

Angular momentum of a strongly focused Gaussian beam

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Abstract

A circularly polarized paraxial Gaussian laser beam carries $\pm\hbar$ angular momentum per photon as spin, with zero orbital angular momentum. Focusing the beam with a rotationally symmetric lens cannot change this angular momentum flux, yet the focused beam must have spin $|S_z| < \hbar$ per photon. The remainder of the original spin is converted to orbital angular momentum, manifesting itself as a longitudinal optical vortex at the focus. We investigate the nature of this orbital angular momentum.

Keywords: optical vortices, orbital angular momentum, non-paraxial beams

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1. Introduction

Although there has been an ongoing controversy about the angular momentum density of circularly polarized electromagnetic waves for nearly a century (Henriot 1934, Atkinson 1935, Humblet 1943, Crichton and Marston 2000, Stewart 2005, Zambrini and Barnett 2005, Nieminen *et al* 2007), it is generally accepted that there is no doubt about the total angular momentum flux of a beam of finite power. In particular, it is well known that the angular momentum flux of a circularly polarized paraxial Gaussian beam of power P and optical angular frequency ω is $\pm P/\omega$, with the sign given by the handedness of the circular polarization. This corresponds to the equally well-known quantum result of photons having \hbar spin.

A more problematic controversy is the separation of the angular momentum flux or density into spin and orbital components. This controversy is not unrelated to the older one referred to above. Essentially, one can choose either of two expressions for the angular momentum density of an electromagnetic wave. Since the linear momentum density is equal to $\mathbf{p} = \mathbf{E} \times \mathbf{H}/c$, where \mathbf{E} and \mathbf{H} are the instantaneous electric and magnetic fields respectively, it is natural to assume that the angular momentum density \mathbf{j} is the moment of this,

$$\mathbf{j} = \mathbf{r} \times \mathbf{p}, \quad (1)$$

where \mathbf{r} is the position vector of the point of interest. However, in this case, the angular momentum of a circularly polarized plane wave would be zero. Notably, the assumption that the angular momentum density is $\mathbf{r} \times \mathbf{p}$ is implicitly a statement that the spin—the part of the angular momentum density that is independent of the choice of origin—of the electromagnetic field is zero, as readily seen if we choose an origin such that $\mathbf{r} = 0$.

However, if we instead begin with the conservation laws resulting from invariance under spatial rotations (Noether 1918, Tavel 1971, Soper 1976, Jauch and Rohrlich 1976), we can write the angular momentum density as a sum of spin and orbital terms

$$\mathbf{j} = \mathbf{l} + \mathbf{s} = \frac{\epsilon_0}{c} \sum_{i=x}^{y,z} E_i \mathbf{r} \times \nabla A_i + \frac{\epsilon_0}{c} \mathbf{E} \times \mathbf{A}, \quad (2)$$

where \mathbf{A} is the vector potential. Since the term $\epsilon_0 \mathbf{E} \times \mathbf{A}/c$ is manifestly coordinate system independent, it can be unambiguously identified as the spin density. This expression for the total angular momentum density was shown by Humblet (1943) to be equivalent to (1), integrating the density (1) by parts and dropping terms that vanish if the fields vanish sufficiently quickly as $r \rightarrow \infty$. Conversely, (1) can be obtained from (2) by symmetrizing the canonical energy momentum tensor (Jackson 1999). That this is only valid,

again, if the fields vanish sufficiently quickly as $r \rightarrow \infty$ is often left unstated or understated; Jauch and Rohrlich (1976) carefully point out this requirement. Bromberg (2006) gives a lucid discussion of the equivalence of the two approaches.

More problematic, and rightly so, is the apparent lack of gauge invariance of the spin density. Consequently, it is typically concluded that the separation of the angular momentum of a general electromagnetic wave into spin and orbital components cannot be made in a physically meaningful way—the result is either gauge dependent or not Lorentz invariant. However, if we consider the fact that the (quasi-)monochromaticity of an electromagnetic wave, at least for a physically achievable wave (i.e., not an infinite plane wave) is also not Lorentz invariant, it does not come as a great surprise to find that the spin density of a monochromatic wave is both gauge independent and physically meaningful (van Enk and Nienhuis 1994, Crichton and Marston 2000). Since this is in accord with both the correspondence principle—the quantum theory must yield the classical theory in an appropriate limit—and Noether’s theorem, there is every reason to accept (2) as the *correct* expression for the angular momentum density, rather than the naïve (1).

This leads to an interesting problem. A rotationally symmetric system cannot alter the angular momentum state of an electromagnetic wave, and thus focusing by a rotationally symmetric optical system cannot alter the angular momentum state of a laser beam. If we consider a focused circularly polarized beam, following Crichton and Marston (2000) we can measure the spin angular momentum in the far field. In the far field, the beam is a spherical wave, and, locally, we measure the radial component of the spin angular momentum. However, only the component along the beam axis can contribute to the total angular momentum of the beam. Since the maximum magnitude of the spin density is \hbar per photon, the total spin angular momentum must be less than this amount (see figure 1). On the other hand, the total angular momentum flux cannot have been changed by the act of focusing the beam. Therefore, an orbital angular momentum flux must have been introduced into the beam.

We investigate the nature of this orbital angular momentum flux, and show that it is associated with the optical vortex nature of the axial electromagnetic field. We also clearly demonstrate an orbital motion of energy within the beam.

2. Angular momentum of a focused beam

The simple result of spin angular momentum flux equal to P/ω is only valid in the paraxial approximation, as it depends on E_z being zero. If we consider a beam of finite width in its focal plane, then the beam will spread through diffraction, and will, at a sufficiently large distance, be propagating in a purely radial direction. That is, for large r , we must have $E_r = 0$. In this case, the electric field is purely tangential, and the spin angular momentum density in polar spherical coordinates is

$$s_r = \epsilon_0 \text{Im}(E_\theta E_\phi^*)/\omega, \quad (3)$$

with the other vector components being zero. For a rotationally symmetric beam of the type we consider here, s_r will be independent of the azimuthal angle ϕ .

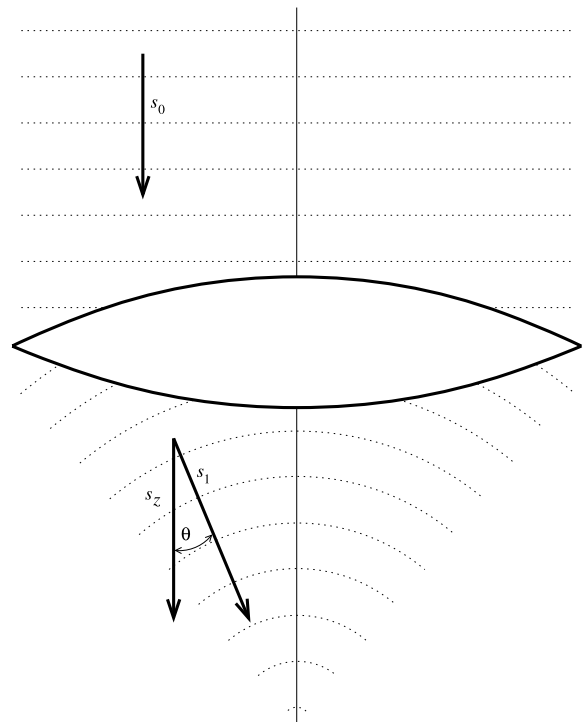


Figure 1. Reduction of spin about the beam axis by a lens. If a circular polarized paraxial beam is incident on a lens, the initial spin flux density vector, s_0 , is parallel to the beam axis. After focusing, it will no longer be parallel. At the location shown in the figure, the final spin density vector s_1 is at an angle of θ (this angle will vary across the focused beam), and only the component s_z parallel to the beam axis will contribute to the total spin flux of the beam. Thus, the *total* spin angular momentum flux is reduced by focusing.

Therefore, the maximum possible contribution to the total spin angular momentum, of which, by symmetry, only the z component is non-zero, is $s_r \cos \theta$, where θ is the angle measured from the z axis. Integrating this over the beam must result in $|S_z| < \hbar$ per photon. If we consider a non-paraxial beam with a Gaussian profile, we can write the amplitude in the far field as

$$U = U_0 \exp(-\tan^2 \theta / \tan^2 \theta_0), \quad (4)$$

where θ_0 is the angle at which the amplitude of the field drops to $1/e$ of the value at $\theta = 0$. This angle is the beam convergence angle (Nieminen *et al* 2003). For maximum possible spin, we have $s_r = \epsilon_0 U^2/\omega$, and the total spin angular momentum of the beam, in units of \hbar per photon, can be found by integrating over a hemisphere:

$$S_z = A/P \quad (5)$$

where

$$A = \int_0^{\pi/2} \exp(-2 \tan^2 \theta / \tan^2 \theta_0) \sin \theta \cos \theta \, d\theta \quad (6)$$

and

$$P = \int_0^{\pi/2} \exp(-2 \tan^2 \theta / \tan^2 \theta_0) \sin \theta \, d\theta. \quad (7)$$

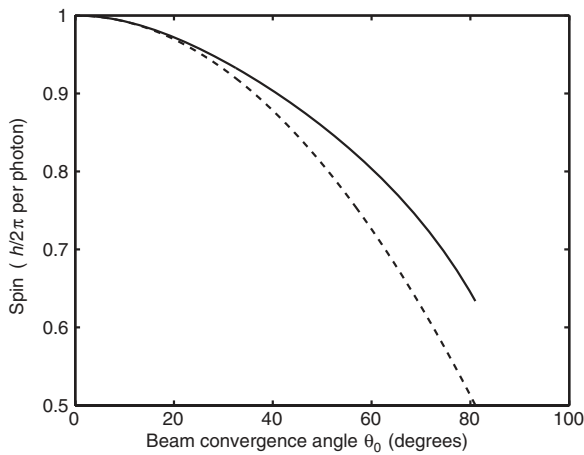


Figure 2. Spin angular momentum in \hbar per photon for a non-paraxial TEM₀₀ Gaussian beam. The solid line shows the exact result, the dotted line shows the small angle approximation.

This can be readily integrated numerically, and the result for all practically realizable Gaussian beams is shown in figure 2. The qualitative behaviour seen here is expected since $S_z = 0.5P/\omega$ for a dipole radiation field (Humblet 1943, Crichton and Marston 2000).

For $\theta_0 \ll 1$, small angle approximations can be made, giving

$$S_z = 1 - \theta_0^2/4. \quad (8)$$

The relative error in the change in spin ($1 - S_z$) is less than 0.01 for beam convergence angles $\theta_0 < 6.25^\circ$, and less than 0.1 for $\theta_0 < 21^\circ$. In the paraxial limit ($\theta \rightarrow \infty$), we recover the usual result for paraxial beams, namely $S_z = 1$. A quadratic fit to the ratio of the exact result to the small angle approximate result yields a simple correction factor giving an improved simple formula,

$$S_z = (1 - \theta_0^2/4) \times (0.3106 \cos^2 \theta_0 - 0.6818 \cos \theta_0 + 1.3696), \quad (9)$$

which is accurate to better than $0.006\hbar$ per photon.

Although the spin angular momentum flux of the beam is reduced by the beam being focused, a lossless rotationally symmetric optical system cannot change the total angular momentum flux of an electromagnetic field (Waterman 1971, Nieminen 2004). Therefore, there must be a corresponding increase in the orbital angular momentum. This is in remarkable contrast to the usual methods of generating optical vortices which employ astigmatic or cylindrical lenses or holograms designed to break the rotational symmetry. The key difference is that orbital angular momentum generation by focusing depends on the initial presence of spin angular momentum, whereas astigmatic systems do not.

An interesting question that remains to be answered is in what way the focused beam carries the orbital angular momentum. This is best addressed by considering a rigorous electromagnetic model of the beam. Orbital angular momentum about a beam axis is typically associated with an optical vortex, and accompanied by an azimuthal flow of energy.

2.1. Multipole expansion

A time-harmonic electromagnetic beam can be represented as a sum of electric and magnetic multipoles:

$$\mathbf{E}(\mathbf{r}) = \sum_n \sum_{m=-n}^n a_{nm} \mathbf{M}_{nm} + b_{nm} \mathbf{N}_{nm} \quad (10)$$

where \mathbf{M}_{nm} and \mathbf{N}_{nm} are the TE and TM regular multipole fields, or vector spherical wavefunctions (Nieminen *et al* 2003). Not only are these wavefunctions a complete orthogonal set of divergence-free solutions of the vector Helmholtz equation (and hence solutions to the Maxwell equations), they are also eigenfunctions of the angular momentum operator J^2 , with eigenvalues $[n(n+1)]^{1/2}$, and J_z , with eigenvalues m . The spin and orbital contributions to the angular momentum can be calculated from the expansion coefficients a_{nm} and b_{nm} (Crichton and Marston 2000, Nieminen *et al* 2007). The reader interested in the mathematical details and derivation should refer to Crichton and Marston (2000).

The only non-zero multipole coefficients for a left-circularly polarized rotationally symmetric beam are those with $m = 1$. Thus, the total angular momentum about the z axis is \hbar per photon. The multipole expansion coefficients for the beam can be determined by an overdetermined point-matching method (Nieminen *et al* 2003). For a beam of finite width, it is found that the total spin is less than \hbar per photon (spin calculated in this way exactly reproduces the curve in figure 2). The remainder of the angular momentum is orbital.

Since the multipole expansion of the beam is known, the fields can be calculated at any point in space. The components of the electric field in the focal plane are shown in figure 3. For a strongly focused beam such as is shown in figure 3, the longitudinal (i.e. z) component of the field is significant, with a magnitude of ≈ 0.3 times the transverse components. All components of the electric field show secondary diffraction rings (the radial dependence of multipole fields includes a spherical Bessel function). The phases of the x and y components are uniform, except for an increment of π between successive diffraction rings, and, due to the circular polarization, differ by $\pi/2$ from each other. The phase of the z component, however, shows a clear azimuthal dependence identical to that seen in $l = 1$ paraxial vortex modes. Since this vortex behaviour is only possessed by the longitudinal component of the field, this can be called a *longitudinal optical vortex*.

Calculation of the time-averaged Poynting vector (see figure 4) shows that there is indeed a transverse component, which, since its handedness is uniform, is responsible for the transport of the orbital angular momentum.

So far, we have only considered the vector components of the complex amplitude of the electric field. Calculation of the instantaneous energy density of the beam yields a striking demonstration of the azimuthal energy flow; this is shown in figure 5 and movie 1 (available at stacks.iop.org/JOptA/10/115005). Instantaneously, the beam has the same fields in the focal plane as a plane polarized beam, and therefore, if tightly focused, shows the expected elongation

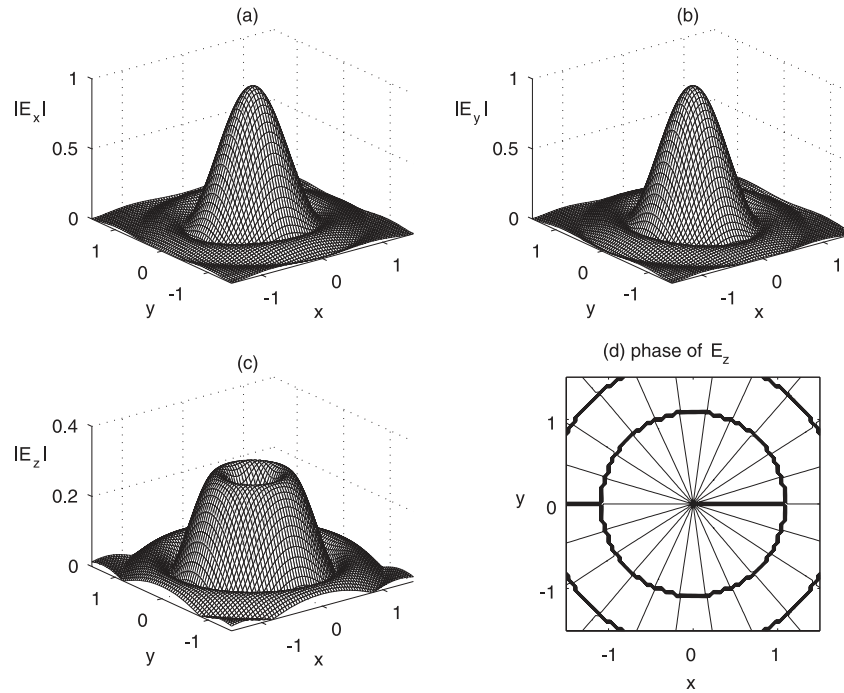


Figure 3. Electric field components of a strongly focused circularly polarized Gaussian beam, with a convergence angle of 45° . The x , y , and z components of the electric field in the focal plane are shown in (a), (b), and (c), while (d) shows phase contours (with a spacing of $2\pi/20$) for the z component, showing azimuthal variation of phase as seen in vortex beams.

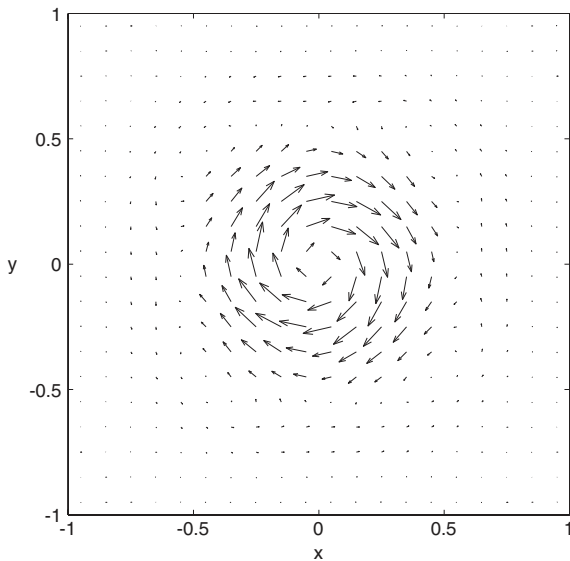


Figure 4. Poynting vector of a strongly focused circularly polarized Gaussian beam, with a convergence angle of 45° . The transverse part (x and y components) of the Poynting vector in the focal plane are shown.

of the focal spot along the direction of the electric field. As this elongated focal spot rotates at the optical frequency, this shows that there is an unambiguous azimuthal flow of energy. This movement of field energy is accompanied by a momentum flux (Umov 1950, Nieminen *et al* 2007), which in this case is azimuthal about the beam axis, and therefore results in orbital angular momentum about the beam axis.

As the beam is more strongly focused, the magnitude of the longitudinal (z) component of the field increases, and the orbital angular momentum increases as a result. The same increase can also be considered to result from the decrease of spin angular momentum, along with the conservation of total angular momentum. The change in the angular momentum and the growth of the longitudinal optical vortex is smooth and well behaved as the convergence angle of the beam is increased, with no sudden qualitative or quantitative changes. As the beam is more strongly focused, the diffraction rings also become more prominent, but this does not affect the angular momentum of the beam.

3. Discussion

3.1. Coupling between spin and orbital angular momenta

Due to the dependence of orbital angular momentum density on the choice of origin about which moments are taken, and the independence of spin density on this choice, the conversion of spin to orbital angular momentum must be accompanied by a torque. Therefore, conversion from one type of angular momentum to the other cannot occur in free space, or in media which can be electromagnetically represented by a uniform scalar permittivity. At the interface between two media, such as, for example, the surface of a lens, coupling between spin and orbital angular momenta can occur.

Bomzon *et al* (2006) claimed that the angular momentum per photon actually *increases* when a circularly polarized beam is focused. However, this apparently paradoxical result simply further demonstrates the incorrectness of the expression used for the angular momentum flux.

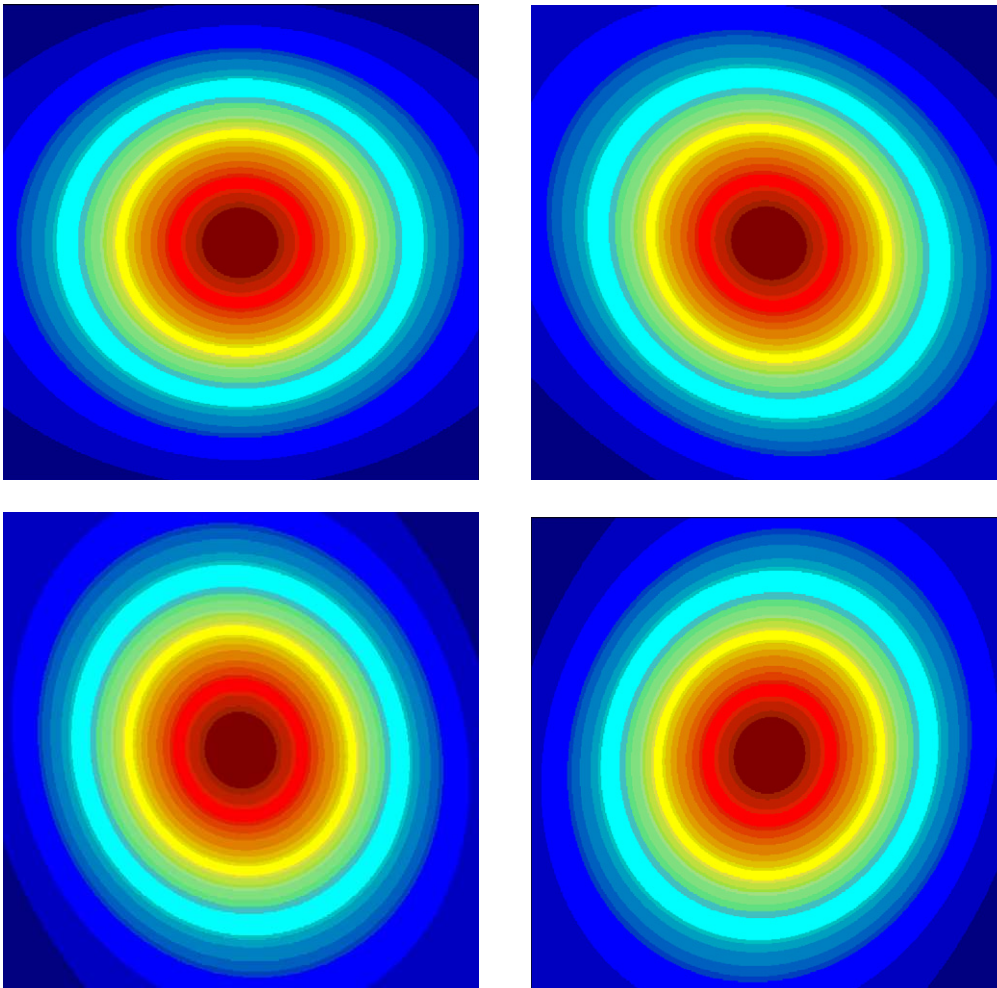


Figure 5. Instantaneous energy density of a strongly focused circularly polarized Gaussian beam, with a convergence angle of 45° . The energy density in the focal plane is shown, and clearly demonstrates the rotation of the beam. See movie 1 (available at stacks.iop.org/JOptA/10/115005) for an animation.

(This figure is in colour only in the electronic version)

More recently, Zhao *et al* (2007) have also stated that spin-to-orbital conversion can occur in a homogeneous and isotropic medium, in the context of a tightly focused optical vortex beam. Again, this cannot be the case, since no torque can act on the medium. The conversion does *not* occur within the homogeneous medium, but when the beam is focused by the lens—at the interfaces between the various media, where the media concerned are inhomogeneous.

In more general media, where a torque can be exerted, conversion between spin and orbital angular momenta may be possible. For example, Marrucci *et al* (2006) have considered such conversion in inhomogeneous anisotropic media.

To achieve the special case of conversion between spin and orbital angular momenta while maintaining a constant total angular momentum flux, the *total* torque must be zero, despite the torque density being non-zero.

3.2. Torque density acting on lens

In light of the above considerations, the transformation of spin angular momentum to orbital angular momentum by the

lens must result in a reaction torque density about the z -axis acting on the lens that depends on the choice of origin. While, as noted above, such a torque acting on empty space is unacceptable on physical grounds, it is not only entirely reasonable, but expected, in the case of a lens.

One need only to consider a simple ray picture of the action of a lens where the z -axis and the axis of the lens do not coincide. If we choose a z -axis parallel to the lens axis, but laterally displaced from it, incident rays will be parallel to the z -axis, but the focused rays will generally not pass through the z -axis. Thus, each focused ray carries orbital angular momentum about the z -axis, while the incident rays do not. Consequently, there must be a reaction torque density acting on the lens; this torque density depends on the choice of origin. The component about any axis parallel to the beam axis of the total torque acting on the lens is, of course, zero.

3.3. Rotation in optical traps

The presence of orbital angular momentum in the focal region suggests that orbital motion of absorbing or reflective spherical

particles should be observable in optical tweezers. However, since particles trapped in a Gaussian beam trap will be located on the beam axis, all that will be observed will be spinning of the particle about its axis. In order to observe this orbital angular momentum, it would be necessary to use a multiringed beam with initial zero orbital angular momentum, for example a Laguerre–Gauss mode LG_{p0} , with $p > 1$, or a Bessel beam, as the input to the objective lens of the trap. In this case, particles trapped in one of the rings rather than the central spot can be expected to undergo orbital motion.

An experiment intended to measure the conversion of spin to orbital angular momentum has been carried out by Zhao *et al* (2007), who reported an observed effect. However, since they used an LG_{01} beam, initially carrying \hbar orbital angular momentum per photon as well as spin due to its polarization, the results are not as unambiguously clear as they would have been had the beam had an initial orbital angular momentum of zero. However, their work does appear to be a valid experimental detection of this effect.

4. Conclusion

We note that focusing a circularly polarized beam preserves the total angular momentum flux of the beam about its axis. However, the spin component of the angular momentum flux is necessarily reduced as the beam is more strongly focused. Due to the conservation of total angular momentum when the beam is focused by a rotationally symmetric optical system, there must be a corresponding increase in the orbital angular momentum flux. This result is remarkable in that it predicts the generation of orbital angular momentum by a rotationally symmetric optical system, in apparent contradiction with common expectation.

This orbital angular momentum is associated with the axial component of the electric field, E_z , which has the typical

$\exp(i\phi)$ dependence of charge 1 optical vortices; we call this a longitudinal optical vortex.

References

- Atkinson R D 1935 *Phys. Rev.* **47** 623–7
 Bomzon Z, Gu M and Shamir J 2006 *Appl. Phys. Lett.* **89** 241104
 Bromberg J L 2006 *Isis* **97** 237–59
 Crichton J H and Marston P L 2000 *Electron. J. Differ. Equations Conf.* **04** 37–50
 Henriot E 1934 *C. R. Hebd. Séances Acad. Sci.* **198** 1146–8
 Humblet J 1943 *Physica* **10** 585–603
 Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley)
 Jauch J M and Rohrlich F 1976 *The Theory of Photons and Electrons* 2nd edn (New York: Springer)
 Marrucci L, Manzo C and Paparo D 2006 *Phys. Rev. Lett.* **96** 163905
 Nieminen T A 2004 *Opt. Commun.* **235** 227–9
 Nieminen T A, Knöner G, Heckenberg N R and Rubinsztein-Dunlop H 2007 *Laser Manipulation of Cells and Tissues (Methods in Cell Biology vol 82)* ed M W Berns and K O Greulich (Amsterdam: Elsevier) chapter 6 pp 207–36
 Nieminen T A, Loke V L Y, Stilgoe A B, Knöner G, Brańczyk A M, Heckenberg N R and Rubinsztein-Dunlop H 2007 *J. Opt. A: Pure Appl. Opt.* **9** S196–203
 Nieminen T A, Rubinsztein-Dunlop H and Heckenberg N R 2003 *J. Quant. Spectrosc. Radiat. Transfer* **79/80** 1005–17
 Noether E 1918 *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Klasse* **1918** 235–57
 Soper D E 1976 *Classical Field Theory* (New York: Wiley)
 Stewart A M 2005 *Eur. J. Phys.* **26** 635–41
 Tavel M A 1971 *Trans. Theory Stat. Mech.* **1** 183–207 (Engl. Transl.) (arXiv:physics/0503066)
 Umov N A 1950 *Izbrannye Sochineniya (Selected Works)* (Moscow: Gostexizdat) (in Russian)
 van Enk S J and Nienhuis G 1994 *J. Mod. Opt.* **41** 963–77
 Waterman P C 1971 *Phys. Rev. D* **3** 825–39
 Zambrini R and Barnett S M 2005 *J. Mod. Opt.* **52** 1045–52
 Zhao Y, Edgar J S, Jeffries G D M, McGloin D and Chiu D T 2007 *Phys. Rev. Lett.* **99** 073901