:

$$(23) \qquad \Delta_1 = \frac{\sin^2 \frac{N}{2} (\delta+\xi)}{\sin^2\left(\frac{\delta+\xi}{2}\right)} - \frac{\sin^2 \frac{N}{2} (\delta-\xi)}{\sin^2\left(\frac{\delta-\xi}{2}\right)}$$

The following identities were used in summing the series of Eq. 18:

$$(24) \qquad \sum_1^N \sin k\alpha = \frac{\cos \frac{\alpha}{2} - \cos (N+\frac{1}{2})\alpha}{2 \sin \frac{\alpha}{2}}$$

$$\sum_1^N \cos k\alpha = \frac{\sin (N+\frac{1}{2})\alpha - \sin \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2}}$$

Differentiating the expression for error signal, Δ_1, of Eq. 23 gives the following value for error slope at cross-over:

$$(25) \qquad \left.\frac{\partial \Delta_1}{\partial \delta}\right|_{\delta=o} = \frac{N \sin (N\xi) \sin^2 (\frac{\xi}{2}) - \sin (\xi) \sin^2 (\frac{N\xi}{2})}{\sin^4 (\frac{\xi}{2})}$$

B. DERIVATION OF σ_{Δ_1} -- SQUARE LAW DETECTOR

Next we obtain an equation for the mean square error in Δ_1 due to noise, i.e.:

$$(26) \qquad \sigma_{\Delta_1}^{\;2} = \overline{(\Delta_1 - \bar{\Delta}_1)^2}$$

From Eqs. 17, 19, and 20:

$$(27) \qquad \Delta_1 - \bar{\Delta}_1 = 2\left[U_1(a_1-a_2) + V_2(a_1+ a_2) + U_2(b_1+ b_2) + V_1(b_2-b_1)\right] + 4(U_1V_2- U_2V_1)$$

For the large signal-to-noise ratio case considered here, the second term of Eq. 27 which consists of noise cross-modulation products, can be neglected. The equation for $\sigma_{\Delta_1}^{\;2}$ is then:

$$(28) \qquad \sigma_{\Delta_1}^{\;2} = 4\Big[(a_1-a_2)^2\overline{U_1^{\;2}} + (a_1+a_2)^2\overline{V_2^{\;2}} + (b_1+b_2)^2\overline{U_2^{\;2}} + (b_2-b_1)^2\overline{V_1^{\;2}}$$
$$+ 2(a_1-a_2)(b_1+b_2)\overline{U_1U_2} + 2(a_1+a_2)(b_2-b_1)\overline{V_1V_2}\Big]$$

Terms of the form $\overline{U_i\ V_j}$ are zero since the two quadrature components of noise in the individual channels are independent.

Next, consider the individual quantities occurring in Eq. 28. The quantity $\overline{U_1^{\;2}}$ can be expressed as follows:

$$\overline{U_1^{\;2}} = \overline{\left(\sum_1^N u_k \cos k\xi\right)\left(\sum_1^N u_\ell \cos \ell\,\xi\right)}$$

$$(29) \qquad = \overline{\sum_1^N u_k^2 \cos^2 k\xi}$$

$$= \sigma^2 \sum_1^N \cos^2 k\xi = \frac{\sigma^2}{4}\left[(2N-1)+ \frac{\sin\,(2N+1)\ \xi}{\sin\ \xi}\right]$$

Since the noise components in the different channels are independent, terms of the form $\overline{u_k u_\ell}$ in Eq. 29 were replaced with:

$$(30) \qquad \begin{array}{lll} \overline{u_k u_\ell} = 0 & & k \neq \ell \\ \overline{u_k^2} = \sigma^2 & & \text{all } k \end{array}$$

The corresponding expressions for the other terms in Eq. 28 involving U_i or V_i are:

$$(31) \qquad \overline{U_2^{\ 2}} = \overline{V_2^{\ 2}} = \frac{\sigma^2}{4}\left[(2N+1) - \frac{\sin (2N+1)\ \xi}{\sin \xi}\right]$$

$$(32) \qquad \overline{V_1^{\ 2}} = \overline{U_1^{\ 2}}$$

and:

$$(33) \qquad \overline{U_1U_2} = \overline{V_1V_2} = \frac{\sigma^2}{4}\left[\frac{\cos \xi - \cos (2N+1)\ \xi}{\sin \xi}\right]$$

Terms of the form $(a_1 \pm a_2)$ or $(b_1 \pm b_2)$ in Eq. 28 are functions of δ. For simplicity, only the $\delta = 0$ case was considered here, which would be closely approximated in many cases with closed-loop tracking systems. Substituting the expressions of Eq. 18 for a_1, b_1, a_2, and b_2 into Eq. 28 gives:

$$(34) \qquad \sigma_{\Delta_1}^{\ 2} = 16\left[\overline{U_1^{\ 2}}\left(\sum_1^N \sin k\xi\right)^2 + \overline{U_2^{\ 2}}\left(\sum_1^N \cos k\xi\right)^2 - 2\overline{U_1U_2}\left(\sum_1^N \sin k\xi\right)\left(\sum_1^N \cos k\xi\right)\right]$$

which after substitution of Eqs. 29, 31, 32, 33 and further simplification becomes:

$$(35) \qquad \sigma_{\Delta_1}^{\ 2} = 2\sigma^2\left[\frac{4N \sin^2 \frac{N\xi}{2} \sin \xi + \sin 2N\xi - 2 \sin N\xi}{\sin \xi \sin^2 \frac{\xi}{2}}\right]$$

C. ERROR IN ESTIMATING δ -- SQUARE LAW DETECTOR

Expressions have now been obtained for the error slope (Eq. 25) and the mean square error in Δ_1 due to noise (Eq. 35). These can be combined using Eq. 21 to obtain the rms error in measuring incremental phase shift, δ. The incremental phase shift ξ, which determines the monopulse squint angle has been retained as a parameter, and it is interesting to see how the error in δ varies with this quantity. Combining Eqs. 21, 25, and 35, the mean square error in δ can be expressed as follows when N is large:

$$(36) \qquad \sigma_{\delta 1}^{2} = \frac{\sigma_{\Delta_1}^{2}}{\left(\frac{\partial \Delta_1}{\partial \delta}\right)^2_{\delta=0}} \doteq \frac{\sigma^2}{2N^3} F(\psi)$$

where: $\psi = N\xi$

and:

$$(37) \qquad F(\psi) = \psi^3 \frac{(4\psi \sin^2 \frac{\psi}{2} + \sin 2\psi - 2 \sin \psi)}{(\psi \sin \psi - 4 \sin^2 \frac{\psi}{2})^2}$$

The function $F(\psi)$ is plotted in Fig. 3. For $\psi = \pi$, the squint angle is 1/2 beamwidth, i.e., the centers of the two monopulse beams are separated by one beamwidth.

Note that the function $F(\psi)$ is constant for small ψ, i.e., the accuracy of angular measurement is independent of squint angle in this region. This is explained by noting that as the squint angle decreases, the error slope decreases, while the correlation between the noise components in the two beams increases. The noise therefore cancels more completely in the difference signal. The error slope and noise in the difference signal, Δ, decrease proportionately so that the resulting error in δ remains constant.

For squint angles less than 1/2 beamwidth (i.e., $\psi = \pi$), the value of $F(\psi)$ is 24, and the corresponding rms error in measurement of δ is:

$$(38) \qquad \sigma_{\delta 1} \doteq \frac{\sqrt{12}\sigma}{N^{3/2}}$$

This equation is valid for large N, and in this case the rms error in δ approaches the theoretical limit discussed in Section III. When N is small, the mean square error in δ approaches the limit given in Eq. 14 for small squint angles (i.e., a term of the form $N^3 - N$ occurs in the denominator).

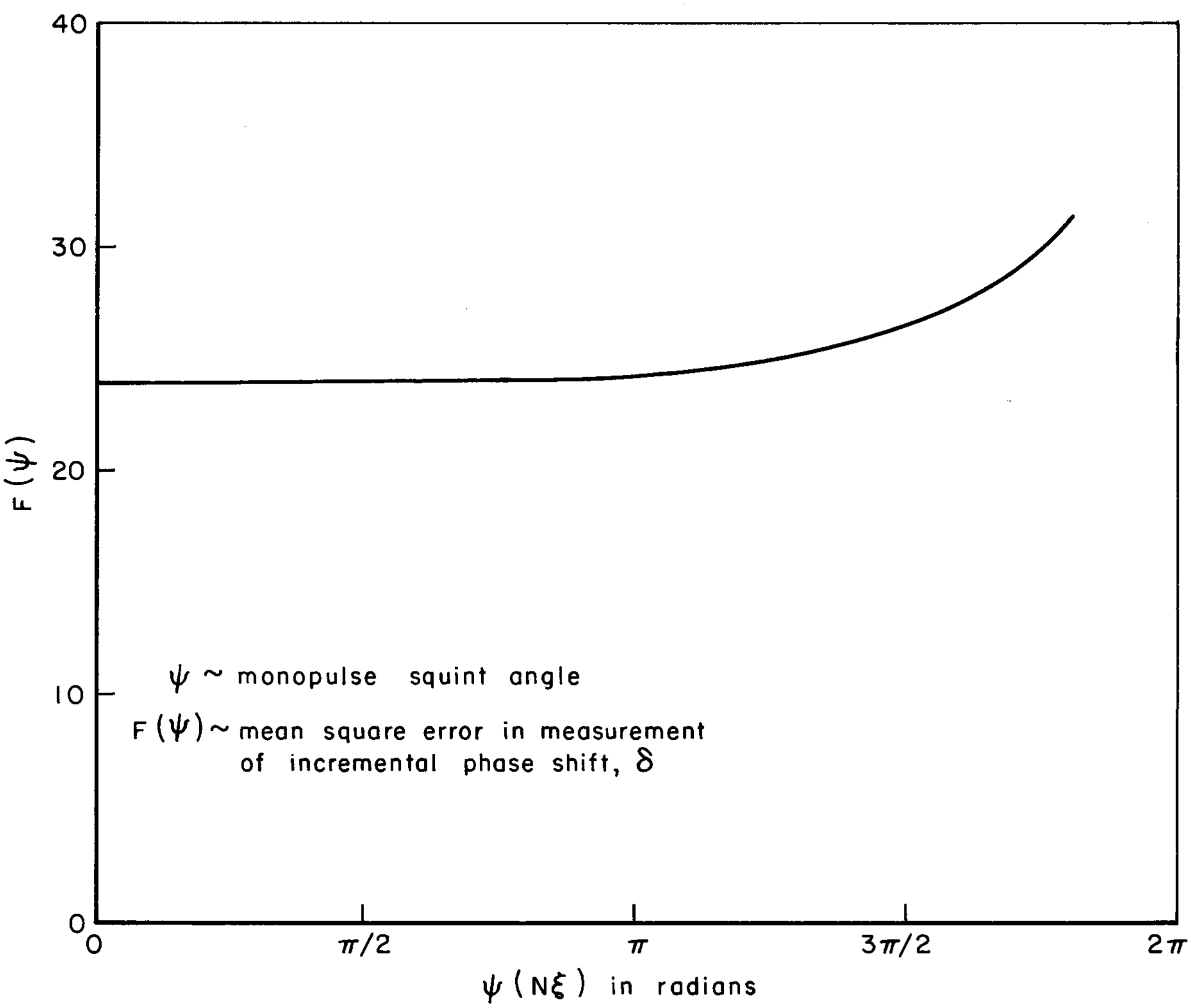

Fig. 3 — The function F(ψ)

D. LINEAR ENVELOPE DETECTOR

The corresponding analysis for the case of a linear envelope detector will be outlined here. Again a large signal-to-noise ratio is assumed and the rms error in the difference signal is computed only for the case of $\delta = 0$. With a linear envelope detector, the difference signal, which is denoted by Δ_2 in this case, is:

$$(39) \qquad \Delta_2 = \rho_1 - \rho_2$$

where ρ_1 and ρ_2 are defined as before (Eqs. 15 through 19).

In the absence of noise, the difference signal is:

$$(40) \qquad \Delta_2 = \frac{\sin \frac{N(\delta+\xi)}{2}}{\sin \frac{\delta+\xi}{2}} - \frac{\sin \frac{N(\delta-\xi)}{2}}{\sin \frac{\delta-\xi}{2}}$$

Differentiating this expression to obtain the error slope gives:

$$(41) \qquad \left.\frac{\partial \Delta_2}{\partial \delta}\right|_{\delta=0} = \frac{N \sin \frac{\xi}{2} \cos \frac{N\xi}{2} - \sin \frac{N\xi}{2} \cos \frac{\xi}{2}}{\sin^2 \frac{\xi}{2}}$$

In addition to the error slope, one must also evaluate the rms error in Δ_2 due to noise. Using Eqs. 39, 15, 19, and noting that for the large signal-to-noise ratio case cross-modulation products of noise can be neglected, gives:

$$(42) \qquad \Delta_2 = \sqrt{a_1^2 + b_1^2 + 2a_1(U_1+V_2) + 2b_1(U_2-V_1)} - \sqrt{a_2^2 + b_2^2 + 2a_2(U_1-V_2) - 2b_2(U_2+V_1)}$$

For the case of $\delta = 0$ and large signal-to-noise ratio this equation reduces to:

$$(43) \qquad \Delta_2 - \overline{\Delta_2} = \frac{U_1(a_1-a_2) + V_1(b_2-b_1) + U_2(b_1+b_2) + V_2(a_1+a_2)}{\sqrt{a_1^{\,2} + b_1^{\,2}}}$$

As before, this expression is squared and averaged to obtain $\sigma_{\Delta_2}^{\;2}$.

Again the rms error in δ, denoted by $\sigma_{\delta 2}$ in this case, is the ratio of σ_{Δ_2} and the error slope. The same end result for rms error in δ is obtained in this case, namely, for the case of large signal-to-noise ratio and a linear detector:

$$(44) \qquad \sigma_{\delta 2}^{\;2} = \frac{\sigma^2}{2N^3} F(\psi)$$

where $F(\psi)$ is defined in Eq. 37.

V. COHERENT METHODS OF ANGLE MEASUREMENT

Two coherent or linear signal-processing techniques are discussed in this section. In both cases, a difference signal proportional to the incremental phase shift, δ, is formed by combining the individual signals, e_k, coherently (i.e., at r.f. or i.f. where phase is preserved). In the first case, the signals from each half of the array are summed individually with equal weights. The difference between these two partial sums is proportional to the incremental phase shift, δ. This is essentially a phase comparison monopulse technique, the difference signal being proportional to the difference in phase of the two partial sums. The second technique, discussed in Subsection V-B, is similar, but weighs the different signals, e_k, unequally.

A. CONSTANT WEIGHTS

First consider the case where the signals are weighted equally and the difference signal has the form:

$$(45) \qquad D_3 = \sum_{N/2+1}^{N} e_k - \sum_{1}^{N} e_k$$

For convenience, we assume the array contains an even number of elements, N. Again, the approximate angle of incidence must be known to steer the beams formed by the two halves of the array. From Eqs. 1 and 45, the signal component of the difference signal is given by:

$$(46) \qquad D_3 = \left(-2 \frac{\sin^2 \frac{N\delta}{4}}{\sin \delta/2}\right) \sin (\omega t+\gamma) = \Delta_3 \sin (\omega t+\gamma)$$

where: $\gamma = \phi + \frac{N+1}{2} \delta$ = reference phase at center of array

The amplitude of the difference signal, Δ_3, is a function of the magnitude of the incremental phase shift, δ, while the sense of the error signal is determined by the r.f. phase of D_3. In this case, a sum signal must also be formed to determine the sense of the error signal:

$$(47) \qquad S = \sum_{1}^{N} e_k = K \cos (\omega t + \gamma)$$

For small pointing errors, i.e., small values of δ, the quantity K is approximately equal to the number of channels, N. Of course, the sum signal S also contains a noise term, so that at very low signal-to-noise ratios, errors in sign of the error signal may occur frequently. Again we will consider only the large signal-to-noise ratio case, and assume that the amplitude of the sin $(\omega t + \gamma)$ component of D_3 is obtained coherently.

In this case, the rms noise in the difference signal, Δ_3, is:

$$(48) \qquad \sigma_{\Delta_3} = \sqrt{N}\, \sigma$$

The error slope in this case is:

$$(49) \qquad \left(\frac{\partial \Delta_3}{\partial \delta}\right) = \frac{\sin^2 \frac{N\delta}{4} \cos \delta/2 - \frac{N}{2} \sin \frac{\delta}{2} \sin \frac{N\delta}{2}}{\sin^2 \delta/2}$$

For small values of δ, this reduces to:

$$(50) \qquad \left(\frac{\partial \Delta_3}{\partial \delta}\right)_{\delta=0} = - \frac{N^2}{4}$$

The rms error in δ is again obtained by dividing the rms error in the difference signal by the error slope, i.e.:

$$(51) \qquad \sigma_{\delta 3} = \frac{\sigma_{\Delta_3}}{\left(\frac{\partial \Delta_3}{\partial \delta}\right)_{\delta=0}} = \frac{4\sigma}{N^{3/2}}$$

The limiting accuracy of Eq. 14 is not achieved in this case when N is greater than 2. For large N, the rms error obtained with the coherent technique described here is 13 per cent greater than the error for amplitude comparison monopulse.

B. VARIABLE WEIGHTS*

In the preceding subsection, it was shown that when the signals are all weighted equally, the limiting accuracy (Eq. 14) is not achieved. From the following heuristic argument, one would expect an improvement in accuracy when a set of unequal weights is used. The error signal obtained by subtracting the two end signals from a multi-element array ($e_N - e_1$) is greater for a given δ than the error signal from the two middle elements of the array. However, the rms noise in these two difference signals is the same. Hence, a set of variable weights which favor the end elements of an array should improve the ratio of error-signal to noise. In the following paragraphs it will be shown that this is indeed the case, and that with an optimum set of weights the limiting accuracy is achieved.

In the general case of variable weights, the difference signal, D_4, is:

$$
\begin{aligned}
D_4 &= \sum_{1}^{N/2} w_k \left[e_{N-k+1} - e_k \right] \\
&= \left(\sum_{1}^{N/2} 2w_k \sin\left(k - \frac{N+1}{2}\right) \delta \right) \sin(\omega t + \gamma) \qquad (52) \\
&= \Delta_4 \sin(\omega t + \gamma)
\end{aligned}
$$

Again, N is assumed to be an even number, and the large signal-to-noise ratio case is considered. As before, the sum signal is used to determine the sense of the error signal and to extract coherently only one quadrature component of signal plus noise.

*The possibility of improving accuracy with a set of variable weights was suggested by D. L. Margerum, of Systems Laboratories Corporation.

The mean square noise in the error signal Δ_4, is then:

$$(53) \qquad \sigma_{\Delta_4}^{\ 2} = 2\sigma^2 \sum_{1}^{N/2} w_k^{\ 2}$$

Again considering the case of small δ, the error slope obtained by differentiating Δ_4 of Eq. 52 is given by:

$$(54) \qquad \frac{\partial \Delta_4}{\partial \delta} = 2 \sum_{1}^{N/2} w_k \left(k - \frac{N+1}{2}\right)$$

Next, one would like to find the set of weights w_k which minimizes the ratio of σ_{Δ_4} to the error slope, i.e., the set for which:

$$(55) \qquad \frac{\partial}{\partial w_k} \left[\frac{\sigma_{\Delta_4}}{\left(\frac{\partial \Delta_4}{\partial \delta}\right)} \right] = 0 \qquad\qquad k = 1, 2, .., N/2$$

This optimum set of weights is proportional to distance from the center of the array, i.e.:

$$(56) \qquad w_k = c\left(k - \frac{N+1}{2}\right)$$

where c is a constant.

From Eqs. 53, 54, and 56, the corresponding mean square error in δ is:

$$(57) \qquad \left(\sigma_{\delta 4}\right)^2 = \frac{12\ \sigma^2}{N^3 - N}$$

This is the same mean square error as was obtained in Section III for the limiting accuracy and again in Section IV for amplitude comparison monopulse.

VI. CONCLUSIONS

A lower bound was obtained for the rms error in angle measurement with a linear phased array for the single sample, receiver-noise-limited case. It was shown that this limiting accuracy is achieved with amplitude comparison monopulse for the more tractable case of large signal-to-noise ratios. It was also shown that with amplitude comparison monopulse, angular accuracy is independent of squint angle over a wide range of squint angles -- from zero to approximately 1/2 beamwidth.

When the individual signals are processed coherently, as discussed in Section V, and weighted equally, the angular error exceeds the theoretical limit by 13 per cent. However, when an optimum set of weights is used, the limiting accuracy is again achieved for large signal-to-noise ratios. It was shown that optimum weighting in this case implies weights proportional to distance from the center of the array.

Some of the results obtained here are also valid at low signal-to-noise ratios. The expression for limiting accuracy derived in Section III is valid at all signal-to-noise ratios. However, the results for amplitude comparison monopulse are invalid at low signal-to-noise ratio, since the noise cross-modulation terms in the envelope detector output were neglected. Noise cross-modulation would predominate at low signal-to-noise ratios, resulting in rapid degradation of angular accuracy. The analysis of Section V for the coherent signal processing case is valid at low signal-to-noise ratios, provided an accurate reference or sum signal is available. However, at low signal-to-noise ratios the reference signal would also contain a large noise component, resulting in frequent errors in the sign of the error signal.

When closed-loop angle-tracking is used, the accuracy would exceed that estimated here for the single sample case. In effect, with closed-loop tracking many samples are integrated to obtain a current estimate of angle. Although the closed-loop case was not considered explicitly, the results obtained here provide an essential input for the analysis of linear arrays employing this technique.

REFERENCE

1. Cramér, Harald, Mathematical Methods of Statistics, Princeton University Press, Princeton, 1946, Chap. 32.