Angular processes related to Cauchy random walks

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Abstract

We study the angular process related to random walks in the Euclidean and in the non-Euclidean space where steps are Cauchy distributed.

This leads to different types of non-linear transformations of Cauchy random variables which preserve the Cauchy density. We give the explicit form of these distributions for all combinations of the scale and the location parameters.

Continued fractions involving Cauchy random variables are analyzed. It is shown that the n-stage random variables are still Cauchy distributed with parameters related to Fibonacci numbers. This permits us to show the convergence in distribution of the sequence to the golden ratio.

Keywords: hyperbolic trigonometry, arcsine law, continued fractions, Fibonacci numbers, non-linear transformations of random variables.

AMS Classification 60K99

1 Introduction

We consider a particle starting from the origin O of $\mathbb{R}^2$ which takes initially a horizontal step of length 1 and a vertical one, say $\mathbf{C}_1$, with a standard Cauchy distribution. It reaches therefore the position $(1, \mathbf{C}_1)$. The line l_1 joining the origin with $(1, \mathbf{C}_1)$ forms a random angle Θ_1 with the horizontal axis (See Figure 1(a) below).

On l_1 the traveller repeats the same movement with a step of unit length (either forward or backward) along l_1 and a standard Cauchy distributed step, say $\mathbf{C}_2$, on the line orthogonal to l_1. The right triangle obtained with the last two displacements has an hypothenuse belonging to the line l_2 with random inclination Θ_2 on l_1.

After n steps the sequence of random angles $\Theta_1, \cdots, \Theta_n$ describes the rotation of the moving particle around the starting point, their partial sums describe an angular random walk which can be written as

$$S_n = \Theta_1 + \cdots + \Theta_n = \sum_{j=1}^{n} \arctan \mathbf{C}_j \tag{1.1}$$

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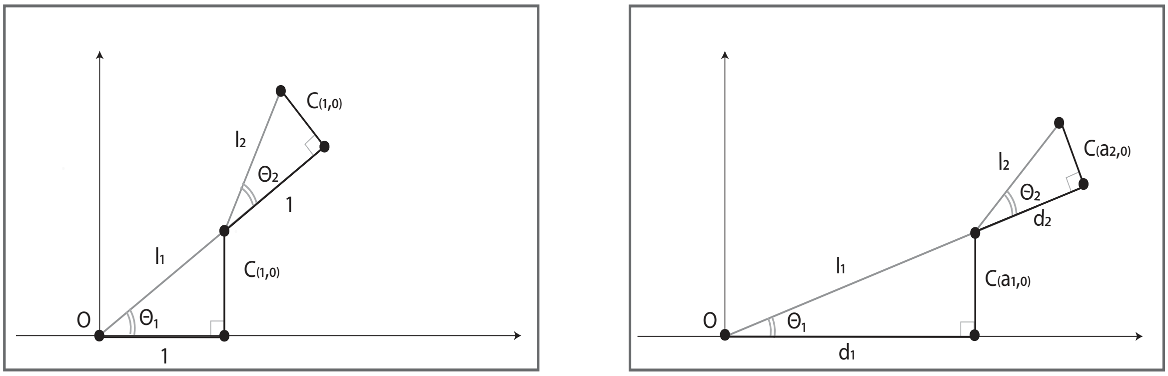


Figure 1: The angular process in the Euclidean plane.

where $\mathbf{C}_j$ are independent standard Cauchy random variables. If the random steps of the planar random walk above were independent Cauchy random variables with scale parameter a_j and location parameter b_j then the process (1.1) must be a little bit modified and rewritten as

$$S_n = \Theta_1 + \cdots + \Theta_n = \sum_{j=1}^{n} \arctan C_j, \tag{1.2}$$

where $C_j \sim C_{(a_j,b_j)}$. The model (1.2) can be extended also to the case where the first step has length d_j and the second one is Cauchy distributed with scale parameter a_j and position parameter b_j (see Figure 1(b)), then

$$\tan \Theta_j = C_{\left(\frac{a_j}{d_j}, \frac{b_j}{d_j}\right)}.$$

The same random walk can be generated if the two orthogonal steps, at each displacement, are represented by two independent Gaussian random variables X_j and Y_j. In this case, for each right triangle, we can write

$$\tan \Theta_j = \frac{X_j}{Y_j}.$$

If X_j and Y_j are standard Gaussian random variables then $\tan \Theta_j = \frac{X_j}{Y_j}$ possesses standard Cauchy distribution and we get the model in (1.1). The model (1.2) can be obtained by considering orthogonal Gaussian steps with different variances and in this case the parameter a_j of the random variables C_j is the ratio $\frac{\sigma_y^j}{\sigma_x^j}$.

The model (1.1) describing the angular random process has an hyperbolic counterpart. We consider a particle starting from the origin O of the Poincaré half-plane $\mathbb{H}_2^+ = \{(x, y) : \ y > 0\}$. At the j-th displacement, $j = 1, 2 \dots$, the particle makes two steps of random hyperbolic length η_j and $\hat{\eta}_j$ on two orthogonal geodesic lines. The j-th displacement leads to a right triangle T_j with sides of length η_j and $\hat{\eta}_j$ and a random acute angle Θ_j. In each triangle T_j the first step is taken on the extension of the hypotenuse of the triangle T_{j-1} (see Figure 2). From hyperbolic trigonometry (for basic results on hyperbolic geometry see, for example, [5]) we have that

$$\sin \Theta_j = \frac{\sinh \hat{\eta}_j}{\sqrt{\cosh^2 \eta_j \cosh^2 \hat{\eta}_j - 1}}, \quad \cos \Theta_j = \frac{\sinh \eta_j \cosh \hat{\eta}_j}{\sqrt{\cosh^2 \eta_j \cosh^2 \hat{\eta}_j - 1}}.$$

From the above expressions we have that

$$\tan \Theta_j = \frac{\tanh \hat{\eta}_j}{\sinh \eta_j}.$$

If we take independent random hyperbolic displacements η_j and $\hat{\eta}_j$ such that the random variables $E_j = \frac{\tanh \hat{\eta}_j}{\sinh \eta_j}$ are standard Cauchy distributed then $\Theta_j = \arctan \mathbf{C}_j$. If the triangles T_j were isosceles then $\tan \Theta_j = \frac{1}{\cosh \eta_j}$ and the angle $\Theta_j \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ so that in this case the Cauchy distribution cannot be attributed to $\tan \Theta_j$.

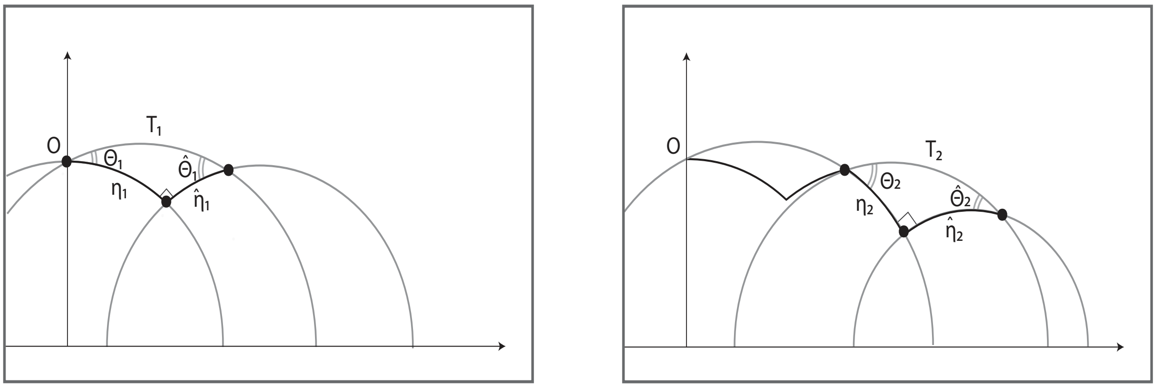


Figure 2: The angular random process in the Poincaré half-plane.

In the model described here the random steps (and therefore the random angular windings Θ_j) are independent. If we consider the model of papers [2] and [3], where the displacements are taken orthogonally to the geodesic lines joining the origin O of H_2^+ with the positions occupied at deviation instants, the angular displacements Θ_j must be such that

$$\sin \Theta_j = \frac{\sinh \eta_j}{\sqrt{1 + \prod_{r=1}^{j} \cosh^2 \eta_r}} = \sinh \eta_j \cos \left(\arctan \prod_{r=1}^{j} \cosh^2 \eta_r \right)$$

and therefore involve dependent random variables.

The process $A_n = \sum_{j=1}^{n} \text{area}(T_j)$, describing the sum of areas of the hyperbolic triangles T_j, has a much more complicated structure. For the area of the random hyperbolic triangle T_j we note that

$$\begin{aligned} \text{area}(T_j) &= \frac{\pi}{2} - \Theta_j - \hat{\Theta}_j = \frac{\pi}{2} - \left[\arctan \left(\frac{\tanh \hat{\eta}_j}{\sinh \eta_j} \right) + \arctan \left(\frac{\tanh \eta_j}{\sinh \hat{\eta}_j} \right) \right] \\ &= \frac{\pi}{2} - \arctan \left(\frac{\coth \eta_j}{\sinh \hat{\eta}_j} + \frac{\coth \hat{\eta}_j}{\sinh \eta_j} \right) = \text{arcotan} \left(\frac{\coth \eta_j}{\sinh \hat{\eta}_j} + \frac{\coth \hat{\eta}_j}{\sinh \eta_j} \right). \end{aligned}$$

Since each acute angle inside T_j is linked to both sides of the triangle the analysis of the random process $A_n = \sum_{j=1}^{n} \text{area}(T_j)$ is much more complicated and we drop it.

Let $C_j \sim C_{(a_j, b_j)}$, $j = 1, 2 \dots$ be independent Cauchy random variables with scale parameter a_j and location parameter b_j. In the study of the angular random walk (1.1) and (1.2) we must analyze the distribution of the following non-linear transformations of Cauchy random variables

$$U = \frac{C_1 + C_2}{1 - C_1 C_2} \tag{1.3}$$

since

$$\arctan C_1 + \arctan C_2 = \arctan \frac{C_1 + C_2}{1 - C_1 C_2}.$$

We will show that the random variable (1.3) is endowed with Cauchy distribution but the value of the parameters a_1, b_1 of C_1 and a_2, b_2 of C_2 heavily influence the structure of the parameters of U.

In particular, if $b_1 = b_2 = 0$ and $a_1 = a_2 = 1$, then U is still distributed as a standard Cauchy distribution and therefore in (1.1) we have that

$$S_n \overset{i.d.}{=} \arctan \mathbf{C}.$$

Since also $\frac{1}{\mathbf{C}}$ is a standard Cauchy, from (1.3), a number of related random variables preserving the form of the Cauchy distribution can be considered. For example, the following random variables

$$Z_1 = \frac{\mathbf{C}_1\mathbf{C}_2 + 1}{\mathbf{C}_1 - \mathbf{C}_2}, \qquad Z_2 = \frac{1 - \mathbf{C}_1\mathbf{C}_2}{\mathbf{C}_1 + \mathbf{C}_2}, \qquad Z_3 = \frac{\mathbf{C}_1 + \mathbf{C}_2}{\mathbf{C}_1\mathbf{C}_2 - 1},$$

also possess standard Cauchy distribution. We can also derive much more complicated random variables by suitably combining three (or more) independent standard Cauchy $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ as

$$Z_4 = \frac{\mathbf{C}_1 + \frac{\mathbf{C}_2+\mathbf{C}_3}{1-\mathbf{C}_2\mathbf{C}_3}}{1 - \mathbf{C}_1\frac{\mathbf{C}_2+\mathbf{C}_3}{1-\mathbf{C}_2\mathbf{C}_3}} = \frac{\mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3 - \mathbf{C}_1\mathbf{C}_2\mathbf{C}_3}{1 - \mathbf{C}_1\mathbf{C}_2 - \mathbf{C}_1\mathbf{C}_3 - \mathbf{C}_2\mathbf{C}_3}$$

and so on.

If $b_1 = b_2 = 0$ and the scale parameters a_1, a_2 are different, then (1.3) still preserves the Cauchy distribution but with scale parameter equal to $\frac{a_1+a_2}{1+a_1a_2}$ and location parameter equal to zero. This can be grasped by means of the following relationship

$$\arctan C_1 + \arctan C_2 \overset{i.d.}{=} \arctan\left\{\frac{a_1 + a_2}{1 + a_1a_2}\mathbf{C}\right\}, \tag{1.4}$$

where $C_j \sim C_{(a_j,0)}$. Result (1.4) is illustrated in Figure 3.

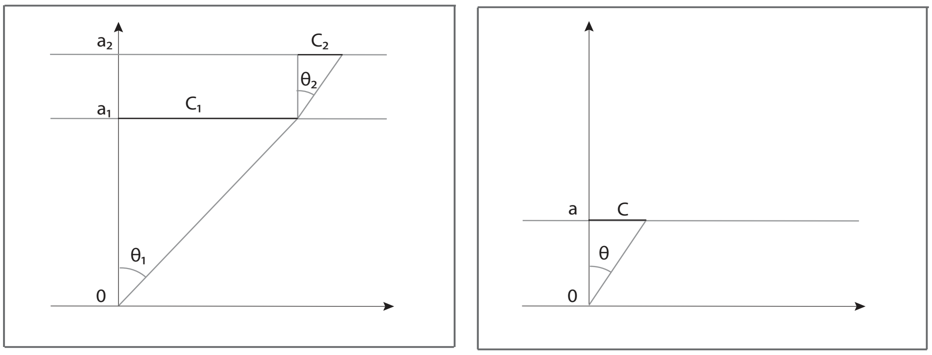


Figure 3: The figure shows that shooting a ray with inclination Θ_1, uniformly distributed, against the line at distance a_1 and then shooting a ray with a uniformly distributed angle Θ_2 on the line at distance a_2 is equivalent to shooting on the barrier at the distance $a = \frac{a_1+a_2}{1+a_1a_2}$ with a uniformly distributed angle Θ.

By iterating the process (1.4) we arrive at the formula

$$\sum_{j=1}^{3} \arctan C_j \overset{i.d.}{=} \arctan\left\{\frac{\sum_{j=1}^{3} a_j + a_1a_2a_3}{1 + \sum_{i\neq j} a_ia_j}\mathbf{C}\right\}$$

which gives an insight into further extensions of the process outlined above.
Many other relationships can be produced by combining the above results and we can observe that if $C_1 \sim C_{(a_1,0)}$ and $C_2 \sim C_{(a_2,0)}$ are independent Cauchy random variables, then

$$W = a_1 a_2 \frac{C_1 + C_2}{C_1 C_2 - (a_1 a_2)^2}$$

also is a centered Cauchy random variable with scale parameter equal to $\frac{a_1+a_2}{1+a_1 a_2}$.

Much more complicated are the cases where the location parameters of the Cauchy distributions are different from zero. For the special case where C_1 and C_2 are independent Cauchy such that $C_1 \sim C_{(1,b)}$ and $C_2 \sim C_{(1,b)}$, the random variable (1.3) still possesses Cauchy density with scale parameter $\frac{2b^2+4}{b^4+4}$ and location parameter $\frac{2b^3}{b^4+4}$.

We have obtained the general distribution of (1.3) where C_1 and C_2 are independent Cauchy such that $C_1 \sim C_{(a_1,b_1)}$ and $C_2 \sim C_{(a_2,b_2)}$ and also the distribution of

$$\widehat{U} = \frac{\alpha C_1 + \beta C_2}{\gamma + \delta C_1 C_2}$$

for arbitrary real numbers $\alpha, \beta, \gamma, \delta$. In particular, if C_1 and C_2 are independent standard Cauchy then $\widehat{U}$ is Cauchy with scale parameter equal to $\frac{\gamma+\delta}{\alpha+\beta}$ and location parameter equal to zero.

In the last section we have examined continued fractions involving Cauchy random variables. In particular we have studied

$$V_n = \frac{1}{1 + \frac{1}{1+\frac{1}{1+\cdots\frac{1}{1+\mathbf{C}}}}} \tag{1.5}$$

and

$$U_n = \frac{1}{1 + \frac{1}{1+\frac{1}{1+\cdots\frac{1}{1+\mathbf{C}^2}}}} \tag{1.6}$$

which generalize the random variables $V_1 = \frac{1}{1+\mathbf{C}}$ and $U_1 = \frac{1}{1+\mathbf{C}^2}$. Continued fractions involving random variables have been analyzed in [4] and more recently in [1]. The random variable U_1 has the arcsine distribution in $[0,1]$, while $U_t = tU_1$, with $t > 0$, has distribution

$$\Pr\{U_t \in ds\} = \frac{\mathrm{d}s}{\pi\sqrt{s(t-s)}}, \qquad 0 < s < t.$$

For each $n \geq 1$, the random variables V_n, are Cauchy distributed with scale parameter a_n and position parameter b_n that can be expressed in terms of Fibonacci numbers. This permits us to prove the monotonicity of a_n and b_n and that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \phi - 1$ where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Finally we obtain that the sequence of random variables $1 + V_n$ and $1 + U_n$, $n \geq 1$, converges in distribution to the number $\phi = \frac{1+\sqrt{5}}{2}$. This should be expected since it has the infinite fractional expansion

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1+\cdots}} \tag{1.7}$$

which is related to (1.5) and (1.6).

2 Centered Cauchy random variables

Our first task is the study of the distribution of the random variable

$$U = \frac{C_1 + C_2}{1 - C_1 C_2} \tag{2.1}$$

where $C_j \sim C_{(a_j,0)}$ are independent Cauchy random variables with location parameter equal to zero and scale parameter a_j, $j = 1, 2$. We can state our first result.

Theorem 2.1. *The random variable U in (2.1) possesses Cauchy distribution with scale parameter $\frac{a_1+a_2}{1+a_1a_2}$ and position parameter equal to zero. We can also restate the result in symbols as*

$$U \overset{i.d.}{=} \frac{a_1 + a_2}{1 + a_1 a_2} \mathbf{C}.$$

PROOF 1
We can prove Theorem 2.1 in two different and independent ways. The first one is rather technical and starts with

$$\Pr\left\{ \frac{C_1 + C_2}{1 - C_1 C_2} < w \right\} = \mathrm{E}\left\{ \Pr\left\{ \frac{C_1 + C_2}{1 - C_1 C_2} < w \middle| C_2 = y \right\} \right\} = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{a_1}{y^2 + a_1^2} \mathrm{d}y \int_{-\infty}^{\frac{w-y}{1+wy}} \frac{a_1}{x^2 + a_1^2} \mathrm{d}x$$

then

$$\Pr\left\{ \frac{C_1 + C_2}{1 - C_1 C_2} \in \mathrm{d}w \right\} = \frac{a_1 a_2 \mathrm{d}w}{\pi^2} \int_{-\infty}^{\infty} \frac{1 + y^2}{y^2 + a_2^2} \frac{\mathrm{d}y}{y^2(w^2 a_1^2 + 1) + 2yw(a_1^2 - 1) + w^2 + a_1^2}.$$

The integral can be conveniently rewritten as

$$\Pr\left\{ \frac{C_1 + C_2}{1 - C_1 C_2} \in \mathrm{d}w \right\} = \frac{a_1 a_2 \mathrm{d}w}{\pi^2} \int_{-\infty}^{\infty} \left[\frac{Ay + B}{y^2 + a_2^2} + \frac{Cy + D}{y^2(w^2 a_1^2 + 1) + 2yw(a_1^2 - 1) + w^2 + a_1^2} \right] \mathrm{d}y \quad (2.2)$$

where

$$\begin{cases} A = \frac{2w(a_1^2-1)(a_2^2-1)}{[w^2(1-a_1a_2)^2+(a_1-a_2)^2][w^2(1+a_1a_2)^2+(a_1+a_2)^2]}, \\ B = -\frac{(a_2^2-1)[a_1^2-a_2^2+w^2(1-a_2^2a_1^2)]}{[w^2(1-a_1a_2)^2+(a_1-a_2)^2][w^2(1+a_1a_2)^2+(a_1+a_2)^2]}, \\ C = -A(a_1^2 w^2 + 1), \\ D = \frac{(1-a_1^2)[w^4(1-a_2^2a_1^2)+w^2(1-a_1^2)(3-a_2^2)-(a_1^2-a_2^2)]}{[w^2(1-a_1a_2)^2+(a_1-a_2)^2][w^2(1+a_1a_2)^2+(a_1+a_2)^2]}. \end{cases}$$

Two terms of (2.2) can be developed in the following manner

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[\frac{Ay}{y^2 + a_2^2} + \frac{Cy}{y^2(w^2 a_1^2 + 1) + 2yw(a_1^2 - 1) + w^2 + a_1^2} \right] \mathrm{d}y \\ = \; &\frac{A}{2} \int_{-\infty}^{\infty} \left[\frac{2y}{y^2 + a_2^2} - \frac{2y(a_1^2 w^2 + 1) \pm 2w(a_1^2 - 1)}{y^2(w^2 a_1^2 + 1) + 2yw(a_1^2 - 1) + w^2 + a_1^2} \right] \mathrm{d}y \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[\frac{2y}{y^2 + a_2^2} - \frac{2y(a_1^2 w^2 + 1) + 2w(a_1^2 - 1)}{y^2(w^2 a_1^2 + 1) + 2yw(a_1^2 - 1) + w^2 + a_1^2} \right] \mathrm{d}y \\ = \; &\lim_{d \to \infty, c \to -\infty} \log \frac{a_2^2 + y^2}{y^2(w^2 a_1^2 + 1) + 2yw(a_1^2 - 1) + w^2 + a_1^2} \Bigg|_c^d = 0, \end{aligned}$$

and by means of the change of variable

$$y\sqrt{w^2 a_1^2 + 1} + \frac{w(a_1^2 - 1)}{\sqrt{w^2 a_1^2 + 1}} = z\sqrt{w^2 + a_1^2 - \frac{w^2(a_1^2 - 1)^2}{w^2 a_1^2 + 1}} = z\frac{a_1(w^2 + 1)}{\sqrt{w^2 a_1^2 + 1}},$$

the last integral in (2.3) reduces to the form

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}y}{y^2(w^2a_1^2+1)+2yw(a_1^2-1)+w^2+a_1^2} = \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\left[y\sqrt{w^2a_1^2+1}+\frac{w(a_1^2-1)}{\sqrt{w^2a_1^2+1}}\right]^2+w^2+a_1^2-\frac{w^2(a_1^2-1)^2}{w^2a_1^2+1}}$$

$$= \frac{1}{a_1(w^2+1)}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{z^2+1} = \frac{\pi}{a_1(w^2+1)}. \tag{2.4}$$

Result (2.4) leads us to the final expression of the probability

$$\Pr\left\{\frac{C_1+C_2}{1-C_1C_2}\in \mathrm{d}w\right\} = \frac{a_1a_2}{\pi^2}\left[\frac{2w^2(a_2^2-1)(a_1^2-1)^2}{[w^2(1-a_1a_2)^2+(a_1-a_2)^2][w^2(1+a_1a_2)^2+(a_1+a_2)^2]}\frac{\pi}{a_1(w^2+1)} + \frac{B\pi}{a_2}+\frac{D\pi}{a_1(w^2+1)}\right]\mathrm{d}w. \tag{2.5}$$

The most clumsy part of the proof consists in summing up the three terms of (2.5), after some algebra, (2.5) takes the form

$$\begin{aligned}
&\Pr\left\{\frac{C_1+C_2}{1-C_1C_2}\in \mathrm{d}w\right\}\\
=\ &\frac{a_1a_2}{\pi}\frac{(1+w^2)(a_2+a_1)(1+a_2a_1)[w^2(1-a_2a_1)^2+(a_1-a_2)^2]}{[w^2(1-a_1a_2)^2+(a_1-a_2)^2][w^2(1+a_1a_2)^2+(a_1+a_2)^2]a_1a_2(1+w^2)}\mathrm{d}w\\
=\ &\frac{1}{\pi}\frac{\frac{a_2+a_1}{1+a_2a_1}}{\left(\frac{a_2+a_1}{1+a_2a_1}\right)^2+w^2}\mathrm{d}w.
\end{aligned}$$

This concludes the first proof.

PROOF 2

An alternative proof is based on the properties of the standard Cauchy distribution:

- If $\mathbf{C}$ is a standard Cauchy random variable, then $\frac{1}{\mathbf{C}}$ is also a standard Cauchy random variable.
- If $\mathbf{C}_1$ and $\mathbf{C}_2$ are independent standard Cauchy, then $\frac{\mathbf{C}_1+\mathbf{C}_2}{1-\mathbf{C}_1\mathbf{C}_2}$ is also a standard Cauchy as a direct proof easily shows.

We therefore have the following identities in distribution

$$\begin{aligned}
U = \frac{C_1+C_2}{1-C_1C_2} &\overset{i.d.}{=} \frac{(a_1+a_2)\mathbf{C}}{1-a_1a_2\mathbf{C}_1\mathbf{C}_2} \overset{i.d.}{=} \frac{(a_1+a_2)\mathbf{C}}{1+a_1a_2[1-\mathbf{C}_1\mathbf{C}_2-1]}\\
&\overset{i.d.}{=} \frac{(a_1+a_2)\mathbf{C}}{\frac{1-a_1a_2}{\mathbf{C}_1+\mathbf{C}_2}+\frac{a_1a_2(1-\mathbf{C}_1\mathbf{C}_2)}{\mathbf{C}_1+\mathbf{C}_2}}\frac{1}{\mathbf{C}_1+\mathbf{C}_2}\\
&\overset{i.d.}{=} \frac{a_1+a_2}{\frac{1-a_1a_2}{2\mathbf{C}}+\frac{a_1a_2}{\mathbf{C}}}\frac{1}{2} \overset{i.d.}{=} \frac{a_1+a_2}{1-a_1a_2+2a_1a_2}\mathbf{C}
\end{aligned}$$

and this confirms our result. ■

Remark 2.1. Since for $a_1 = a_2 = 1$ we have that U is still a standard Cauchy random variable, it follows that if $a_j = 1$ for $j = 1, \dots n$, we have

$$\sum_{j=1}^{n} \arctan C_j \overset{i.d.}{=} \arctan \mathbf{C}.$$

Remark 2.2. A simple byproduct of Theorem 2.1 is that

$$
\begin{aligned}
Ee^{i\beta U} &= \frac{1}{\pi^2}\int_{\mathbb{R}^2} e^{i\beta\frac{x+y}{1-xy}}\frac{a_1a_2\mathrm{d}x\mathrm{d}y}{(a_1^2+x^2)(a_2^2+y^2)} = \frac{1}{\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta\frac{a_1\tan\theta_1+a_2\tan\theta_2}{1-a_1a_2\tan\theta_1\tan\theta_2}}\mathrm{d}\theta_1\mathrm{d}\theta_2 \\
&= e^{-\frac{a_1+a_2}{1+a_1a_2}|\beta|}. \qquad (2.6)
\end{aligned}
$$

In (2.6) we have used the transformations $x = a_1\tan\theta_1$ and $y = a_2\tan\theta_2$. In the special case $a_1 = a_2 = 1$ the relationship (2.6) yields

$$
\begin{aligned}
e^{-|\beta|} &= \frac{1}{\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta\frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}}\mathrm{d}\theta_1\mathrm{d}\theta_2 = \frac{1}{\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta\tan(\theta_1+\theta_2)}\mathrm{d}\theta_1\mathrm{d}\theta_2 \\
&= \frac{2}{\pi^2}\int_0^{\pi} x\cos(\beta\tan x)\mathrm{d}x. \qquad (2.7)
\end{aligned}
$$

In the last step of (2.7) we have used the transformations $\theta_1+\theta_2 = x$ and $\theta_2 = y$. The integral (2.7) shows that, if (Θ_1, Θ_2) is uniform in the square $S = \{(\theta_1,\theta_2) : -\frac{\pi}{2} < |\theta_i| < \frac{\pi}{2}\}$, $i = 1, 2$, then the random variable $W = \tan(\Theta_1+\Theta_2)$ has characteristic function $e^{-|\beta|}$.

Remark 2.3. It is well-known that for a planar Brownian motion $(B_1(t), B_2(t))$ starting from (x, y) the random variable $B_1(T_y)$ possesses Cauchy distribution with parameters (x, y) where

$$T_y = \inf\{t > 0 : \; B_2(t) = 0\}.$$

If the starting points of two planar Brownian motions $(B_1^i(t), B_2^i(t))$, for $i = 1, 2$, are located on the y axis as in the Figure 4 below we have therefore that

$$
\begin{aligned}
\Theta = \Theta_1+\Theta_2 &= \arctan B_1^1(T_{a_1}) + \arctan B_1^2(T_{a_2}) \\
&\overset{i.d.}{=} \arctan C_1 + \arctan C_2 \\
&\overset{i.d.}{=} \arctan\frac{a_1+a_2}{1+a_1a_2}\mathbf{C}.
\end{aligned}
$$

where C_1 and C_2 are two independent Cauchy random variables with scale parameters a_1 and a_2 respectively and position parameter equal to zero. Therefore if the starting point of a third Brownian motion is $\left(0, \frac{a_1+a_2}{1+a_1a_2}\right)$ then $B\left(T_{\frac{a_1+a_2}{1+a_1a_2}}\right)$ represents its hitting position on the x-axis. This point forms with $(0, 1)$ and the origin a right triangle with an angle $\Theta = \Theta_1 + \Theta_2$.

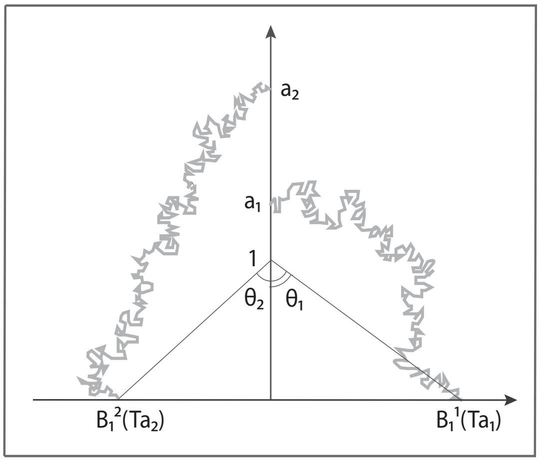
Theorem 2.2. *If $\mathbf{C}_1$ and $\mathbf{C}_2$ are two standard, independent Cauchy random variables, then*

$$\widehat{U} = \frac{\gamma\mathbf{C}_1+\delta\mathbf{C}_2}{\alpha-\beta\mathbf{C}_1\mathbf{C}_2}$$

has Cauchy distribution with scale parameter $\frac{\gamma+\delta}{\alpha+\beta}$ and location parameter equal to zero.

Proof
By applying the arguments of Theorem 2.1 we have that

$$
\begin{aligned}
\Pr\left\{\frac{\gamma\mathbf{C}_1+\delta\mathbf{C}_2}{\alpha-\beta\mathbf{C}_1\mathbf{C}_2}\in\mathrm{d}\omega\right\} &= \frac{\mathrm{d}}{\mathrm{d}\omega}\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{1+y^2}\int_{-\infty}^{\frac{\alpha\omega-\delta y}{\gamma+\omega\beta y}}\frac{\mathrm{d}x}{1+x^2}\mathrm{d}\omega \\
&= \frac{\mathrm{d}\omega}{\pi^2}\int_{-\infty}^{\infty}\left[\frac{1}{1+y^2}\frac{\alpha\gamma+\delta\beta y^2}{y^2(\omega^2\beta^2+\delta^2)+2y\omega(\beta\gamma-\delta\alpha)+\omega^2\alpha^2+\gamma^2}\right]\mathrm{d}y \qquad (2.8) \\
&= \frac{\mathrm{d}\omega}{\pi^2}\int_{-\infty}^{\infty}\left[\frac{Ay+B}{1+y^2}+\frac{Cy+D}{y^2(\omega^2\beta^2+\delta^2)+2y\omega(\beta\gamma-\delta\alpha)+\omega^2\alpha^2+\gamma^2}\right]\mathrm{d}y
\end{aligned}
$$



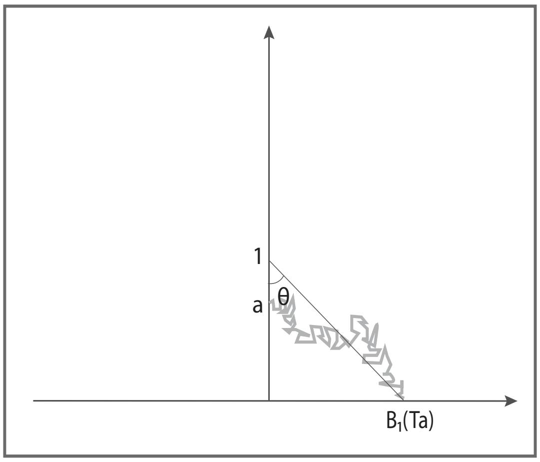


Figure 4: The hitting position on the x-axis of a planar Brownian motion is Cauchy distributed. In the figure the random angles Θ_1, Θ_2 and $\Theta = \Theta_1 + \Theta_2$ are shown.

where

$$\begin{cases} A = \frac{2\omega(\beta\gamma-\alpha\delta)(\beta\delta-\alpha\gamma)}{[\omega^2(\alpha-\beta)^2+(\gamma-\delta)^2][\omega^2(\alpha+\beta)^2+(\gamma+\delta)^2]}, \\ B = \frac{(\alpha\gamma-\beta\delta)[(\gamma^2-\delta^2)+\omega^2(\alpha^2-\beta^2)]}{[\omega^2(\alpha-\beta)^2+(\gamma-\delta)^2][\omega^2(\alpha+\beta)^2+(\gamma+\delta)^2]}, \\ C = -A(\omega^2\beta^2+\delta^2), \\ D = \frac{(\beta\gamma-\alpha\delta)[\omega^4\alpha\beta(\beta^2-\alpha^2)+\omega^2(\beta\gamma-\alpha\delta)(3\alpha\gamma-\beta\delta)+\gamma\delta(\gamma^2-\delta^2)]}{[\omega^2(\alpha-\beta)^2+(\gamma-\delta)^2][\omega^2(\alpha+\beta)^2+(\gamma+\delta)^2]}. \end{cases}$$

We start by evaluating the first part of the integral (2.8) as

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\frac{Ay}{1+y^2} + \frac{Cy}{y^2(\omega^2\beta^2+\delta^2)+2y\omega(\beta\gamma-\delta\alpha)+\omega^2\alpha^2+\gamma^2} \right] \mathrm{d}y \\ = \; & \frac{A}{2} \int_{-\infty}^{\infty} \left[\frac{2y}{1+y^2} - \frac{2y(\omega^2\beta^2+\delta^2) \pm 2\omega(\beta\gamma-\delta\alpha)}{y^2(\omega^2\beta^2+\delta^2)+2y\omega(\beta\gamma-\delta\alpha)+\omega^2\alpha^2+\gamma^2} \right] \mathrm{d}y \\ = \; & \frac{A}{2} \lim_{d\to\infty, c\to-\infty} \log\left(\frac{1+y^2}{y^2(\omega^2\beta^2+\delta^2)+2y\omega(\beta\gamma-\delta\alpha)+\omega^2\alpha^2+\gamma^2} \right)\Bigg|_c^d \\ & + A\omega(\beta\gamma-\delta\alpha) \int_{-\infty}^{\infty} \frac{1}{y^2(\omega^2\beta^2+\delta^2)+2y\omega(\beta\gamma-\delta\alpha)+\omega^2\alpha^2+\gamma^2} \mathrm{d}y \\ = \; & A\omega(\beta\gamma-\delta\alpha) \int_{-\infty}^{\infty} \frac{1}{y^2(\omega^2\beta^2+\delta^2)+2y\omega(\beta\gamma-\delta\alpha)+\omega^2\alpha^2+\gamma^2} \mathrm{d}y \\ = \; & A\omega(\beta\gamma-\delta\alpha) \frac{\pi}{\omega^2\alpha\beta+\gamma\delta}, \end{aligned} \tag{2.9}$$

where the last integral is obtained by means of the change of variable

$$y\sqrt{\omega^2\beta^2+\delta^2} + \frac{\omega(\beta\gamma-\delta\alpha)}{\sqrt{\omega^2\beta^2+\delta^2}} = z\sqrt{\omega^2\alpha^2+\gamma^2-\frac{\omega^2(\beta\gamma-\delta\alpha)^2}{\omega^2\beta^2+\delta^2}} = z\frac{\omega^2\alpha\beta+\gamma\delta}{\sqrt{\omega^2\beta^2+\delta^2}}.$$

In view of result (2.9) and inserting the values of A, B and D we have that

$$\begin{aligned}
&\frac{\mathrm{d}\omega}{\pi^2}\int_{-\infty}^{\infty}\left[\frac{Ay+B}{1+y^2}+\frac{Cy+D}{y^2(\omega^2\beta^2+\delta^2)+2y\omega(\beta\gamma-\delta\alpha)+\omega^2\alpha^2+\gamma^2}\right]\mathrm{d}y\\
=&\ \frac{\mathrm{d}\omega}{\pi}\frac{1}{\omega^2\alpha\beta+\gamma\delta}[A\omega(\beta\gamma-\alpha\delta)+B(\omega^2\alpha\beta+\gamma\delta)+D]\\
=&\ \frac{\mathrm{d}\omega}{\pi}\left[\frac{(\gamma\delta+\omega^2\alpha\beta)(\beta\gamma-\alpha\delta)[\omega^2(\beta^2-\alpha^2)+(\gamma^2-\delta^2)]}{(\omega^2\alpha\beta+\gamma\delta)[\omega^2(\alpha-\beta)^2+(\gamma-\delta)^2][\omega^2(\alpha+\beta)^2+(\gamma+\delta)^2]}\right.\\
&\left.+\frac{(\gamma\delta+\omega^2\alpha\beta)(\alpha\gamma-\beta\delta)[\omega^2(\alpha^2-\beta^2)+(\gamma^2-\delta^2)]}{(\omega^2\alpha\beta+\gamma\delta)[\omega^2(\alpha-\beta)^2+(\gamma-\delta)^2][\omega^2(\alpha+\beta)^2+(\gamma+\delta)^2]}\right]\\
=&\ \frac{\mathrm{d}\omega}{\pi}\frac{\omega^2(\beta^2-\alpha^2)(\beta-\alpha)(\gamma+\delta)+(\gamma^2-\delta^2)(\alpha+\beta)(\gamma-\delta)}{[\omega^2(\alpha-\beta)^2+(\gamma-\delta)^2][\omega^2(\alpha+\beta)^2+(\gamma+\delta)^2]}\\
=&\ \frac{\mathrm{d}\omega}{\pi}\frac{(\alpha+\beta)(\gamma+\delta)[\omega^2(\alpha-\beta)^2+(\gamma-\delta)^2]}{[\omega^2(\alpha-\beta)^2+(\gamma-\delta)^2][\omega^2(\alpha+\beta)^2+(\gamma+\delta)^2]}\\
=&\ \frac{\mathrm{d}\omega}{\pi}\frac{(\alpha+\beta)(\gamma+\delta)}{\omega^2(\alpha+\beta)^2+(\gamma+\delta)^2}.
\end{aligned}$$

■

Remark 2.4. The result of Theorem 2.2 implies that for independent, centered, Cauchy random variables with scale parameters a_1, a_2, we have that

$$\widehat{U}=\frac{\gamma C_1+\delta C_2}{\alpha-\beta C_1C_2}\overset{i.d.}{=}\frac{\gamma a_1\mathbf{C}_1+\delta a_2\mathbf{C}_2}{\alpha-\beta a_1a_2\mathbf{C}_1\mathbf{C}_2}\overset{i.d.}{=}\frac{\gamma a_1+\delta a_2}{\alpha+\beta a_1a_2}\mathbf{C}.$$

3 Non-Centered Cauchy random variables

For independent Cauchy random variables C_1 and C_2, with location parameters b_1 and b_2 and scale parameters equal to one, the random variable U is still Cauchy distributed with both parameters affected by the values of the location parameters b_1 and b_2.

Theorem 3.1. *If C_1 and C_2 are two independent Cauchy random variables with location parameter b and scale parameter equal to one, then the random variable U is still Cauchy distributed with scale parameter $\frac{2b^2+4}{b^4+4}$ and position parameter $\frac{2b^3}{b^4+4}$.*

PROOF
Since $C_i\overset{i.d.}{=}\mathbf{C}_i+b$, $i=1,2$, we have the following relationships which hold in distribution:

$$\begin{aligned}
U\ &=\ \frac{C_1+C_2}{1-C_1C_2}\overset{i.d.}{=}\frac{\mathbf{C}_1+\mathbf{C}_2+2b}{1-\mathbf{C}_1\mathbf{C}_2-b(\mathbf{C}_1+\mathbf{C}_2)-b^2}\overset{i.d.}{=}\frac{\mathbf{C}_1+\mathbf{C}_2+2b}{\frac{1-\mathbf{C}_1\mathbf{C}_2}{\mathbf{C}_1+\mathbf{C}_2}-b-\frac{b^2}{\mathbf{C}_1+\mathbf{C}_2}}\frac{1}{\mathbf{C}_1+\mathbf{C}_2}\\
&\overset{i.d.}{=}\frac{1+\frac{2b}{\mathbf{C}_1+\mathbf{C}_2}}{\frac{1-\mathbf{C}_1\mathbf{C}_2}{\mathbf{C}_1+\mathbf{C}_2}-b-\frac{b^2}{\mathbf{C}_1+\mathbf{C}_2}}\overset{i.d.}{=}\frac{1+\frac{2b}{2\mathbf{C}}}{\frac{1}{\mathbf{C}}-b-\frac{b^2}{2\mathbf{C}}}\overset{i.d.}{=}\frac{1+b\mathbf{C}}{\mathbf{C}(1-\frac{b^2}{2})-b}.
\end{aligned}$$

In the steps above we repeatedly used the properties of the standard Cauchy distribution and also Theorem (2.2).

These transformations permit us to write down the distribution of U as

$$\begin{aligned}
\frac{1}{\mathrm{d}\omega}\Pr\left\{\frac{C_1+C_2}{1-C_1C_2}\in \mathrm{d}\omega\right\} &= \frac{\mathrm{d}}{\mathrm{d}\omega}\Pr\left\{\frac{1+b\mathbf{C}}{\mathbf{C}(1-\frac{b^2}{2})-b}<w\right\}=\frac{\mathrm{d}}{\mathrm{d}\omega}\Pr\left\{\mathbf{C}>\frac{1+bw}{w(1-\frac{b^2}{2})-b}\right\}\\
&= \frac{\mathrm{d}}{\mathrm{d}\omega}\int_{\frac{1+bw}{w(1-\frac{b^2}{2})-b}}^{\infty}\frac{\mathrm{d}x}{\pi(x^2+1)}=\frac{1}{\pi}\frac{\frac{b^2}{2}+1}{[w(1-\frac{b^2}{2})-b]^2+(1+wb)^2}\\
&= \frac{1}{\pi}\frac{\frac{b^2}{2}+1}{w^2(\frac{b^4}{4}+1)+wb^3+b^2+1}=\frac{1}{\pi}\frac{\frac{2b^2+4}{b^4+4}}{w^2+w\frac{b^3}{\frac{b^4}{4}+1}+\frac{b^2+1}{\frac{b^4}{4}+1}\pm\left(\frac{\frac{b^2}{2}+1}{\frac{b^4}{4}+1}\right)^2}\\
&= \frac{1}{\pi}\frac{\frac{2b^2+4}{b^4+4}}{w^2+w\frac{b^3}{\frac{b^4}{4}+1}+\frac{b^2+1}{\frac{b^4}{4}+1}-\left(\frac{\frac{b^2}{2}+1}{\frac{b^4}{4}+1}\right)^2+\left(\frac{2b^2+4}{b^4+4}\right)^2}\\
&= \frac{1}{\pi}\frac{\frac{2b^2+4}{b^4+4}}{\left[w+\frac{2b^3}{b^4+4}\right]^2+\left(\frac{2b^2+4}{b^4+4}\right)^2}.
\end{aligned}$$

■

Remark 3.1. The result of Theorem 3.1 shows that U has center of symmetry on the positive half-line if $b>0$ and on the negative half-line if $b<0$, therefore the non linear transformation U preserves the sign of the mode.

We have now the following generalization of Theorem 3.1.

Theorem 3.2. *If C_i, $i=1,2$, are two independent, Cauchy random variables with location parameters b_i and scale parameters a_i, then the random variable U is still Cauchy distributed with scale parameter*

$$a_U=\frac{(a_1+a_2)(1+a_1a_2-b_1b_2)+(b_1+b_2)(a_1b_2+a_2b_1)}{(1+a_1a_2-b_1b_2)^2+(a_1b_2+a_2b_1)^2},$$

and position parameter

$$b_U=\frac{(a_1+a_2)(a_1b_2+a_2b_1)-(b_1+b_2)(1+a_1a_2-b_1b_2)}{(1+a_1a_2-b_1b_2)^2+(a_1b_2+a_2b_1)^2}.$$

Proof

Observing that $C_i\overset{i.d.}{=}a_i\mathbf{C}_i+b_i$ and taking into account result of Theorem 2.2, it follows that

$$\begin{aligned}
U &= \frac{C_1+C_2}{1-C_1C_2}\overset{i.d.}{=}\frac{a_1\mathbf{C}_1+a_2\mathbf{C}_2+b_1+b_2}{1-a_1a_2\mathbf{C}_1\mathbf{C}_2-a_1b_2\mathbf{C}_1-a_2b_1\mathbf{C}_2-b_1b_2}\\
&\overset{i.d.}{=} \frac{(a_1+a_2)\mathbf{C}+b_1+b_2}{\frac{1-a_1a_2\mathbf{C}_1\mathbf{C}_2}{\mathbf{C}_1+\mathbf{C}_2}-\frac{(a_1b_2+a_2b_1)\mathbf{C}}{\mathbf{C}_1+\mathbf{C}_2}-\frac{b_1b_2}{\mathbf{C}_1+\mathbf{C}_2}}\frac{1}{\mathbf{C}_1+\mathbf{C}_2}\overset{i.d.}{=}\frac{\frac{a_1+a_2}{2}+\frac{b_1+b_2}{2\mathbf{C}}}{\frac{1+a_1a_2}{2\mathbf{C}}-\frac{a_1b_2+a_2b_1}{2}-\frac{b_1b_2}{2\mathbf{C}}}\\
&\overset{i.d.}{=} \frac{a_1+a_2+(b_1+b_2)\mathbf{C}}{(1+a_1a_2-b_1b_2)\mathbf{C}-(a_1b_2+a_2b_1)}.
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{\mathrm{d}\omega}\Pr\left\{\frac{C_1+C_2}{1-C_1C_2}\in \mathrm{d}w\right\} = \frac{\mathrm{d}}{\mathrm{d}w}\Pr\left\{\frac{a_1+a_2+(b_1+b_2)\mathbf{C}}{(1+a_1a_2-b_1b_2)\mathbf{C}-(a_1b_2+a_2b_1)}<w\right\}\\
=\;& \frac{\mathrm{d}}{\mathrm{d}w}\Pr\left\{\mathbf{C}>\frac{w(a_1b_2+a_2b_1)+a_1+a_2}{w(1+a_1a_2-b_1b_2)-(b_1+b_2)}\right\}\\
=\;& \frac{1}{\pi}\frac{(1+a_1a_2-b_1b_2)[w(a_1b_2+a_2b_1)+a_1+a_2]-(a_1b_2+a_2b_1)[w(1+a_1a_2-b_1b_2)-(b_1+b_2)]}{[w(1+a_1a_2-b_1b_2)-(b_1+b_2)]^2+[w(a_1b_2+a_2b_1)+a_1+a_2]^2}\\
=\;& \frac{\frac{(a_1+a_2)(1+a_1a_2-b_1b_2)+(b_1+b_2)(a_1b_2+a_2b_1)}{(1+a_1a_2-b_1b_2)^2+(a_1b_2+a_2b_1)^2}}{\left[w+\frac{(a_1+a_2)(a_1b_2+a_2b_1)-(b_1+b_2)(1+a_1a_2-b_1b_2)}{(1+a_1a_2-b_1b_2)^2+(a_1b_2+a_2b_1)^2}\right]^2+\left[\frac{(a_1+a_2)(1+a_1a_2-b_1b_2)+(b_1+b_2)(a_1b_2+a_2b_1)}{(1+a_1a_2-b_1b_2)^2+(a_1b_2+a_2b_1)^2}\right]^2}.
\end{aligned}
$$

■

Remark 3.2. From Theorem 3.2 it is possible to obtain as a particular case the result of Theorem 3.1 by assuming that $a_1 = a_2 = 1$ and $b_1 = b_2 = b$. For $a_1 = a_2 = a$ and $b_1 = b_2 = b$ we have that U is still Cauchy distributed with scale parameter

$$a_U = \frac{2a(1+a^2+b^2)}{(1+a^2-b^2)^2+(2ab)^2},$$

and position parameter

$$b_U = \frac{2b(a^2+b^2-1)}{(1+a^2-b^2)^2+(2ab)^2}.$$

We note that a_U and b_U depend simultaneously from the scale and location parameters of the random variables involved in U.

4 Continued Fractions

The property that the reciprocal of a Cauchy random variable has still a Cauchy distribution has a number of possible extensions which we deal with in this section.

We start by considering the sequence

$$V_1 = \frac{1}{1+\mathbf{C}}, \qquad V_2 = \frac{1}{1+\frac{1}{1+\mathbf{C}}}, \dots \qquad V_n = \frac{1}{1+\frac{1}{1+\dots\frac{1}{1+\mathbf{C}}}}, \tag{4.1}$$

and show the following theorem.

Theorem 4.1. *The random variables defined in (4.1) have Cauchy distribution $V_n \sim C_{(a_n,b_n)}$ where the scale parameters a_n and the location parameters b_n satisfy the recursive relationships*

$$a_{n+1} = \frac{a_n}{(1+b_n)^2+a_n^2}, \qquad n = 1,2,\dots \tag{4.2}$$

and

$$b_{n+1} = \frac{b_n+1}{(1+b_n)^2+a_n^2}, \qquad n = 1,2,\dots \tag{4.3}$$

Proof

Let us assume that V_n possesses Cauchy density with parameters a_n and b_n, therefore V_{n+1} writes

$$\Pr\{V_{n+1}<v\} = \Pr\left\{\frac{1}{1+V_n}<v\right\} = \Pr\left\{\frac{1}{1+a_n+b_n\mathbf{C}}<v\right\}.$$

After some computations the density of V_{n+1} can be written as

$$f_{V_{n+1}}(v) = \frac{\frac{a_n}{(1+a_n)^2+b_n^2}}{\pi\left[v-\frac{b_n+1}{(1+b_n)^2+a_n^2}\right]^2+\frac{a_n^2}{[(1+b_n)^2+a_n^2]^2}}, \qquad v\in\mathbb{R}.$$

It can be directly ascertained that V_1 possesses Cauchy distribution with parameters $a_1 = 1/2$ and $b_1 = 1/2$. ■

Remark 4.1. We have evaluated the following table of parameters a_n and b_n:

n	1	2	3	...	10^2
a_n	1/2	1/5	1/13	...	$5.77e^{-42}$
b_n	1/2	3/5	8/13	...	0.618034

For $n = 1, 2, 3$ we can observe that the scale parameters a_n coincide with the inverse of the odd-indexed Fibonacci numbers while the sequence b_n has the numerators coinciding with the even-indexed Fibonacci numbers and the denominators correspond to the odd-indexed Fibonacci numbers.

In light of the previous considerations we can show that for $n \geq 1$

$$b_n = \frac{F_{2n}}{F_{2n+1}}, \qquad a_n = \frac{1}{F_{2n+1}}, \tag{4.4}$$

where F_n, $n \geq 0$ is the Fibonacci sequence. Recalling that the Fibonacci numbers admit the following representation (it can be easily checked by induction)

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} \tag{4.5}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, we now prove that if a_n and b_n have the representation in (4.4), then also a_{n+1} and b_{n+1} can be expressed in the same form. From (4.2) and (4.3) we have

$$\begin{aligned} b_{n+1} &= \frac{\frac{F_{2n}}{F_{2n+1}} + 1}{\left(\frac{F_{2n}}{F_{2n+1}} + 1\right)^2 + \frac{1}{F_{2n+1}^2}} = \frac{F_{2n+2}F_{2n+1}}{F_{2n+2}^2 + 1} \\ &= F_{2n+2}\left[\frac{\phi^{2n+1} - (1-\phi)^{2n+1}}{\phi^{4n+4} + (1-\phi)^{4n+4} - 2\phi^{2n+2}(1-\phi)^{2n+2} + 5}\right]\sqrt{5} \\ &= F_{2n+2}\left[\frac{\phi^{2n+1} - (1-\phi)^{2n+1}}{[\phi^{2n+1} - (1-\phi)^{2n+1}]\,[\phi^{2n+3} - (1-\phi)^{2n+3}]}\right]\sqrt{5} \\ &= \frac{F_{2n+2}}{F_{2n+3}}. \end{aligned}$$

Similar calculations prove that $a_{n+1} = \frac{1}{F_{2n+3}}$. In view of representation (4.4) and (4.5), it is easy to show that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{F_{2n}}{F_{2n+1}} = \lim_{n\to\infty} \frac{1 - \left(\frac{1-\phi}{\phi}\right)^n}{\phi - (1-\phi)\left(\frac{1-\phi}{\phi}\right)^n} = \frac{1}{\phi} = \phi - 1, \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{F_{2n+1}} = 0.$$

Otherwise, observing that the sequence b_n, $n \geq 1$ is increasing, because

$$\frac{b_{n+1}}{b_n} = \frac{F_{2n+2}F_{2n+1}}{F_{2n+3}F_{2n}} = \frac{\phi^{4n+3} + (1-\phi)^{4n+3} + 1}{\phi^{4n+3} + (1-\phi)^{4n+3} - 2} \geq 1$$

and taking the limits in (4.2) and (4.3) we have that

$$L = \frac{L}{(1+H)^2 + L^2}, \qquad H = \frac{H+1}{(1+H)^2 + L^2}, \tag{4.6}$$

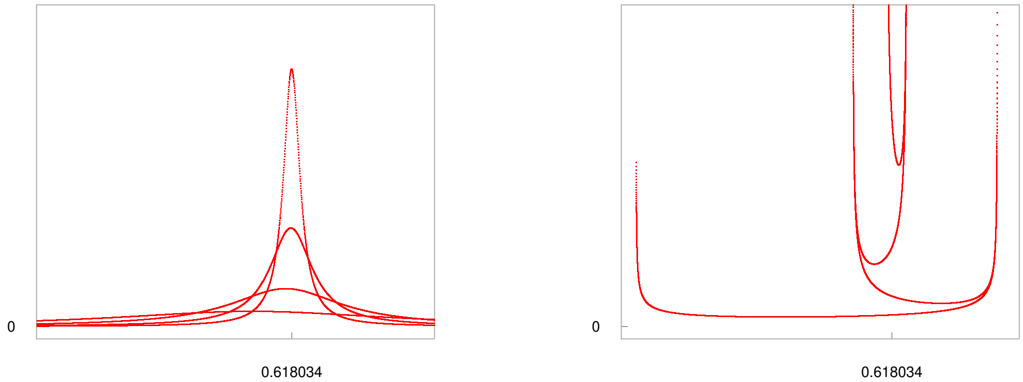


Figure 5: In the first figure are shown the densities of the Cauchy random variables V_1, V_2, V_3 and V_4. In the second figure the densities of U_1, U_2, U_3 and U_4 are plotted.

where $H = \lim_{n\to\infty} a_n$ and $L = \lim_{n\to\infty} b_n$. From the relationships in (4.6) we derive the equality

$$\frac{L}{H} = \frac{L}{H+1}$$

that implies $L = 0$. In fact, for $L \neq 0$, we arrive at the absurd that $H = H + 1$. Substituting $L = 0$ in the second formula of (4.6) we obtain

$$H = \frac{1}{H+1},$$

since H satisfies the algebraic equation $H^2 + H - 1 = 0$ it follows that $H = \phi - 1$ where ϕ is the golden ratio (see Figure 5(a)).

Remark 4.2. A slightly more general case concerns the sequence

$$W_1 = \frac{1}{c_1 + d_1 C_{(a_0,b_0)}} = \frac{1}{c_1 + a_0 d_1 + b_0 d_1 \mathbf{C}}, \qquad W_2 = \frac{1}{c_2 + d_2 W_1}, \qquad W_3 = \frac{1}{c_3 + d_3 W_2}, \dots$$

By performing calculations similar to those of Theorem 4.1 we have that W_1 has Cauchy distribution with scale parameter a_1 and position parameter b_1 such that

$$a_1 = \frac{d_1 a_0}{(c_1 + d_1 b_0)^2 + d_1^2 a_0^2}, \qquad b_1 = \frac{c_1 + d_1 b_0}{(c_1 + d_1 b_0)^2 + d_1^2 a_0^2}.$$

Similarly, if $W_n \sim C_{(a_n,b_n)}$, than $W_{n+1} \sim C_{(a_{n+1},b_{n+1})}$ where

$$a_{n+1} = \frac{d_{n+1} a_n}{(c_{n+1} + d_{n+1} b_n)^2 + d_{n+1}^2 a_n^2}, \qquad b_{n+1} = \frac{c_{n+1} + d_{n+1} b_n}{(c_{n+1} + d_{n+1} b_n)^2 + d_{n+1}^2 a_n^2} \tag{4.7}$$

for every $n \geq 2$. The sequences in (4.7) for $c_n = d_n = 1$ coincide with (4.2) and (4.3).

Another sequence of continued fractions involving the Cauchy distribution is the following one

$$U_1 = \frac{1}{1 + \mathbf{C}^2}, \qquad U_2 = \frac{1}{1 + \frac{1}{1+\mathbf{C}^2}}, \dots \qquad U_n = \frac{1}{1 + \frac{1}{1 + \dots \frac{1}{1+\mathbf{C}^2}}} \tag{4.8}$$

It is well-known that the random variable U_1 possesses the arcsin law. Unlike the sequence V_n studied above the new sequence U_n, $n \geq 1$, has a density structure changing with n. Some calculations are sufficient to show that U_1, U_2, U_3, U_4 have density, respectively equal to

$$\begin{aligned} f_{U_1}(u) &= \frac{1}{\pi\sqrt{u(1-u)}}, & 0 < u < 1, \\ f_{U_2}(u) &= \frac{1}{\pi u\sqrt{(1-u)(2u-1)}}, & \frac{1}{2} < u < 1, \\ f_{U_3}(u) &= \frac{1}{\pi(1-u)\sqrt{(2u-1)(2-3u)}}, & \frac{1}{2} < u < \frac{2}{3}, \\ f_{U_4}(u) &= \frac{1}{\pi(2u-1)\sqrt{(2-3u)(5u-3)}}, & \frac{3}{5} < u < \frac{2}{3}. \end{aligned}$$

The general result concerning U_n is stated in the next theorem.

Theorem 4.2. *For every $n \geq 1$ the distribution of the random variable U_n is given by*

$$\begin{aligned} \Pr\{U_n \in \mathrm{d}u\} &= \frac{1}{\pi[(-1)^{n+1}\alpha_n + (-1)^n\beta_n u]} \frac{1}{\sqrt{(-1)^n\beta_n + (-1)^{n+1}(\alpha_n+\beta_n)u}} \\ &\quad \times \frac{1}{\sqrt{(-1)^{n+1}(\alpha_n+\beta_n) + (-1)^n(\alpha_n+2\beta_n)u}}\mathrm{d}u, \end{aligned} \tag{4.9}$$

where

$$(-1)^n \frac{\alpha_n+\beta_n}{\alpha_n+2\beta_n} < (-1)^n u < (-1)^n \frac{\beta_n}{\alpha_n+\beta_n},$$

and $\alpha_n, \beta_n \in \mathbb{N}$ satisfy the recursive relationships $\alpha_n = \beta_{n-1}$, $\beta_n = \alpha_{n-1} + \beta_{n-1}$.

PROOF
Since from (4.8) we have that

$$U_{n+1} = \frac{1}{1+U_n},$$

proceeding by induction, that is assuming that U_n has distribution (4.9), we have that

$$\begin{aligned} \Pr\{U_{n+1} \in \mathrm{d}u\} &= \frac{\mathrm{d}}{\mathrm{d}u}\Pr\left\{U_n > \frac{1-u}{u}\right\}\mathrm{d}u = \frac{\mathrm{d}}{\mathrm{d}u}\int_{\frac{1-u}{u}}^{h(\alpha_n,\beta_n)} \Pr\{U_n \in \mathrm{d}u\} \\ &= \frac{1}{\pi}\frac{1}{u^2}\frac{1}{[(-1)^{n+1}\alpha_n + (-1)^n\beta_n(\frac{1-u}{u})]}\frac{1}{\sqrt{(-1)^n\beta_n + (-1)^{n+1}(\alpha_n+\beta_n)(\frac{1-u}{u})}} \\ &\quad \times \frac{1}{\sqrt{(-1)^{n+1}(\alpha_n+\beta_n) + (-1)^n(\alpha_n+2\beta_n)(\frac{1-u}{u})}}\mathrm{d}u \\ &= \frac{1}{\pi[(-1)^n\beta_n + (-1)^{n+1}(\alpha_n+\beta_n)u]}\frac{1}{\sqrt{(-1)^{n+1}(\alpha_n+\beta_n) + (-1)^n(\alpha_n+2\beta_n)u}} \\ &\quad \times \frac{1}{\sqrt{(-1)^n(\alpha_n+2\beta_n) + (-1)^{n+1}(2\alpha_n+3\beta_n)u}}\mathrm{d}u. \end{aligned}$$

In the first integral the function $h(\alpha_n, \beta_n)$ represents the right boundary of the support of U_n. We conclude that U_{n+1} possesses distribution (4.9) by taking $\alpha_{n+1} = \beta_n$ and $\beta_{n+1} = \alpha_n + \beta_n$. ■

Remark 4.3. The sequence β_n is a Fibonacci sequence since we have that $\beta_n = \beta_{n-1} + \alpha_{n-1} = \beta_{n-1} + \beta_{n-2}$. We note that the sequence of coefficients α_n and β_n are such that

$$\lim_{n\to\infty}\frac{\alpha_{n+1}}{\beta_{n+1}} = \lim_{n\to\infty}\frac{\beta_n}{\beta_{n+1}} = \phi - 1.$$

On the base of arguments similar to those of Remark 4.1 it is possible to show that the sequence U_n, $n \geq 1$, converges in distribution to $\phi - 1$. In this case the upper and lower bounds of the domain of definition of the densities $f_{U_n}(u)$, $n \geq 1$ are expressed as ratios of Fibonacci numbers (see Figure 5(b)).

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