

Anisotropic LRS-BI Universe with $f(Q)$ gravity theory

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The possible anisotropic nature in the early phases of the Universe is one of the interesting aspects of study in cosmology. We investigate the evolution of the Universe in terms of few cosmological parameters considering an anisotropic locally rotationally symmetric (LRS) Bianchi type-I spacetime (LRS-BI) under the $f(Q)$ gravity of symmetric teleparallel theory equivalent to the GR (STEGR). In this study we consider two $f(Q)$ gravity models, viz., $f(Q) = -(Q + 2\Lambda)$, a simple model with the cosmological constant Λ and the power law model, $f(Q) = -\alpha Q^n$ with two constant parameters α and n . Considering a proportionality relation between the directional Hubble parameters with a proportionality constant parameter λ we find a significant contribution of the anisotropic factor in the evolution of the early Universe for both models. The power law model shows the more dominating effect of anisotropy in comparison to the simple model depending on model parameters, especially on the parameter n . The power law model also shows some possible effects of anisotropy on the Universe's evolution in the near future. In addition, both models confirm that the anisotropy does not obtain any appreciable signature in the current stage of the Universe. From our analysis we also set rough constraints on our model parameters as $0.5 \leq \lambda \leq 1.25$, $0.75 \leq \alpha \leq 1.5$ and $0.95 \leq n \leq 1.05$.

I. INTRODUCTION

The cosmological principle suggests that the perspective of an observer on the cosmos is independent of both his position and the direction in which he is looking. This very useful assumption implies that the present-day Universe is largely isotropic and homogenous, and can be modelled perfectly by a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime metric. However, this may not have been the case in the very early stage of the Universe or in the distant future. Recent results from the Wilkinson Microwave Anisotropy Probe (WMAP) [1–3] show that the classic isotropic and homogeneous model of the Universe requires some tweaking to account for the observational evidence. The remarkable aspect of the inflationary paradigm [4] that has been supported by observation is that it isotropizes the early Universe into the FLRW geometry we observe today. One must still allow for the possibility of spatial inhomogeneity and anisotropy, and then examine its development into the observed quantity of homogeneity and isotropy, for a complete version. However, the observational evidences of smallness of Cosmic Microwave Background (CMB) quadrupole [5], low anomalies in the large scale angular distribution of CMB power spectrum [6] etc. suggest some sort of symmetries in the Universe. Again, studies show that a plane symmetric large scale geometry of the Universe with the eccentricity of the order 10^{-2} can bring the quadrupole amplitude match to the observational data without affecting the higher order multipoles of the temperature anisotropy in the CMB angular power spectrum [7]. Further, the polarization analysis of electromagnetic radiation propagating over large cosmological distances also indicates the symmetry axis in the large scale geometry of the Universe [8]. To address the symmetric nature of the Universe, one option is the locally symmetric Kantowski-Sachs spacetime, but one may also start with the Bianchi type cosmological models, which together make up a sizable and nearly exhaustive class of relativistic cosmology models that are homogeneous but not necessarily isotropic. Furthermore, the isotropic ones, which may be seen as a specific sub-case [9], can be properly understood by examining a model that is almost like FLRW but has fewer symmetries. The simplest Bianchi model which is very suitable for addressing the Universe with such a symmetry axis is the locally rotationally symmetric (LRS) Bianchi type-I (BI) model. In LRS-BI model, the metric has the spatial section with planar symmetry and an axis of symmetry [10]. Thus it has two equivalent longitudinal directions and a transverse direction. The metric of it in Cartesian coordinates is

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$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)(dy^2 + dz^2). \quad (1)$$

Here, the longitudinal directions are y and z and the transverse direction is along the x -axis. The metric describes an ellipsoidal expansion of space with the cosmic time which supports the CMB observations and hence it is widely used [10]. Also the above metric has a unique feature in the sense that starting from the most general linearly perturbed spatially flat FLRW metric, under synchronous gauge and the homogeneity bound, it is possible to obtain such an anisotropic metric (1) [11]. Furthermore, Bianchi cosmologies have gained popularity in observational cosmology in recent decades, as WMAP data [1] suggest that the standard cosmological model with a positive cosmological constant resembles the Bianchi morphology [7, 12–15]. Furthermore, according to these findings, the Universe should have been able to develop a somewhat anisotropic spatial geometry in spite of the inflation, which runs counter to the predictions of standard inflationary theories [16–22]. Very recently, a wide range of Bianchi cosmology with the observational data have been studied (see details in [23–28]). Investigations on various aspects of anisotropic cosmology in classical GR by using the LRS-BI metric have been done extensively so far. In Ref. [10] the connection between deceleration parameter and cosmic shear, and jerk parameter and ellipsoidal Universe has been studied in GR theory by using the LRS-BI metric. Another work containing the study of the ellipsoidal Universe in Braneworld cosmology has been found in Ref. [29]. L. Campanellie et al. in 2011 had tested the isotropy of the Universe with type Ia supernovae data in the LRS-BI framework [30]. The work on LRS-BI model to examine the anisotropy in the Universe by using the concept of dynamically anisotropic dark energy and constant deceleration parameter for perfect fluid is found in Ref. [8]. A wide range of studies have also been carried out in Modified Theories of Gravity (MTGs) like $f(R)$, $f(R, T)$ etc. by using the LRS-BI model of the Universe. X. Liu et al. in 2017 had studied the cosmological dynamics of a uniform magnetic field in the LRS-BI Universe for the viable $f(R)$ models [31]. The study of anisotropic cosmological model in $f(R, T)$ gravity with variable deceleration parameter for the LRS-BI metric has been found in Ref. [32]. Moreover, the study of de-Sitter and bounce solution in $f(R, T)$ cosmology for LRS-BI Universe has been carried out in the Ref. [33]. Recently the works on anisotropy cosmology by using the LRS-BI metric have been also performed in the $f(Q)$ -gravity theory. Various cosmological profiles, like energy density, equation of state, skewness parameters have been studied in $f(Q)$ -gravity by using LRS-BI metric in Ref. [34]. Another work on the anisotropic Universe in $f(Q)$ -gravity with the consideration of Hybrid expansion law (HEL) for the average scale factors has been accomplished in Ref. [35].

In addition, the accelerated expansion of the Universe was confirmed in 1998 by the Supernovae Cosmology Project, which utilised IA Supernovae data [36]. This gives rise to a variety of explanations for the acceleration. In general relativity (GR), the existence of a yet undetected unknown kind of energy in the Universe, known as dark energy (DE), with exotic properties such as negative pressure results in a negative equation of state (EoS) parameter. To circumvent the mysterious DE, as an alternative to GR, modified gravity theories were investigated by altering the Einstein-Hilbert action while maintaining the geometry. $f(R)$ -gravity theories are the simplest and most successful in this regard. Other types of theories, such as the teleparallel theory equivalent to the GR (TEGR) and the symmetric teleparallel theory equivalent to the GR (STEGR), have been addressed in the past by modifying the underlying geometry without disrupting the Lagrangian. In these theories, flat space is considered, and the extremely special (symmetric and metric-compatible) Levi-Civita connection used in GR is replaced by an affine connection with either non-vanishing torsion (in TEGR) or non-metricity (in STEGR) as the guiding force of gravity, and its extension $f(Q)$ theory (Q is the non-metricity scalar) was proposed in [37].

Recently, the $f(Q)$ -gravity becomes the point of wide attention, see [34, 38–58] and the references therein for the recent theoretical studies and cosmological and astrophysical applications. However, the majority of the aforementioned cosmological studies have focused on exploring the present interests of the Universe by considering the spatially flat isotropic and homogeneous FLRW metric in Cartesian coordinates. To also note that, the particular line element, so chosen, is automatically a coincident gauge, which reduces the covariant derivative to merely a partial derivative and simplifies the calculation considerably. In our present work we are aiming to study the evolution of the Universe in $f(Q)$ -gravity starting from an anisotropic but homogeneous background metric which offers a broader outlook in this new gravity theory and its cosmological application.

The present article is organized as follows: In the next section the basic formalism of $f(Q)$ -gravity has been discussed. The equations of motion for the LRS-BI model defined by the metric (1) in $f(Q)$ -gravity are derived in section III. In section IV the expressions of DE EoS and other cosmological parameters are developed for the LRS-BI

model. For two different $f(Q)$ models we analyse the cosmological parameters using the numerical calculations and corresponding graphical representations in section V. Finally, in section VI summary and conclusion of the work are drawn with the future prospects of the study.

II. BASIC FORMALISM OF $f(Q)$ GRAVITY

In $f(Q)$ gravity theory, the spacetime is constructed by using the symmetric teleparallelism and non-metricity condition, that is $R^\rho{}_{\sigma\mu\nu} = 0$ and $Q_{\lambda\mu\nu} := \nabla_\lambda g_{\mu\nu} \neq 0$. The associated connection coefficient is given by

$$\Gamma^\lambda{}_{\mu\nu} = \mathring{\Gamma}^\lambda{}_{\mu\nu} + L^\lambda{}_{\mu\nu}, \quad (2)$$

where $\mathring{\Gamma}^\lambda{}_{\mu\nu}$ is the Levi-Civita connection and $L^\lambda{}_{\mu\nu}$ is the disformation tensor. The disformation tensor is expressed as

$$L^\lambda{}_{\mu\nu} = \frac{1}{2}(Q^\lambda{}_{\mu\nu} - Q_\mu{}^\lambda{}_\nu - Q_\nu{}^\lambda{}_\mu).$$

In addition, we define the superpotential tensor

$$P^\lambda{}_{\mu\nu} := \frac{1}{4} \left(-2L^\lambda{}_{\mu\nu} + Q^\lambda g_{\mu\nu} - \tilde{Q}^\lambda g_{\mu\nu} - \frac{1}{2}\delta_\mu^\lambda Q_\nu - \frac{1}{2}\delta_\nu^\lambda Q_\mu \right), \quad (3)$$

and using it the non-metricity scalar is defined as

$$Q = -Q_{\lambda\mu\nu} P^{\lambda\mu\nu}. \quad (4)$$

The action of $f(Q)$ gravity is given by

$$S = \int \left[\frac{1}{2\kappa} f(Q) + \mathcal{L}_m \right] \sqrt{-g} d^4x,$$

where $\kappa = 8\pi G_N$, G_N being the usual Newtonian gravitational constant, g is the determinant of the metric tensor and \mathcal{L}_m is the matter Lagrangian. By varying the action with respect to the metric, we obtain the field equations of $f(Q)$ gravity as

$$\frac{2}{\sqrt{-g}} \nabla_\lambda (\sqrt{-g} f_Q P^\lambda{}_{\mu\nu}) + \frac{1}{2} f(Q) g_{\mu\nu} + f_Q (P_{\nu\rho\sigma} Q_\mu{}^{\rho\sigma} - 2P_{\rho\sigma\mu} Q^{\rho\sigma}{}_\nu) = -\kappa T_{\mu\nu}, \quad (5)$$

where f_Q represents the derivative of $f(Q)$ with respect to non-metricity scalar Q and $T_{\mu\nu}^m$ is the energy-momentum tensor of the matter field.

On the other hand, using the connection coefficient (2), we can have the following relations between the curvature tensors corresponding to Γ and $\mathring{\Gamma}$:

$$R^\rho{}_{\sigma\mu\nu} = \mathring{R}^\rho{}_{\sigma\mu\nu} + \mathring{\nabla}_\mu L^\rho{}_{\nu\sigma} - \mathring{\nabla}_\nu L^\rho{}_{\mu\sigma} + L^\rho{}_{\mu\lambda} L^\lambda{}_{\nu\sigma} - L^\rho{}_{\nu\lambda} L^\lambda{}_{\mu\sigma} \quad (6)$$

and so

$$\begin{aligned} R_{\sigma\nu} &= \mathring{R}_{\sigma\nu} + \frac{1}{2} \mathring{\nabla}_\nu Q_\sigma + \mathring{\nabla}_\rho L^\rho{}_{\nu\sigma} - \frac{1}{2} Q_\lambda L^\lambda{}_{\nu\sigma} - L^\rho{}_{\nu\lambda} L^\lambda{}_{\rho\sigma}, \\ R &= \mathring{R} + \mathring{\nabla}_\lambda Q^\lambda - \mathring{\nabla}_\lambda \tilde{Q}^\lambda - \frac{1}{4} Q_\lambda Q^\lambda + \frac{1}{2} Q_\lambda \tilde{Q}^\lambda - L_{\rho\nu\lambda} L^{\lambda\rho\nu}. \end{aligned}$$

Therefore, by using the symmetric teleparallelism condition, we can rewrite the field equations in (5) as

$$f_Q \mathring{G}_{\mu\nu} + \frac{1}{2} g_{\mu\nu} (f(Q) - Q f_Q) + 2f_{QQ} \mathring{\nabla}_\lambda Q P^\lambda{}_{\mu\nu} = -\kappa T_{\mu\nu}, \quad (7)$$

where

$$\mathring{G}_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{R}$$

is the Einstein tensor. Being a metric-affine theory, the connection acts as a dynamic variable in $f(Q)$ theory, however it has been shown in Refs. [59, 60] that in the coincident gauge choice adopted in the present setting, the affine connection's field equation is trivially satisfied in a model-independent manner. Hence, in this article our sole attention is devoted to the metric field equation (7).

III. EQUATIONS OF MOTION IN THE LRS-BI MODEL

For the current scenario, the LRS-BI metric in Cartesian coordinates given by equation (1) is also a coincident gauge. We have the directional Hubble parameters $H_x = \dot{A}/A$, $H_y = \dot{B}/B$ and the average Hubble parameter

$$(H_x + 2H_y)/3 = H = \dot{a}/a, \quad (8)$$

where a is considered to be the average scale factor in the discussed anisotropic Universe. The anisotropic evolution of the Universe is characterised by the shear scalar σ defined by

$$\sigma^2 = \frac{1}{3}(H_x - H_y)^2 = 3(H - H_y)^2. \quad (9)$$

We can calculate the non-metricity scalar Q in this spacetime metric as

$$Q = 2H_y^2 + 4H_xH_y. \quad (10)$$

From the field equations (7), we derive the following equations of motion:

$$\kappa\rho = \frac{f(Q)}{2} - 2f_Q [2H_xH_y + H_y^2], \quad (11)$$

$$\kappa p_x = -\frac{f(Q)}{2} + 2f_Q [3HH_y + \dot{H}_y] + 2H_y\dot{Q}f_{QQ}, \quad (12)$$

$$\kappa p_y = -\frac{f(Q)}{2} + f_Q [3H(3H - H_y) + \dot{H}_x + \dot{H}_y] + (H_x + H_y)\dot{Q}f_{QQ}, \quad (13)$$

where ρ is the density of the matter field, and p_x and p_y are the pressures of the field along x and y directions respectively. Under a situation of isotropic pressure, $p_x = p_y = p$ we have

$$f_Q(H - H_y) = k a^{-3}, \quad (14)$$

for a constant k . Therefore, using equation (9), alternately equation (10) can be expressed as

$$Q = 6H^2 - \frac{6k^2}{f_Q^2 a^6} = 6H^2 - 2\sigma^2. \quad (15)$$

This is a handy expression to calculate the non-metricity scalar once we have the expressions of the average Hubble parameter and the shear scalar of the considered anisotropic Universe.

IV. DARK ENERGY EOS AND OTHER COSMOLOGICAL PARAMETERS

The field equations (7) can be equivalently written in the following effective form:

$$\mathring{G}_{\mu\nu} = -\frac{\kappa}{f_Q} T_{\mu\nu}^{eff} = -\frac{\kappa}{f_Q} T_{\mu\nu} + T_{\mu\nu}^{DE}, \quad (16)$$

where $-\kappa/f_Q$ is the effective gravitational constant and for its positivity in our construction, we assume $f_Q < 0$. The DE component emerged from the modification of STEGR and is given by

$$T_{\mu\nu}^{DE} = -\frac{1}{f_Q} \left[\frac{1}{2} g_{\mu\nu} (f(Q) - Qf_Q) + 2f_{QQ} \mathring{\nabla}_\lambda Q P^\lambda{}_{\mu\nu} \right]. \quad (17)$$

Moreover, the Friedmann like equations in the STEGR determine the cosmology of an anisotropic Universe given by the metric (1) and can be calculated from equations (11) and (12) as (for simplicity we assume $\kappa = 1$)

$$\begin{aligned} \rho^{eff} &= \rho - \frac{f(Q)}{2} + 3H^2 f_Q - \frac{3k^2}{f_Q a^6} \\ &= \rho - \frac{f(Q)}{2} + 3H^2 f_Q - \sigma^2 f_Q, \end{aligned} \quad (18)$$

and

$$p^{eff} = p + \frac{f(Q)}{2} - 3H^2 f_Q - 2H\dot{Q}f_{QQ} + \frac{2k\dot{Q}f_{QQ}}{f_Q a^3}. \quad (19)$$

The additional terms arising in these two equations are due to the non-metricity of spacetime which behave like a fictitious DE fluid. Thus these DE fluid components evolving due to the non-metricity can be given by

$$\rho^{DE} = -\frac{f(Q)}{2} + 3H^2 f_Q - \frac{3k^2}{f_Q a^6}, \quad (20)$$

$$p^{DE} = \frac{f(Q)}{2} - 3H^2 f_Q - 2H\dot{Q}f_{QQ} + \frac{2k\dot{Q}f_{QQ}}{f_Q a^3}. \quad (21)$$

On a comparison with the spatially flat FLRW case [52], we could identify two novel terms $-3k^2/f_Q a^6$ and $2k\dot{Q}f_{QQ}/f_Q a^3$ appearing in the energy and pressure equations, respectively. These terms are arised from the anisotropy of the considered spacetime and these can be noted as the ρ_σ and p_σ , respectively.

Now the effective EoS can be written by using equations (18) and (19) as follows:

$$\omega^{eff} := \frac{p^{eff}}{\rho^{eff}} = \frac{p + \frac{f(Q)}{2} - 3H^2 f_Q - 2H\dot{Q}f_{QQ} + \frac{2k\dot{Q}f_{QQ}}{f_Q a^3}}{\rho - \frac{f(Q)}{2} + 3H^2 f_Q - \sigma^2 f_Q} \quad (22)$$

Further, the DE EoS can also be written by using equations (20) and (21) as given by

$$\omega^{DE} := \frac{p^{DE}}{\rho^{DE}} = -\frac{\frac{f(Q)}{2} - 3H^2 f_Q - 2H\dot{Q}f_{QQ} + \frac{2k\dot{Q}f_{QQ}}{f_Q a^3}}{\frac{f(Q)}{2} - 3H^2 f_Q + \sigma^2 f_Q}. \quad (23)$$

The effective EoS and DE EoS will help us to understand the different stages of the evolution of the Universe. Further, we have derived the expression of Hubble parameter in $f(Q)$ gravity for LRS-BI model by using the temporal component of field equation i.e. equation (11) along with equations (10) and (15). The expression of Hubble parameter takes the form:

$$H = \sqrt{\frac{1}{6f_Q} \left[\frac{f(Q)}{2} - k\rho \right] + \frac{\sigma^2}{3}}. \quad (24)$$

With the help of this Hubble parameter expression, we can also derive other cosmological parameters like deceleration parameters, luminosity distance, distance modulus in $f(Q)$ gravity for the model.

Now, the present scenario gives us a system of three equations (11,12,13) with five unknowns H_x, H_y, ρ, p_x, p_y and the model $f(Q)$. So even for model specific analysis, we need two additional assumptions to close the system. We take the first condition in form of the barotropic EOS, $p = \omega\rho$ and the second as the simplest physically relatable condition $\sigma^2 \propto \theta^2$. This latter condition is well-known in anisotropy-based cosmology literature [61–67] and also very recently being utilised in probing $f(Q)$ theory in early universe [11], in the same article it is also showed that in radiation era, a quadratic model $f(Q) = f_0 Q^2$ reproduces the above condition. Under this proportionality, we have the relation

$$H_x = \lambda H_y, \quad (25)$$

where λ is a constant. Therefore, the average Hubble parameter becomes

$$H = \frac{(2 + \lambda)}{3} H_y. \quad (26)$$

Accordingly, equations (10, 11, 12 and 13) take the forms as follows:

$$Q = \frac{18(1+2\lambda)}{(2+\lambda)^2} H^2, \quad (27)$$

$$k\rho = \frac{f(Q)}{2} - \frac{18(1+2\lambda)}{(2+\lambda)^2} H^2 f_Q, \quad (28)$$

$$kp_x = -\frac{f(Q)}{2} + \frac{6}{2+\lambda} f_Q [3H^2 + \dot{H}] + \frac{216(1+2\lambda)}{(2+\lambda)^3} H^2 \dot{H} f_{QQ}, \quad (29)$$

$$kp_y = -\frac{f(Q)}{2} + 3f_Q \left(\frac{1+\lambda}{2+\lambda} \right) [3H^2 + \dot{H}] + \frac{108(1+\lambda)(1+2\lambda)}{(2+\lambda)^3} H^2 \dot{H} f_{QQ}. \quad (30)$$

For equation (25), the condition derived for the pressure isotropy i.e. equation (14) reduce to

$$\frac{k}{a^3} = \left(\frac{\lambda-1}{\lambda+2} \right) f_Q H. \quad (31)$$

Similarly, the effective EoS and the DE EoS given by equations (22) and (23) respectively can be written as

$$\omega^{eff} = \frac{p + \frac{f(Q)}{2} - 3H^2 f_Q - \left(\frac{6}{2+\lambda} \right) H \dot{Q} f_{QQ}}{\rho - \frac{f(Q)}{2} + 3H^2 f_Q - \sigma^2 f_Q}, \quad (32)$$

$$\omega^{DE} = -\frac{\frac{f(Q)}{2} - 3H^2 f_Q - \left(\frac{6}{2+\lambda} \right) H \dot{Q} f_{QQ}}{\frac{f(Q)}{2} - 3H^2 f_Q + \sigma^2 f_Q}. \quad (33)$$

Further, the value of \dot{Q} can be written in the following form:

$$\dot{Q} = 36H(1+2\lambda) \left[\frac{p(2+\lambda) + \frac{f(Q)(2+\lambda)}{2} - 18f_Q H^2}{6f_Q(2+\lambda)^2 + 216(1+2\lambda)H^2 f_{QQ}} \right]. \quad (34)$$

The above expression of \dot{Q} leads to the required form of effective EoS to be used for the graphical analysis purpose for various $f(Q)$ models.

For the relation between the directional Hubble parameters in equation (25), the relation between directional scale factors can be obtained as

$$A = cB^\lambda. \quad (35)$$

Here, c is a constant and for simplification we have taken $c = 1$. Thus, the average scale factor takes the form: $a = B^{\frac{\lambda+2}{3}}$ and hence the total energy density of the Universe can be found as

$$\rho = \rho_0 B^{-(1+\omega)(2+\lambda)}, \quad (36)$$

which follows from the energy-momentum conservation in the studied setting as shown in [34]. However, the scale factor parameter is not an observational parameter. Therefore it is better to express all the important cosmological parameters in terms of cosmological redshift. As the direct observational data of various parameters like Hubble parameter, distance modulus etc. against cosmological redshift are available, so by evaluating the theoretical expressions of all the cosmological parameters in terms of cosmological redshift, one can compare theoretical results with the observational data. So, if z be the cosmological redshift along the y -direction in the LRS-BI Universe, then the average scale factor a can be derived from the metric (1) in terms of z as

$$a = (AB^2)^{\frac{1}{3}} = (1+z)^{-\frac{(2+\lambda)}{3}}, \quad (37)$$

where $B = (1+z)^{-1}$ and $A = (1+z)^{-\lambda}$. All the cosmological parameters in the following sections will be expressed in terms of cosmological redshift z .

V. LRS-BI UNIVERSE WITH TWO DIFFERENT $f(Q)$ GRAVITY MODELS

In this section, we try to obtain various cosmological parameters for two different form of $f(Q)$ -gravity models to develop basic understanding about the LRS-BI Universe and also try to find out the role of anisotropy in the early stage of the Universe, or in the near past as follows:

A. $f(Q) = -(Q + 2\Lambda)$ model

First we take a simple form of $f(Q)$, i.e. $f(Q) = -(Q + 2\Lambda)$, where Λ is the cosmological constant. This model is just an extension of the $f(Q) = -Q$ model, which mimics the classical GR results in $f(Q)$ -gravity [34, 58]. Here we have added the additional cosmological constant part to get more physical and realistic results in conformity with the present day cosmology. It is to be noted here that this model in isotropic cosmology is nothing but the Λ CDM model [58, 68]. Here we are interested to see its role in LRS-BI Universe. Now, the expressions of effective EoS, DE EoS and Hubble parameter given by equations (32), (33) and (24) respectively take the forms for this model as

$$\omega^{eff} = \frac{\frac{1}{3}\Omega_{r0}(1+z)^{\frac{4(2+\lambda)}{3}} - \Omega_{\Lambda0} + \Omega_{\sigma0}(1+z)^{2(2+\lambda)}}{\Omega_{m0}(1+z)^{(2+\lambda)} + \Omega_{r0}(1+z)^{\frac{4(2+\lambda)}{3}} + \Omega_{\Lambda0}}, \quad (38)$$

$$\omega^{DE} = -1 + \frac{\Omega_{\sigma0}}{\Omega_{\Lambda0}}(1+z)^{2(2+\lambda)}, \quad (39)$$

$$H = H_0 \sqrt{\Omega_{m0}(1+z)^{(2+\lambda)} + \Omega_{r0}(1+z)^{\frac{4(2+\lambda)}{3}} + \Omega_{\Lambda0} + \Omega_{\sigma0}(1+z)^{2(2+\lambda)}}. \quad (40)$$

Here H_0 is the current Hubble parameter, $\Omega_{m0} = 8\pi\rho_{m0}/3H_0^2$ is the density parameter for the matter content, $\Omega_{r0} = 8\pi\rho_{r0}/3H_0^2$ is the density parameter for the radiation content, $\Omega_{\Lambda0} = \Lambda/3H_0^2$ is the density parameter for the vacuum energy and $\Omega_{\sigma0} = \sigma_0^2/3H_0^2$ is the density parameter for the anisotropy of the present Universe [69]. The term ρ_{m0} and ρ_{r0} are current values of matter density and radiation density of the Universe respectively. Again, the term σ_0^2 in $\Omega_{\sigma0}$ is the current value of shear scalar which is related to the shear scalar σ^2 by the relation $\sigma^2 = \sigma_0^2 a^{-6}$ and $\Omega_{\sigma0} \leq 10^{-15}$ [69, 70].

Similarly, the deceleration parameter (q) and luminosity distance (d_L) for the given $f(Q)$ model can be written as

$$q = 1 - \frac{\lambda}{2} + \left(\frac{2+\lambda}{2}\right) \left[\frac{\Omega_{r0}(1+z)^{\frac{4(2+\lambda)}{3}} - \Omega_{\Lambda0} + \Omega_{\sigma0}(1+z)^{2(2+\lambda)}}{\Omega_{m0}(1+z)^{(2+\lambda)} + \Omega_{r0}(1+z)^{\frac{4(2+\lambda)}{3}} + \Omega_{\Lambda0} + \Omega_{\sigma0}(1+z)^{2(2+\lambda)}} \right], \quad (41)$$

$$d_L = \frac{(\lambda+2)(1+z)^{\frac{2+\lambda}{3}}}{3H_0} \int_0^\infty \left[\frac{(1+z)^{\frac{\lambda-1}{3}}}{\sqrt{\Omega_{m0}(1+z)^{(2+\lambda)} + \Omega_{r0}(1+z)^{\frac{4(2+\lambda)}{3}} + \Omega_{\Lambda0} + \Omega_{\sigma0}(1+z)^{2(2+\lambda)}}} \right] dz. \quad (42)$$

With the help of equation (42) we can calculate the values of the plot reduce to distance modulus for different values of z by using the relation,

$$D_m = 5 \log d_L + 25. \quad (43)$$

To see and understand the behaviours of all these cosmological parameters at different stages of evolution of the Universe, we have to plot these parameters against the cosmological redshift z . For this purpose we have to choose a reliable range of values of the model parameter i.e. λ . To estimate this range of λ we have plotted the Hubble parameter for different values of λ along with four sets of observational data, viz. HKP and SVJO5 data [71], SJVKS10 data [72] and GCH09 data [73] on the Hubble parameter and also with the Λ CDM model prediction up to $z = 2$ as shown in Fig. 1. The plot shows consistent results with the observational data for values of λ within the range of 0.5 to 1.25 with $\lambda \neq 1$. It is to be noted that we have used Planck 2018 results [74] on the cosmological parameters for all numerical calculations.

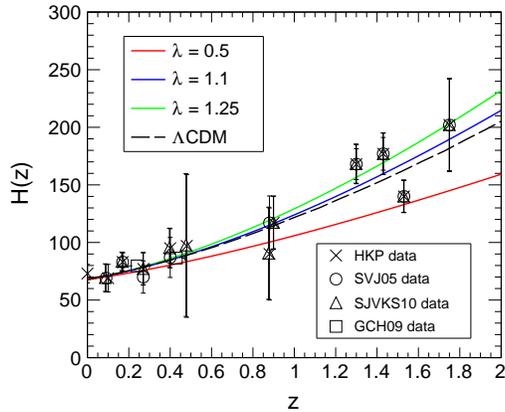


FIG. 1. Behaviour of Hubble parameter $H(z)$ with respect to z for different values of the parameter λ as predicted by the model $f(Q) = -(Q + 2\Lambda)$. The model predictions are in comparison with the four sets of Hubble parameter data, viz, HKP and SVJ05 data [71], SJVKS10 data [72] and GCH09 data [73].

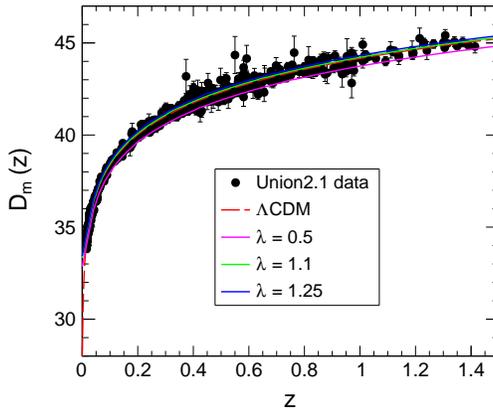


FIG. 2. Behaviour of distance Modulus D_m with respect to cosmological redshift z for different values of the parameter λ as predicted by the model $f(Q) = -(Q + 2\Lambda)$. The model predictions are in comparison with the SCP Union 2.1 observational data [75].

Similarly, have plotted distance modulus D_m against cosmological redshift z in Fig. 2 for the same set of λ values as considered in the Hubble parameter plot, along with the SCP Union 2.1 observational data [75] and the corresponding Λ CDM model prediction. This plot also shows the consistent agreement with the observational data. Based on the above two plots we have constrained the values of the model parameter λ within the range 0.5 to 1.25. And hence using the above considered set of values of λ , we have plotted the effective EoS parameter ω^{eff} , DE EoS parameter ω^{DE} and deceleration parameter q against the cosmological redshift z in Fig. 3. We have seen that both ω^{eff} and q deviate significantly from the Λ CDM case at higher values of z for all the three anisotropic cases. However the deviation decreases to some level as the λ tends to unity as expected. In the case of ω^{DE} the deviation from the standard cosmology is more vivid, prominently abrupt and takes place at smaller values of z (depending on the value of λ) in comparison to the cases of ω^{eff} and q . In this case the model agrees with Λ CDM upto $z \sim 10^2$, but for $z > 10^2$ the model shows sharp increase of the value of ω^{DE} towards the positive side. This sharp and abrupt behaviour of ω^{DE} is due to the contribution of the anisotropic factor in the early Universe as clear from equation (39). For higher z values this factor becomes substantial than the Λ CDM part. Thus this model shows that the anisotropy was prominent at the early stage of the Universe as the λ deviated from one.

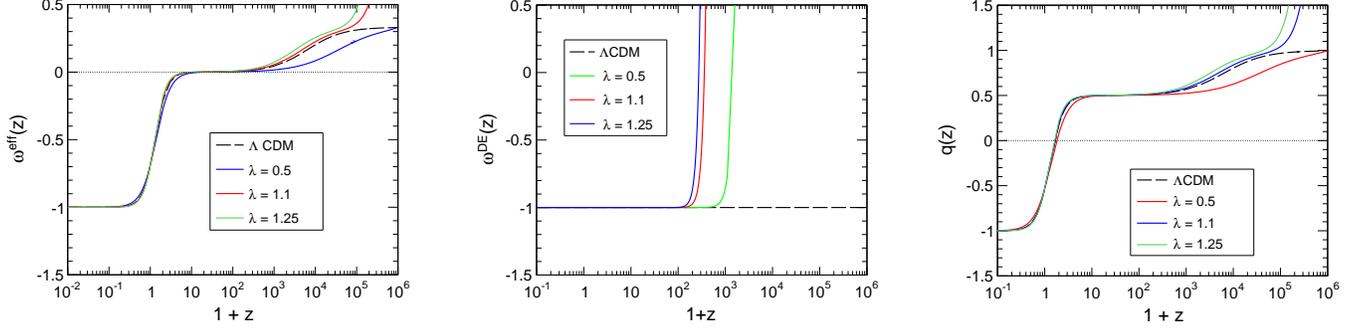


FIG. 3. Behaviour of effective EoS parameter ω^{eff} (left), DE EoS parameter ω^{DE} (middle) and deceleration parameter q (right) against the cosmological redshift z for different values of parameter λ as predicted by the model $f(Q) = -(Q + 2\Lambda)$ in comparison with that of Λ CDM model.

B. $f(Q) = -\alpha Q^n$ model

This is the second model we consider in this study, which is a power law model with a constant multiplier α and with the exponent n . The major advantage of the power law model is that it is mathematically simple yet powerful [76]. Also the power law model can explain the late time cosmic acceleration and is also consistent with BBN constrains [77]. The study of isotropic cosmology in $f(Q)$ theory while considering power law model is found in Refs. [58, 78, 79]. For this model we can express $f(Q)$ and its derivatives in terms of cosmological density parameters as follows

$$f(Q) = -\alpha Q^n = -\alpha \left[\frac{3^{\frac{1}{n}} H_0^{\frac{2}{n}} [\Omega_{mo}(1+z)^{(2+\lambda)} + \Omega_{r0}(1+z)^{\frac{4(2+\lambda)}{3}} + \Omega_{\Lambda 0}]}{(n - \frac{1}{2})^{\frac{1}{n}} \alpha^{\frac{1}{n}}} \right]^n, \quad (44)$$

$$f_Q = -\alpha n \left[\frac{3^{\frac{1}{n}} H_0^{\frac{2}{n}} [\Omega_{mo}(1+z)^{(2+\lambda)} + \Omega_{r0}(1+z)^{\frac{4(2+\lambda)}{3}} + \Omega_{\Lambda 0}]}{(n - \frac{1}{2})^{\frac{1}{n}} \alpha^{\frac{1}{n}}} \right]^{n-1}, \quad (45)$$

$$f_{QQ} = -\alpha n(n-1) \left[\frac{3^{\frac{1}{n}} H_0^{\frac{2}{n}} [\Omega_{mo}(1+z)^{(2+\lambda)} + \Omega_{r0}(1+z)^{\frac{4(2+\lambda)}{3}} + \Omega_{\Lambda 0}]}{(n - \frac{1}{2})^{\frac{1}{n}} \alpha^{\frac{1}{n}}} \right]^{n-2}. \quad (46)$$

In above equations we have used the expression of Hubble parameter for this model, which takes the form:

$$H = \sqrt{\frac{3^{\frac{1}{n}} H_0^{\frac{2}{n}} [\Omega_{mo}(1+z)^{(2+\lambda)} + \Omega_{r0}(1+z)^{\frac{4(2+\lambda)}{3}} + \Omega_{\Lambda 0}]}{6(n - \frac{1}{2})^{\frac{1}{n}} \alpha^{\frac{1}{n}}} + H_0^2 \Omega_{\sigma 0} (1+z)^{2(2+\lambda)}}. \quad (47)$$

Now, we are in a position to derive and calculate the effective EoS parameter (ω^{eff}) and DE EoS parameter (ω^{DE}) from equations (22) and (23) respectively for this model by using above expressions for $f(Q)$, f_Q and f_{QQ} . Moreover, we can derive the deceleration parameter q and luminosity distance d_L for this model by using the following expressions:

$$q = -\left(1 + \frac{\dot{H}}{H^2}\right), \quad (48)$$

$$d_L = \frac{(\lambda + 2)(1+z)^{\frac{2+\lambda}{3}}}{3} \int_0^\infty \left[\frac{(1+z)^{\frac{\lambda-1}{3}}}{H(z)} \right] dz. \quad (49)$$

Finally, using equation (49) in equation (43), we can derive the expression of the distance modulus for this power law model.

It is clear that all the cosmological parameters for this power law model depend on three model parameters, viz. λ , α and n . Similar to the case of previous model, to constrain these model parameters we have plotted the Hubble parameter and the distance modulus against cosmological redshift z for different sets of values of these parameters along with the HKP, SVJ05, SJVKS10 and GCH09 observational data and the Λ CDM plot in Figs. 4 and 5 respectively. From these plots we constrained the parameters α and n respectively as $0.75 \leq \alpha \leq 1.5$ and $0.95 \leq n \leq 1.05$ with $n \neq 1$. (2019)

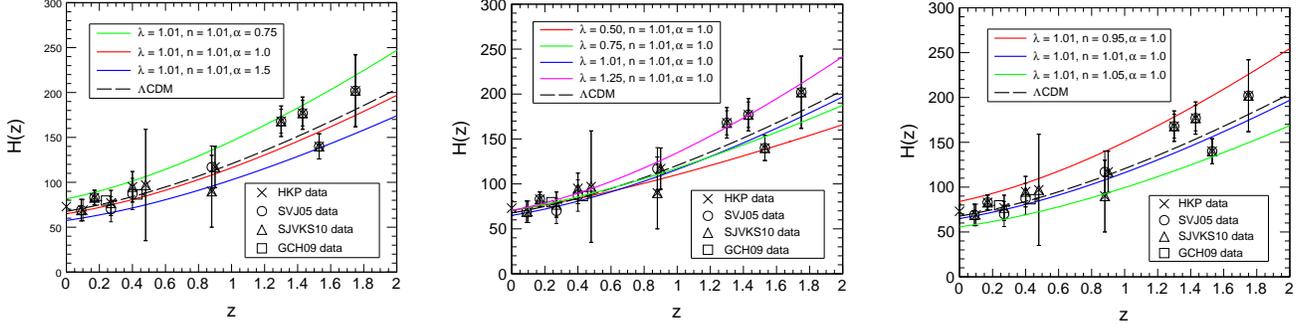


FIG. 4. Hubble parameter $H(z)$ against cosmological redshift z for different sets of values of model parameters λ , α and n as predicted by the power law model. The model predictions are in comparison with the four sets of Hubble parameter data, viz. HKP and SVJ05 data [71], SJVKS10 data [72] and GCH09 data [73].

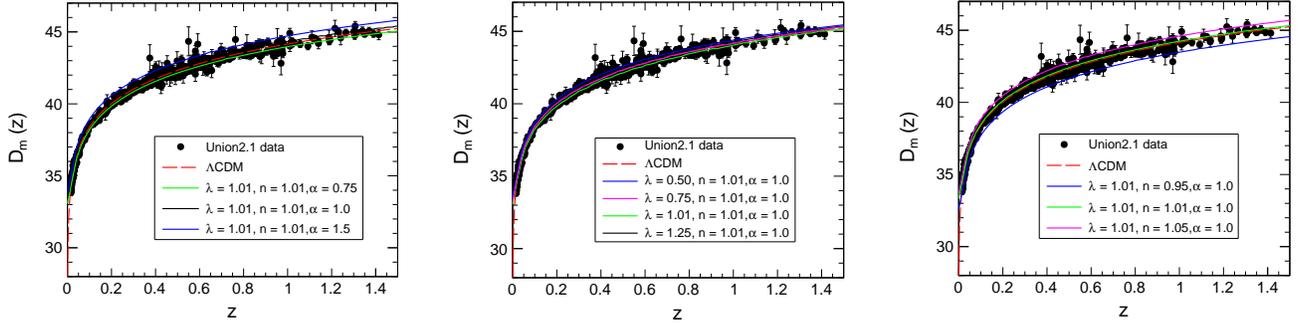


FIG. 5. Distance modulus $D_m(z)$ against cosmological redshift z for different sets of values of model parameters λ , α and n as predicted by the power law model. The model predictions are in comparison with the SCP Union 2.1 observational data [75].

We have selected three specific sets of model parameters from their constrained ranges as mentioned above to plot the effective EoS parameter ω^{eff} , DE EoS parameter ω^{DE} and deceleration parameter q against cosmological redshift z on the basis of the agreements of these cosmological parameters with the corresponding parameters in the Λ CDM Universe. It is found that behaviour of these plots are similar for the same value of n with different values of other two parameters, but the behaviour changes significantly for different values of n . Therefore we have added three plots for three different values of n within its constrained range for ω^{eff} , ω^{DE} and q in Figs. 6, 7 and 8 respectively.

From the above graphical analysis of effective EoS parameter ω^{eff} with respect to the cosmological redshift z (Fig. 6) we have observed that this parameter deviates from the Λ CDM case for higher z values. The pattern and prominence of this deviation depends on the set of model parameters used, specially on the value of the parameter n . It is seen that for higher values of n the deviation becomes prominent even from comparatively smaller values of z . However, for all sets of constrained model parameters this deviation becomes significant for the very large values of z or in the very early Universe. On the other hand the present value of ω^{eff} for all sets of model parameters agrees almost well with its Λ CDM value. From the plots of the deceleration parameter q against z (Fig. 8), one can see that this parameter deviate notably from the Λ CDM result both in the early as well in the near future of the Universe. For

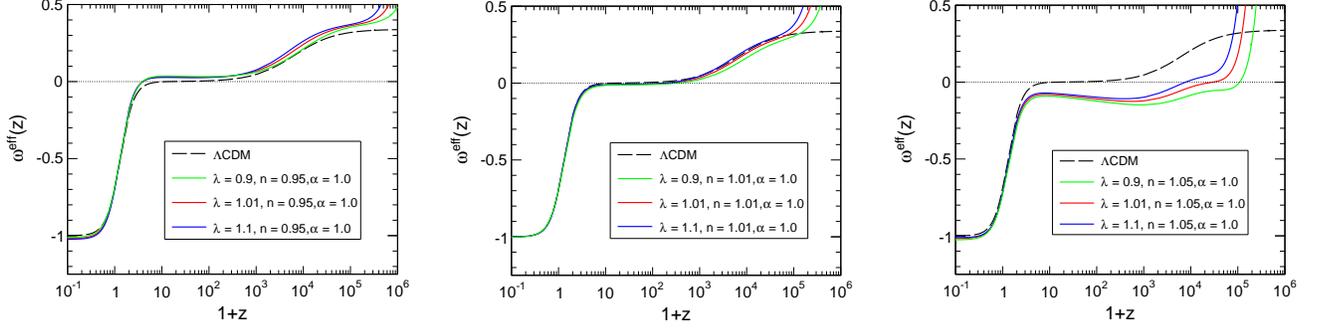


FIG. 6. Effective EoS parameter ω^{eff} against cosmological redshift z for different sets of values of model parameters as predicted by the power law model in comparison with that of Λ CDM model.

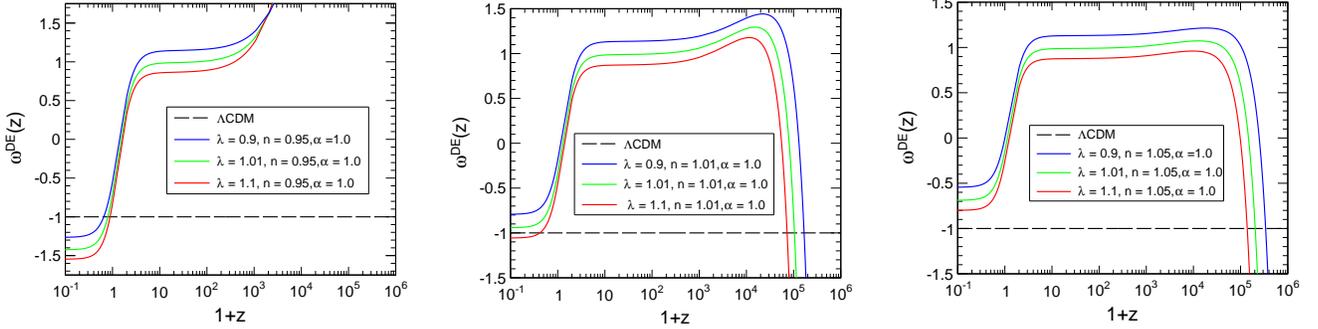


FIG. 7. DE EoS parameter ω^{DE} against cosmological redshift z for different sets of values of model parameters as predicted by the power law model in comparison with that of Λ CDM model.

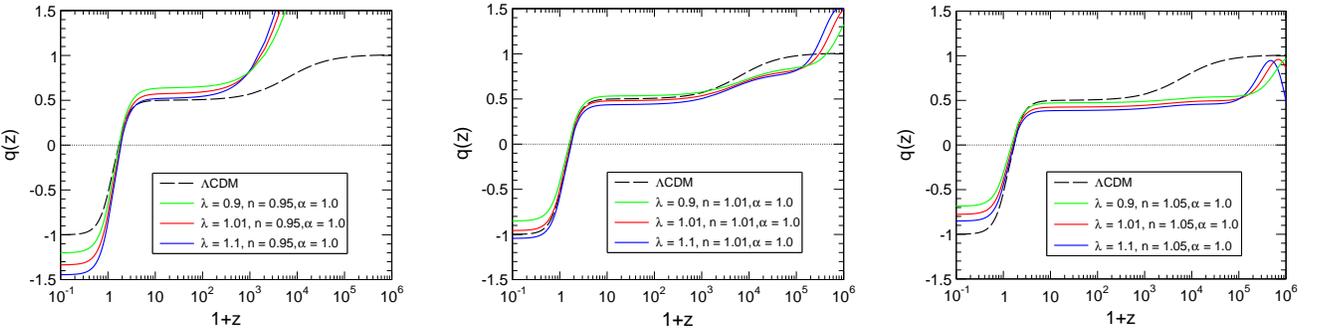


FIG. 8. Deceleration parameter $q(z)$ against cosmological redshift z for different values of model parameters as predicted by the power law model in comparison with that of Λ CDM model.

this parameter such deviations are significantly higher for smaller values of the parameter $n < 1$. For this model the behaviour of ω^{DE} (Fig. 7) is peculiar and interesting throughout the evolution of the Universe. Except for the case with $n = 0.95$, other cases with $n > 1$ within the constraint range, ω^{DE} remains positive for the values of $z < \sim 10^4$. This is also the situation to be in near feature except the case of $\lambda = 1.1$. For the values of $z > \sim 10^4$, ω^{DE} falls back to large negative values, indicating large negative pressure and hence rapid expansion of the Universe. In the case with $n = 0.95$, ω^{DE} always remains positive for $z > 0$. Here, in the near the future part ω^{DE} remains negative and appears different from the expectation of the Λ CDM model, however it agrees in the current phase ($z = 0$). Thus according to the power law model it can be concluded that although in the current scenario the role of anisotropy is not so prominent, but in the early as well as in the future Universe it may have or may play some vital role in the evolution of the Universe.

VI. DISCUSSION AND CONCLUSIONS

In this study, we have considered a symmetric teleparallel theory equivalent to GR (STTEGR), and its extension the modified $f(Q)$ gravity theory and probed it in a Universe with small anisotropy (in terms of the LRS-BI spacetime) to examine the evolution of various cosmological parameters along with the effective equation of state and hence tried to study the evolution of the Universe. Here, we have started from deriving the non metricity scalar and then deriving the field equations for the LRS-BI metric. Since LRS-BI is an anisotropic model, thus we have also derived the shear scalar for our calculations. From the field equations, we have finally derived the effective EoS, DE EoS and other cosmological parameters. To close the system in such anisotropic scenario, we have considered the physically relatable condition $\sigma^2 \propto \theta^2$ which in turn yielded the relation $H_x = \lambda H_y$, λ being a constant. We have finally derived all the field equations and other cosmological parameters in accordance with this assumption using the average scale factor in terms of redshift z .

We have considered two $f(Q)$ models in this study. In the first model, we have taken the form of $f(Q) = -(Q + 2\Lambda)$ in which Λ is the cosmological constant. All important cosmological parameters have been derived using this model and found that all these expressions contained the parameter λ . To constraint the value of λ we have used the observational data. For this purpose, at first we have plotted the expression of the Hubble parameter obtained for the model with the HKP and SVJO5 [71], SJVKS10 [72], and GCH09 [73] data in comparison with Λ CDM model for up to $z = 2$. It is found that the parameter λ shows a good agreement with data within the range 0.5 to 1.25 of its values with the required restriction $\lambda \neq 1$ for the considered anisotropy. We have further plotted the distance modulus for the obtained range of λ values along with SCP Union2.1 data [75] and found that the plots excellently agree with the observational data as well as the Λ CDM model. Based on these two plots, thus we have found that $0.5 \leq \lambda \leq 1.25$ with $\lambda \neq 1$ is a reliable range for the parameter λ in this $f(Q)$ model. The effective EoS parameter ω^{eff} , DE EoS parameter ω^{DE} and deceleration parameter q are plotted using the values of the parameter λ within the constrained range with respect to redshift up to 10^6 . We found that although the model agrees with the current scenario and the near past ($z \leq 10^2$), it deviates significantly from the Λ CDM model in the early Universe. These deviations are due to the contribution of Ω_σ parameter in the expressions. Thus, from our analysis of this considered model of $f(Q)$, we have found that anisotropy may have some contributions in the early Universe. It is to be noted that for $\lambda = 1$, i.e. in the isotropic case, the expressions reduce to that of the Λ CDM model.

The second model considered in this study is in the form of power law as $f(Q) = -\alpha Q^n$ in which α and n are the two model parameters. We have derived all expressions of cosmological parameters for this model as we did in our previous model and found that all the cosmological parameters depend on three model parameters λ , α and n in this case. We have employed the same technique to constrain these model parameters as in the previous case, i.e. using the observational data of Hubble parameter and distance modulus, and plot them with the results from our expressions of Hubble parameter and distance modulus along with the Λ CDM model prediction. It is found that the effective ranges of parameters λ , α and n should be $0.5 \leq \lambda \leq 1.25$ with $\lambda \neq 1$, $0.75 \leq \alpha \leq 1.5$ and $0.95 \leq n \leq 1.05$ with $n \neq 1$ respectively. With these values of model parameters we have plotted the effective EoS parameter ω^{eff} , DE EoS parameter ω^{DE} and deceleration parameter q with respect to the cosmological redshift z and found that behaviours of these cosmological parameters are similar for same value of n with different values of other two parameters, but their behaviours change significantly for different values of n , especially at higher z values. We observed that these cosmological parameters deviate from their Λ CDM model predictions at higher values of redshift. Thus, the power law model also confirms the role of anisotropy in the early Universe. Further, we have found that these deviations are more prominent for higher values of n in the case of effective EoS as in this case even in the lower z values deviations are substantial. ω^{DE} shows prominent deviations from the Λ CDM prediction almost for all values of n with all values of other parameters nearly along the whole range of z of study with some peculiar behaviours at very early phases of the Universe depending on the value of n . In the case of the deceleration parameter q , we have found that it diverges too early for $n = 0.95$ as compared to the other two values of n . Again, for $n = 1.05$, the plot shows downward trend at very high values of z . It needs to be mentioned that all these plots reduce to Λ CDM ones, when all the three model parameters i.e. α , λ and n become equal to unity. Thus this model gives a scenario of our Universe where anisotropy has the dominating effect in its early phases and some possible effect in the near future.

Therefore from our work we have tested the $f(Q)$ theory in LRS-BI metric to understand that the anisotropic nature of the Universe by considering a specific relation between directional Hubble parameters and found the existence

of anisotropy in the early Universe for two different models of $f(Q)$ even though the signature of anisotropy is absent in the current stage of the Universe. These findings could be confirmed by the cosmological data of the early Universe, which may be available in the future from the advanced telescopes such as Thirty Meter Telescope [80], Extremely Large Telescope [81], CTA [82] etc. In this context more rigorous analysis of the anisotropic models of the Universe based on the STEGR would be interesting and necessary in near future.

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