92. *Angular Cluster Sets and Oricyclic Cluster Sets*

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1. Let G be the unit disk $|z|<1$ and Γ be its circumference $|z|=1$. For a point $\zeta\in\Gamma$, let $V=V(\zeta)$ be an angle with vertex at ζ and $K=K(\zeta)$ be an inscribed disk at ζ, that is,

$$K(\zeta)=\{z;\ |z-\rho\zeta|<1-\rho\},$$

where ρ is a constant, $0<\rho<1$.

For a function $f(z)$ given in G, we set

$$C(\zeta,K)=C(\zeta,K,f)$$
$$=\{a;\ \text{there is a sequence } z_\nu\in K(\zeta),\ z_\nu\to\zeta,\ f(z_\nu)\to a\}.$$

$C(\zeta,V)=C(\zeta,V,f)$ is defined similarly.

We put

$$C_{\mathfrak{A}}(\zeta,f)=\bigcup_V C(\zeta,V,f),\quad C_{\mathfrak{O}}(\zeta,f)=\bigcap_K C(\zeta,K,f),$$

where summation and intersection are taken over all $V(\zeta)$ and $K(\zeta)$. $C_{\mathfrak{A}}$ and $C_{\mathfrak{O}}$ are called *angular cluster set* and *oricylic cluster set*, respectively [2].

Obviously $C_{\mathfrak{A}}\subset C_{\mathfrak{O}}$. We will show here that $C_{\mathfrak{A}}(\zeta,f)=C_{\mathfrak{O}}(\zeta,f)$ except on a set of σ-porosity of the order $1/2$ (see the definition below), for any arbitrary function $f(z)$.

If $C_{\mathfrak{F}}(\zeta,f)$ is the fine cluster set at ζ [4], Brelot and Doob [4] proved that $C_{\mathfrak{A}}(\zeta,f)\subset C_{\mathfrak{F}}(\zeta,f)$ for harmonic or holomorphic $f(z)$. Since $K(\zeta)$ is a fine neighborhood of ζ, we have $C_{\mathfrak{A}}\subset C_{\mathfrak{F}}\subset C_{\mathfrak{O}}$. Thus the relation between $C_{\mathfrak{A}}$ and $C_{\mathfrak{O}}$ will suggest some relation between $C_{\mathfrak{A}}$ and $C_{\mathfrak{F}}$.

2. Let us define some notions. A *KK* (or *VV*)*-singular point* is the point $\zeta\in\Gamma$ such that $C(\zeta,K',f)\neq C(\zeta,K'',f)$ (or $C(\zeta,V',f)\neq C(\zeta,V'',f)$) for some pair of inscribed disks $K'(\zeta)$ and $K''(\zeta)$ (or angles $V'(\zeta)$ and $V''(\zeta)$). The set of all *KK* (or *VV*)-singular points is called *KK* (or *VV*)*-singular set* and denoted by $E_{KK}(f)$ (or $E_{VV}(f)$).

A *GK* (or *GV*)-singular point is the point $\zeta\in\Gamma$ such that $C(\zeta,K,f)\neq C(\zeta,f)$ (or $C(\zeta,V,f)\neq C(\zeta,f)$) for some $K(\zeta)$ (or $V(\zeta)$), where $C(\zeta,f)$ is the cluster set at ζ, that is,

$$C(\zeta,f)=\{a;\ \text{there is a sequence } z_\nu\in G,\ z_\nu\to\zeta,\ f(z_\nu)\to a\}.$$

GK (or *GV*)-singular set is denoted by $E_{GK}(f)$ (or $E_{GV}(f)$).

KV-singularity is defined analogously.

For a $\varepsilon>0$, we set $U_\varepsilon(\zeta)=\{z;\ |z-\zeta|<\varepsilon\}$ (ε-neighborhood). Suppose a set $E \subset \Gamma$ and a point $\zeta \in \Gamma$ are given. Let $r(\zeta, \varepsilon)=r(\zeta, \varepsilon, E)$ be the largest of lengths of arcs contained in $U_\varepsilon(\zeta) \cap \Gamma$ and not intersecting with E. The set E is *of porosity of the order* α, $0<\alpha \leqq 1$ (or simply *of porosity* (α)) at ζ, if

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{\varepsilon}(r(\zeta, \varepsilon))^\alpha > 0.$$

E is of porosity (α) on Γ if it is so at each $\zeta \in E$. A set which is a countable sum of sets porosity (α) is said to be *of σ-porosity* (α).

A set of σ-porosity (α) is of the first Baire category. When $\alpha=1$ and E is measurable, it is of measure 0. But when $0<\alpha<1$, it may be of positive measure.

Examples of sets, which are of the first category but not of σ-porosity (α), can be constructed by procedures of the Cantor-type.

A set of (σ-)porosity of the order 1 ($\alpha=1$) is simply said *of (σ-) porosity.*

σ-porosity of the order α can be considered as a precise version of the first Baire category.

Dolzhenko [1] proved the following theorem: *For any arbitrary function $f(z)$, not necessarily one-valued, $E_{VV}(f)$ is of type $G_{\delta\sigma}$ and of σ-porosity. $E_{GV}(f)$ is F_σ and of the first category.*

He also showed that: For any set of σ-porosity there is a bounded holomorphic function $f(z)$ such that $E_{VV}(f) \supset E$. Even for bounded holomorphic $f(z)$, $E_{GV}(f)$ may be of measure 2π.

Now we prove the following theorem by the method of Dolzhenko's paper.

Theorem 1. *For any arbitrary function $f(z)$, $E_{KK}(f)$ is of $G_{\delta\sigma}$ and of σ-porosity.*

Proof. Let $\{\rho_m\}$ be all rational numbers satisfying $0<\rho_m<1$, and $K_m=K_m(\zeta)$ be the inscribed disk $\{z; |z-\rho_m\zeta|<1-\rho_m\}$. Let $\{D_n\}$ be the sequence of all closed disks in the w-plane, having rational radii r_n and having rational points a_n as centers.

$E_{n,m}$ is the set of points $\zeta \in \Gamma$ such that

$$\begin{array}{c}\text{the set } \{w=f(z); z \in K_m(\zeta), \operatorname{dis}(z, \Gamma)<1/m\} \\ \text{lies at a distance} \geqq r_n \text{ from } D_n.\end{array} \tag{1}$$

$F_{n,p,q}$ is the set of points $\zeta \in \Gamma$ such that

$$\begin{array}{c}\text{the set } \{w=f(z); z \in K_p(\zeta), 1/3q<\operatorname{dis}(z, \Gamma)<1/q\} \\ \text{has common points with } D_n.\end{array} \tag{2}$$

Then $E_{n,m}$ is closed and $F_{n,p,q}$ is open. We put

$$F_{n,p}=\bigcap_{s=1}^{\infty} \bigcup_{q=s}^{\infty} F_{n,p,q} \quad \text{and} \quad A_{n,m,p}=E_{n,m} \cap F_{n,p} \tag{3}$$

We will show that

$$E_{KK}(f)=\bigcup_{n,m,p} A_{n,m,p} \tag{4}$$

Take a point $\zeta \in E_{KK}(f)$. There exist $K'(\zeta)$ and $K''(\zeta)$, $K' \supset K''$, for which $C(\zeta, K') \supsetneqq C(\zeta, K'')$. Choose numbers p and s such that $K_p(\zeta) \supset K'(\zeta)$ and

$$D_s \cap C(\zeta, K_p) \neq \phi, \quad \mathrm{dis}(D_s, C(\zeta, K'')) > 5r_s. \tag{5}$$

Then we can find a number m such that $K''(\zeta) \supset K_m(\zeta)$ and

$$\mathrm{dis}(D_s, f(z)) > 4r_s \quad \text{for} \quad z \in K_m(\zeta) \cap \{z\,;\, \mathrm{dis}(z, \Gamma) < 1/m\}.$$

If D_n is a disk with radius $r_n = 2r_s$ and concentric with D_s,

$$\mathrm{dis}(D_n, f(z)) > r_n \quad \text{for} \quad z \in K_m(\zeta) \cap \{z\,;\, \mathrm{dis}(z, \Gamma) < 1/m\},$$

which shows $\zeta \in E_{n,m}$. In view of (5) there exists an infinite number of q such that $D_n \cap \{w = f(z)\,;\, z \in K_p(\zeta),\ 1/3q < \mathrm{dis}(z, \Gamma) < 1/q\} \neq \phi$, which shows $\zeta \in F_{n,p}$. Thus $\zeta \in A_{n,m,p}$ and $E_{KK}(f) \subset A_{n,m,p}$.

Take $\zeta \in A_{n,m,p}$. From (1), $C(\zeta, K_m) \cap D_n = \phi$. On the other hand from (2), $C(\zeta, K_p) \cap D_n \neq \phi$. Thus we have $C(\zeta, K_m) \neq C(\zeta, K_p)$ and $E_{KK}(f) \supset A_{n,m,p}$.

The equality (4) shows that $E_{KK}(f)$ is of type $G_{\delta\sigma}$. It remains to prove that $A = A_{n,m,p}$ is of porosity. If $\rho_m \leqq \rho_p$, $C(\zeta, K_m) \supset C(\zeta, K_p)$ and A must be void. Hence we assume $\rho_p < \rho_m$.

Suppose A is not of porosity at a point $\zeta \in A$. Then for sufficiently small $\varepsilon > 0$, $K_p(\zeta) \cap U_\varepsilon(\zeta)$ is covered by the set $\bigcup_{\xi \in A} K_m(\xi)$. Thus if $z \in K_p(\zeta) \cap U_\varepsilon(\zeta)$, there is a point $\xi \in A$, $z \in K_m(\xi)$. Therefore $w = f(z)$ lies at a distance $\geqq r_n$ from D_n, and $C(\zeta, K_p, f) \cap D_n$ must be void. This contradicts with the definition of $F_{n,p}$ and the porosity of A is proved.

3. Theorem 2. *For any arbitrary function $f(z)$, $E_{KV}(f)$ is $G_{\delta\sigma}$ and of σ-porosity of the order* 1/2.

Proof. Let $\{\rho_m\}$, $\{K_m\}$, $\{D_n\}$ have the same meanings as in the proof of Theorem 1. We denote by $V_{m,k} = V_{m,k}(\zeta)$ the angle of opening $\rho_m \pi/2$ with vertex at ζ and with bisector forming an angle $\rho_k \pi/2$ with inner normal to Γ at ζ.

$E_{n,m,k}$ is the set of points $\zeta \in \Gamma$ such that

the set $\{w = f(z)\,;\, z \in V_{m,k}(\zeta) \cap G,\ \mathrm{dis}(z, \Gamma) < 1/m\}$

lies at a distance $\geqq r_n$ from D_n.

$E_{n,m,k}$ is closed. Put $A_{n,m,k,p} = E_{n,m,k} \cap F_{n,p}$, where $F_{n,p}$ is the one used in proof of the former theorem. Then we can show as before that

$$E_{KV}(f) = \bigcup_{n,m,k,p} A_{n,m,k,p},$$

which shows E_{KV} is $G_{\delta\sigma}$. To see that $A = A_{n,m,k,p}$ is of porosity (1/2), we take G as the upper half plane, Γ as the real axis, and ζ as the origin. Then ∂K is the circle $x^2 + y^2 = 2\rho y$, writing ρ instead of $1 - \rho_p$.

Let $M = \bigcup_{\xi \in A} V_{m,k}(\xi)$. Suppose there exists a sequence $z_\nu = x_\nu + iy_\nu \to 0$, $z_\nu \in K_p \backslash M$. Then the set A must omit intervals $\{I_\nu\}$, where I_ν is the intersection of an angle $\bar{V}^{(\nu)} = -V_{m,k}(0) + z_\nu = \{z\,;\, z = -Z + z_\nu,$

$Z \in V_{m,k}(0)\}$ with the real axis. $|I_\nu|$ (the length of I_ν) is $ay_\nu \geqq \frac{1}{2} a\rho x_\nu^2$, where a is a constant depending on ρ_m and ρ_k. From this we can infer that if $z_\nu \in U_\varepsilon(0)$, $r(0,\varepsilon) \geqq bx_\nu^2$, where b is a constant. Thus we have

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{\varepsilon}(r(0,\varepsilon))^{\frac{1}{2}} > 0,$$

and obtain a contradiction to the assumption that 0 is not a point of porosity (1/2) for A. Therefore, for $\varepsilon > 0$ small enough, $K_p(0) \cap U_\varepsilon(0)$ is covered by the set M. If $z \in K_p \cap U_\varepsilon$ there is a point $\xi \in A$, $z \in V_{m,k}(\xi)$, and $w = f(z)$ lies at a distance $\geqq r_n$ from D_n. Thus $C(0, K_p) \cap D_n = \phi$. This is absurd in view of the definition of $F_{n,p}$.

4. From the Theorems 1, 2 and the Dolzhenko's theorem quoted in §2, we have

Theorem 3. *For any arbitrary function $f(z)$, there holds $C_{\mathfrak{A}}(\zeta, f) = C_{\mathfrak{D}}(\zeta, f)$ at every $\zeta \in \Gamma$ except on a set of porosity* (1/2).

Remark. Let $u = h(t) \geqq 0$, $t \geqq 0$, be a continuous and increasing function. A set $E \subset \Gamma$ can be defined to be *of porosity in the $h(t)$-measure* at ζ if

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{\varepsilon} h(r(\zeta,\varepsilon)) > 0.$$

If $t = h^{-1}(u)$ is the inverse of $h(t)$ and $\int_0 h^{-1}(u)u^{-2}du < \infty$, the set $\{z = re^{i\theta};\ \theta < h(1-r)\}$ is a fine neighborhood of ζ [4]. We can show that the set $F = \{\zeta;\ C_{\mathfrak{A}}(\zeta, f) \backslash C_{\mathfrak{A}}(\zeta, f) \neq \phi\}$ is of σ-porosity in the $h(t)$-measure, where $h(t)$ satisfies the above condition. Probably F would be characterized more precisely.

Theorem 4. *There is a bounded holomorphic function $f(z)$ for which $E_{KV}(f)$ is of measure 2π.*

Proof. Fix an inscribed disk $K(1) = \{z;\ |z-\rho| < 1-\rho\}$, $0 < \rho < 1$. There is a constant β such that an arc $\gamma = \{z = re^{i\theta};\ \theta = \beta\sqrt{1-r}\}$ is contained in $K(1)$. Choose t_n, $0 < t_n < 1$, $t_n \nearrow 1$ such that $\sum\sqrt{1-t_n} < \infty$. We define

$$f(z) = \prod \frac{z^{k_n} - t_n^{k_n}}{(t_n z)^{k_n} - 1},$$

where the integers k_n are determined by $k_n = [3\pi/\beta\sqrt{1-t_n}] + 1$. This product converges since $\sum k_n(1-t_n) < \infty$. For every point $\zeta \in \Gamma$, $K(\zeta)$ contains an infinite number of zeros of $f(z)$ and $C(\zeta, K, f)$ contains 0, but $f(z)$ has angular limits of modulus 1 at almost every point of Γ. Thus $E_{KV}(f)$ is of measure 2π (and of σ-porosity (1/2)).

This gives by the way an example $f(z)$ for which $E_{GV}(f)$ is of measure 2π.

5. Theorem 4 can be sharpened as follows.

Theorem 5. *Let $E \subset \Gamma$ be a closed set of porosity* (1/2). *Then there is a bounded holomorphic function $f(z)$ such that $E_{KV}(f)=E$.*

Proof. E^C consists of a countable number of arcs $I_\nu=(\zeta'_\nu, \zeta''_\nu)$. Let $L(\zeta)=\partial K(\zeta)$ be an inscribed circle at $\zeta: L(\zeta)=\{z;\ |z-\rho\zeta|=1-\rho\}$. Except at most finite number of ν's, $L(\zeta'_\nu)\cap L(\zeta''_\nu)\neq\phi$. Let $z'_{\nu,1}=z''_{\nu,1}$ be the one of intersection points of $L(\zeta'_\nu)$ and $L(\zeta''_\nu)$ which is nearer to Γ. For every $n>1$, $z'_{\nu,n}$ be the point on $L(\zeta'_\nu)$ such that $(1-|z'_{\nu,n}|)/|\zeta'_\nu-z'_{\nu,n}| =\frac{1}{2}(1-|z'_{\nu,n-1}|)/|\zeta'_\nu-z'_{\nu,n-1}|$. The sequence $\{z''_{\nu,n}\}$ on $L(\zeta''_\nu)$ is defined analogously.

Then $\sum_{\nu,n}(1-|z'_{\nu,n}|)+\sum(1-|z''_{\nu,n}|)<\infty$, whence the Blaschke product

$$f(z)=\prod\frac{\bar{z}'_{\nu,n}}{|z'_{\nu,n}|}\,\frac{z-z'_{\nu,n}}{1-\bar{z}'_{\nu,n}z}\prod\frac{\bar{z}''_{\nu,n}}{|z''_{\nu,n}|}\,\frac{z-z''_{\nu,n}}{1-\bar{z}''_{\nu,n}z}$$

converges and defines a bounded holomorphic function in G. Since for each $\zeta\in E$ $|\zeta-z'_{\nu,n}|\geqq|\zeta'_\nu-z'_{\nu,n}|$ and $(1-|z'_{\nu,1}|)/|\zeta'_\nu-z'_{\nu,1}|\leqq K|\zeta'_\nu-z'_{\nu,1}|$ for sufficiently large ν, where K is a constant, we have

$$\sum_{\nu,n}\frac{1-|z'_{\nu,n}|}{|\zeta-z'_{\nu,n}|}+\sum_{\nu,n}\frac{1-|z''_{\nu,n}|}{|\zeta-z''_{\nu,n}|}<\infty,$$

thus $f(z)$ has an angular limit of modulus 1 at each point $\zeta\in E$ (Frostman [3]). But if $1>\rho>\rho'$, $K'(\zeta)$ for $\zeta\in E$ contains an infinite number of points from $\{z'_{\nu,n}, z''_{\nu,n}\}_{\nu,n}$ because of the porosity (1/2) of E. Thus $C(\zeta, K')\ni 0$, and all $\zeta\in E$ belong to $E_{KV}(f)$.

Since $\zeta\in E^C$ is not a limit point of zeros of $f(z)$, $f(z)$ is continuous there. Hence at every $\zeta\in E^C$ $C(\zeta, V)=C(\zeta, K)$ and $\zeta\notin E_{KV}(f)$.

Theorem 6. *If $E=\cup E^{(\mu)}$ where $E^{(\mu)}$ is closed and of porosity* (1/2), *there is a bounded holomorphic function $f(z)$ for which $E_{KV}(f)\supset E$.*

Proof. We can assume that $E^{(\mu)}\cap E^{(\nu)}=\phi$ if $\mu\neq\nu$. For, if not so, set $F^{(1)}=E^{(1)}$. Since $(E^{(1)})^C$ consists of a countable set of open arcs $\{I_k^{(1)}\}$, $E^{(2)}\backslash E^{(1)}=\cup_k(E^{(2)}\cap I_k^{(1)})$ and each $E^{(2)}\cap I_k^{(1)}=P_k^{(2)}$ is closed. As $(E^{(2)}\cup E^{(1)})^C$ is also a countable collection of open arcs, we see that $E^{(3)}\backslash(E^{(2)}\cup E^{(1)})$ can be written as a countable sum of closed sets $P_k^{(3)}$, pairwise disjoint. Repeating this indefinitely and renumbering $\{P_k^{(\kappa)}\}$ as $\{F^{(\mu)}\}$, our assertion follows.

Corresponding to $E^{(\mu)}$ we construct a sequence of zeros $\{z_{\nu,n}^{(\mu)}\}$ and Blaschke product $f_\mu(z)$, as in Theorem 5. Set

$$f(z)=\sum 2^{-\mu}f_\mu(z).$$

If $\zeta\in E_\mu$, all $f_\nu(z)$ $(\nu\neq\mu)$ are continuous at ζ. Put $B=\sum 2^{-\kappa}f_\kappa(\zeta)$, $B_1=B-f_\mu(\zeta)$, where $f_\mu(\zeta)$ is the angular limit of $f_\mu(z)$ at ζ. It is easily seen that $f(z)\to B$ as z approaches ζ angularly, but $f(z_{\nu,n}^{(\mu)})\to B_1$ as $z_{\nu,n}^{(\mu)}\to\zeta$ in the inscribed disk $K'(\zeta)$ (used in Theorem 5). Thus $C(\zeta, V)=\{B\}$

$\neq C(\zeta, K', f)$ and $\zeta \in E_{KV}(f)$.

6. We state the following theorems without proofs.

Theorem 7. *If* $E = \cup E^{(\mu)}$, *where* $E^{(\mu)}$ *is closed and of porosity, there is a bounded holomorphic function* $f(z)$ *for which* $E_{KK}(f) \supset E$.

Theorem 8. *For any arbitrary function* $f(z)$, $E_{GK}(f)$ *is* F_σ *and of the first category.* E_{GK} *may be of measure* 2π *even for bounded holomorphic* $f(z)$.

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