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Calculus of Variations

Anisotropic nonlinear Neumann problems

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Abstract We consider nonlinear Neumann problems driven by the $p(z)$-Laplacian differential operator and with a p-superlinear reaction which does not satisfy the usual in such cases Ambrosetti–Rabinowitz condition. Combining variational methods with Morse theory, we show that the problem has at least three nontrivial smooth solutions, two of which have constant sign (one positive, the other negative). In the process, we also prove two results of independent interest. The first is about the L^{∞}-boundedness of the weak solutions. The second relates $W^{1,p(z)}$ and C^1 local minimizers.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear anisotropic Neumann problem:

$$\begin{cases} -\Delta_{p(z)} u(z) = f(z, u(z)) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

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Here $\Delta_{p(z)}$ denotes the $p(z)$-Laplacian differential operator, defined by

$$\Delta_{p(z)} u = \operatorname{div} \left(\|\nabla u\|^{p(z)-2} \nabla u \right),$$

with $p \in C^1(\overline{\Omega})$, $p_{\min} = \min\limits_{z \in \overline{\Omega}} p(z) > 1$ and f is a Carathéodory reaction, i.e., for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z, \zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z, \zeta)$ is continuous.

The aim of this work is to prove a "three solutions theorem" for problem (1.1), when the potential function

$$F(z, \zeta) = \int_0^\zeta f(z, s)\, ds$$

exhibits a p-superlinear growth at $\pm\infty$. This makes the energy (Euler) functional of the problem (1.1) indefinite, in particular noncoercive. Recently there have been three solutions theorems for Dirichlet problems driven by the p-Laplacian ($p = constant$). We mention the works of Bartsch–Liu [6], Carl–Perera [8], Dancer–Perera [12], Filippakis–Kristaly–Papageorgiou [20], Gasiński–Papageorgiou [23], Liu–Liu [30], Papageorgiou–Papageorgiou [34,35] and Zhang–Chen–Li [38]. From the aforementioned works, the p-superlinear case was investigated by Bartsch–Liu [6] and Filippakis–Kristaly–Papageorgiou [20]. To express the p-superlinearity of the potential $F(z, \cdot)$, they used the well known Ambrosetti–Rabinowitz condition. The other works deal either with coercive or asymptotically p-linear problems. The study of the corresponding Neumann problem (for both the p-Laplacian and the $p(z)$-Laplacian) is in some sense lagging behind. We mention the works of Aizicovici–Papageorgiou–Staicu [4], Fan–Deng [16], Mihăilescu [32]. In Aizicovici–Papageorgiou–Staicu [4] the authors deal with an equation driven by the p-Laplacian and having a potential $F(z, \cdot)$ which is p-superlinear and satisfies the Ambrosetti–Rabinowitz condition. Fan–Deng [16] consider parametric problems driven by the $p(z)$-Laplacian. More precisely, their differential operator (left hand side), has the form

$$-\Delta_{p(z)} u(z) + \lambda\, |u(z)|^{p(z)-2}\, u(z),$$

with $\lambda > 0$ being the parameter. Their reaction (right hand side) $f(z, \zeta)$ is Carathéodory, increasing in $\zeta \in \mathbb{R}$ and satisfying the Ambrosetti–Rabinowitz condition (see Theorem 1.3 of Fan–Deng [16]). They prove certain bifurcation-type results with respect to the parameter $\lambda > 0$. Finally Mihăilescu [32] considers a $p(z)$-Laplacian equation with $\inf\limits_{\Omega} p > N$ (low dimension case) and assumes a reaction with oscillatory behaviour. His approach is based on an abstract three critical points theorem for oscillatory C^1-functionals.

Partial differential equations involving variable exponents and nonstandard growth conditions, arise in many physical phenomena and have been used in elasticity, in fluid mechanics, in image restoration and in the calculus of variations. We mention the works of Acerbi–Mingione [1,2], Cheng–Levine–Rao [10], Marcellini [31], Ruzička [36], Zhikov [39]. A comprehensive survey of equations with nonstandard growth can be found in the recent paper of Harjulehto–Hästö-Lê–Nuortio [26], which has also a detailed bibliography.

Our approach is variational based on critical point theory and Morse theory (critical groups). In the process, we also produce two results of independent interest, which we present in Sect. 3. The first one concerns the boundedness of the solutions of problem (1.1), which is a prerequisite to have smoothness up to the boundary. The second result relates Sobolev and Hölder local minimizers of a large class of C^1-functionals. Our main result (three solutions theorem) is presented in Sect. 4 and produces three nontrivial smooth solutions for problem (1.1), two of which have constant sign.

In the next chapter, for the convenience of the reader, we briefly present the main mathematical tools that will be used in the analysis of the problem (1.1). We also present the main properties of the variable exponent Sobolev and Lebesgue spaces.

2 Mathematical background and hypotheses

Let

$$L_1^\infty(\Omega) = \left\{ p \in L^\infty(\Omega) : \operatorname*{ess\,inf}_{\Omega} p \geqslant 1 \right\}.$$

For $p \in L_1^\infty(\Omega)$, we set

$$p_{min} = \operatorname*{ess\,inf}_{\Omega} p \quad \text{and} \quad p_{max} = \operatorname*{ess\,sup}_{\Omega} p.$$

By $M(\Omega)$ we denote the vector space of all functions $u\colon \Omega \longrightarrow \mathbb{R}$ which are measurable. As usual, we identify two measurable functions which differ on a Lebesgue-null set. For $p \in L_1^\infty(\Omega)$, we define

$$L^{p(z)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{p(z)}\, dz < +\infty \right\}.$$

We furnish $L^{p(z)}(\Omega)$ with the following norm (known as the *Luxemburg norm*):

$$\|u\|_{p(z)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|u|}{\lambda} \right)^{p(z)} dz \leqslant 1 \right\}.$$

Also we introduce the variable exponent Sobolev space

$$W^{1,p(z)}(\Omega) = \left\{ u \in L^{p(z)}(\Omega) : \|\nabla u\| \in L^{p(z)}(\Omega) \right\}$$

and we equip it with the norm

$$\|u\|_{1,p(z)} = \|u\|_{p(z)} + \|\nabla u\|_{p(z)}.$$

An equivalent norm on $W^{1,p(z)}(\Omega)$ is given by

$$\|u\| = \inf \left\{ \lambda > 0 : \int_\Omega \left(\left(\frac{\|\nabla u\|}{\lambda} \right)^{p(z)} + \left(\frac{|u|}{\lambda} \right)^{p(z)} \right) dz \leqslant 1 \right\}.$$

In what follows, we set

$$p^*(z) = \begin{cases} \dfrac{Np(z)}{N - p(z)} & \text{if } p(z) < N, \\ +\infty & \text{if } p(z) \geqslant N. \end{cases}$$

The properties of the variable exponent Sobolev and Lebesgue spaces can be found in the papers of Kováčik–Rákosnik [27] and Fan–Zhao [18].

Proposition 2.1 *If $p \in L_1^{\infty}(\Omega)$ and $1 < p_{min} \leqslant p_{max} < +\infty$, then*

(a) the spaces $L^{p(z)}(\Omega)$ and $W^{1,p(z)}(\Omega)$ are separable reflexive Banach spaces and $L^{p(z)}(\Omega)$ is also uniformly convex;
(b) if $p, q \in C(\overline{\Omega})$, $p_{max} < N$ and $1 \leqslant q(z) \leqslant p^*(z)$ (respectively $1 \leqslant q(z) < p^*(z)$) for all $z \in \overline{\Omega}$, then $W^{1,p(z)}(\Omega)$ is embedded continuously (respectively compactly) in $L^{q(z)}(\Omega)$;
(c) $L^{p(z)}(\Omega)^* = L^{p'(z)}(\Omega)$, where $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$ and for all $u \in L^{p(z)}(\Omega)$ and $v \in L^{p'(z)}(\Omega)$, we have

$$\int\limits_{\Omega} |uv|\,dz \leqslant \left(\frac{1}{p_{min}} + \frac{1}{(p')_{min}}\right) \|u\|_{p(z)} \|v\|_{p'(z)}.$$

We introduce the following modular functions:

$$\varrho(u) = \int\limits_{\Omega} |u|^{p(z)}\,dz \quad \forall u \in L^{p(z)}(\Omega),$$

$$I(u) = \int\limits_{\Omega} \left(\|\nabla u\|^{p(z)} + |u|^{p(z)}\right) dz \quad \forall u \in W^{1,p(z)}(\Omega).$$

Proposition 2.2 *(a) For $u \neq 0$, we have*

$$\|u\|_{p(z)} = \lambda \iff \varrho\left(\frac{u}{\lambda}\right) = 1.$$

(b) We have

$$\|u\|_{p(z)} < 1 \text{ (respectively } = 1, > 1) \iff \varrho(u) < 1 \text{ (respectively} = 1, > 1).$$

(c) If $\|u\|_{p(z)} > 1$, then

$$\|u\|_{p(z)}^{p_{min}} \leqslant \varrho(u) \leqslant \|u\|_{p(z)}^{p_{max}}.$$

(d) If $\|u\|_{p(z)} < 1$, then

$$\|u\|_{p(z)}^{p_{max}} \leqslant \varrho(u) \leqslant \|u\|_{p(z)}^{p_{min}}.$$

(e) We have

$$\lim_{n \to +\infty} \|u_n\|_{p(z)} = 0 \iff \lim_{n \to +\infty} \varrho(u_n) = 0.$$

(f) We have

$$\lim_{n \to +\infty} \|u_n\|_{p(z)} = +\infty \iff \lim_{n \to +\infty} \varrho(u_n) = +\infty.$$

Similarly, we also have

Proposition 2.3 *(a) For $u \neq 0$, we have*

$$\|u\| = \lambda \iff I\left(\frac{u}{\lambda}\right) = 1.$$

(b) We have

$$\|u\| < 1 \text{ (respectively} = 1, > 1) \iff I(u) < 1 \text{ (respectively} = 1, > 1).$$

(c) If $\|u\| > 1$, *then*

$$\|u\|^{p_{min}} \leqslant I(u) \leqslant \|u\|^{p_{max}}.$$

(d) If $\|u\| < 1$, *then*

$$\|u\|^{p_{max}} \leqslant I(u) \leqslant \|u\|^{p_{min}}.$$

(e) We have

$$\lim_{n\to+\infty} \|u_n\| = 0 \quad \Longleftrightarrow \quad \lim_{n\to+\infty} I(u_n) = 0.$$

(f) We have

$$\lim_{n\to+\infty} \|u_n\| = +\infty \quad \Longleftrightarrow \quad \lim_{n\to+\infty} I(u_n) = +\infty.$$

In the study of problem (1.1), we will use the following natural spaces:

$$C_n^1(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \text{ on } \Omega \right\}$$

and

$$W_n^{1,p(z)}(\Omega) = \overline{C_n^1(\Omega)}^{\|\cdot\|},$$

with $\|\cdot\|$ being the norm of $W^{1,p(z)}(\Omega)$. Note that $C_n^1(\overline{\Omega})$ is an ordered Banach space with positive cone, defined by

$$C_+ = \left\{ u \in C_n^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega} \right\}.$$

This cone has a nonempty interior in $C^1(\overline{\Omega})$, given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega} \right\}.$$

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition*, if the following holds:

"Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that $\{\varphi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|x_n\|)\, \varphi'(x_n) \longrightarrow 0 \quad \text{in } X^* \text{ as } n \to +\infty,$$

admits a strongly convergent subsequence."

The condition is more general than the usual in critical point theory "Palais–Smale condition". However, it can be shown (see e.g., Gasiński–Papageorgiou [22]) that the deformation theorem and consequently the minimax theory of the critical values, remains valid if the Palais–Smale condition is replaced by the weaker Cerami condition.

Theorem 2.4 *If* $\varphi \in C^1(X)$ *and satisfies the Cerami condition,* $x_0, x_1 \in X, r > 0$, $\|x_0 - x_1\| > r$,

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = r\} = \eta_r,$$
$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0, 1]; X) : \gamma(0) = x_0, \gamma(1) = x_1\},$$

then $c \geqslant \eta_r$ *and* c *is a critical value of* φ.

If $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, then we defined the following sets:

$$\begin{aligned} \varphi^c &= \{z \in X : \varphi(x) \leqslant c\}, \\ \dot{\varphi}^c &= \{z \in X : \varphi(x) < c\}, \\ K^\varphi &= \left\{x \in X : \varphi'(x) = 0\right\}. \end{aligned}$$

Also, if $Y_2 \subseteq Y_1 \subseteq X$, then for every integer $k \geqslant 0$, by $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group with integer coefficients. The critical groups of φ at an isolated critical point $x_0 \in X$ with $c = \varphi(x_0)$ are defined by

$$C_k(\varphi, x_0) = H_k\left(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}\right) \quad \forall k \geqslant 0,$$

where U is a neighbourhood of x_0, such that $K^\varphi \cap \varphi^c \cap U = \{x_0\}$ (see Chang [9]). The excision property of singular homology implies that the above definition of critical groups is independent of the particular choice of the neighbourhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the Cerami condition and

$$-\infty < \inf \varphi(K^\varphi).$$

For some $c < \inf \varphi(K^\varphi)$, the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \forall k \geqslant 0$$

(see Bartsch–Li [5]). The deformation theorem (see e.g., Gasiński–Papageorgiou [22, p. 626]) implies that the above definition is independent of the particular choice of the level $c < \inf \varphi(K^\varphi)$. In fact, if $\eta < \inf \varphi(K^\varphi)$, then

$$C_k(\varphi, \infty) = H_k(X, \dot{\varphi}^\eta) \quad \forall k \geqslant 0.$$

Indeed, if $\theta < \eta < \inf \varphi(K^\varphi)$, then φ^θ is a strong deformation retract of $\dot{\varphi}^\eta$ (see e.g., Granas–Dugundji [24, p. 407]) and so

$$H_k(X, \varphi^\theta) = H_k(X, \dot{\varphi}^\eta) \quad \forall k \geqslant 0.$$

Assuming that K^φ is finite and defining

$$\begin{aligned} P(t, x) &= \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, x) t^k \quad \forall x \in K^\varphi \\ P(t, \infty) &= \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, \infty) t^k, \end{aligned}$$

we have the *Morse relation*:

$$\sum_{x \in K^\varphi} P(t, x) = P(t, \infty) + (1 + t) Q(t), \tag{2.1}$$

where $Q(t)$ is formal series in $t \in \mathbb{R}$ with integer coefficients (see Chang [9]).

In the sequel we will use the pair $\left(W_n^{1,p(z)}(\Omega), W_n^{1,p(z)}(\Omega)^*\right)$ and by $\langle \cdot, \cdot \rangle$ we will denote the duality brackets for this pair. Let $A\colon W_n^{1,p(z)}(\Omega) \longrightarrow W_n^{1,p(z)}(\Omega)^*$ be the nonlinear map, defined by

$$\langle A(u), y \rangle = \int_\Omega \|\nabla u\|^{p(z)-2} (\nabla u, \nabla y)\, dz \quad \forall u, y \in W_n^{1,p(z)}(\Omega).$$

The following result concerning A is well known (see e.g., Fan [13] or Gasiński–Papageorgiou [22]).

Proposition 2.5 *The map $A\colon W_n^{1,p(z)}(\Omega) \longrightarrow W_n^{1,p(z)}(\Omega)^*$ defined above is continuous, strictly monotone (hence maximal monotone) and of type $(S)_+$, i.e., if $u_n \longrightarrow u$ weakly in $W_n^{1,p(z)}(\Omega)$ and*

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \leqslant 0,$$

then

$$u_n \longrightarrow u \quad \text{in } W_n^{1,p(z)}(\Omega).$$

For every $r \in \mathbb{R}$, we set $r^{\pm} = \max\{\pm r, 0\}$. The notation $\|\cdot\|$ will denote the norm of the Sobolev space $W_n^{1,p(z)}(\Omega)$ and of $\mathbb{R}^N$. It will always be clear from the context which norm we use. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and for $x, y \in \mathbb{R}$, we define $x \wedge y = \min\{x, y\}$.

The hypotheses on the data of (1.1) are the following:

$\underline{H_0}$: $p \in C^1(\overline{\Omega})$ and $1 < p_{min} = \min\limits_{\overline{\Omega}} p \leqslant p_{max} = \max\limits_{\overline{\Omega}} p < N$.

$\underline{H_1}$: $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) $|f(x, \zeta)| \leqslant a(z)+c|\zeta|^{r(z)-1}$ for almost all $z \in \Omega$, all $\zeta \in \mathbb{R}$, with $a \in L^{\infty}(\Omega)_+$, $c > 0$ and $r \in C(\overline{\Omega})$, such that

$$p_{max} = \max_{\overline{\Omega}} p < r_{max} = \max_{\overline{\Omega}} r < \widehat{p}^* = \frac{N p_{min}}{N - p_{min}};$$

(ii) if

$$F(z, \zeta) = \int_0^{\zeta} f(z, s)\, ds,$$

then

$$\lim_{|\zeta|\to+\infty} \frac{F(z, \zeta)}{|\zeta|^{p_{max}}} = +\infty$$

uniformly for almost all $z \in \Omega$ and there exist $\tau \in C(\overline{\Omega})$ with $\tau(z) \in \left((r_{max} - p_{min})\frac{N}{p_{min}}, \widehat{p}^*\right)$ for all $z \in \overline{\Omega}$ and $\beta_0 > 0$, such that

$$\beta_0 \leqslant \liminf_{|\zeta|\to+\infty} \frac{f(z, \zeta)\zeta - p_{max} F(z, \zeta)}{|\zeta|^{\tau(z)}} \tag{2.2}$$

uniformly for almost all $z \in \Omega$;

(iii) there exist $c_0 > 0$ and $\delta_0 > 0$, such that

$$f(z, \zeta)\zeta \geqslant -c_0|\zeta|^{p(z)} \quad \text{for a.a. } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}$$

and

$$F(z, \zeta) \leqslant 0 \quad \text{for a.a. } z \in \Omega, \text{all } |\zeta| \leqslant \delta_0.$$

Remark 2.6 Hypothesis $H_1(ii)$ implies that the potential function $F(z,\cdot)$ is p-superlinear near $\pm\infty$. However, we do not use the usual in such cases Ambrosetti–Rabinowitz condition. Recall that the Ambrosetti–Rabinowitz condition says that there exist $\mu > p_{max}$ and $M > 0$, such that

$$0 < \mu F(z,\zeta) \leqslant f(z,\zeta)\zeta \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \geqslant M. \tag{2.3}$$

Integrating (2.3), we obtain the weaker condition

$$\widehat{c}_0 |\zeta|^{\mu} \leqslant F(z,\zeta) \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \geqslant M, \tag{2.4}$$

for some $\widehat{c}_0 > 0$. Evidently (2.4) dictates for $F(z,\cdot)$ at least μ-growth near $\pm\infty$. In particular it implies the much weaker condition

$$\lim_{|\zeta|\to+\infty} \frac{F(z,\zeta)}{|\zeta|^{p_{max}}} = +\infty \tag{2.5}$$

uniformly for almost all $z \in \Omega$.

In this work we employ (2.4) and (2.2) (see hypothesis $H_1(ii)$). Together they are weaker than the Ambrosetti–Rabinowitz condition (2.3). We mention that Fan–Deng [16] use (2.3) together with the restrictive hypothesis that $f(z,\cdot)$ is increasing. Similar conditions can be found in Costa–Magalhães [11] and Fei [19].

Example 2.7 The following function satisfies hypotheses H_1 (for the sake of simplicity we drop the z-dependence):

$$f(\zeta) = |\zeta|^{p-2}\zeta \left(\ln|\zeta| + \frac{1}{p}\right),$$

where $1 < p < +\infty$. In this case

$$F(\zeta) = \frac{1}{p}|\zeta|^p \ln|\zeta|,$$

which does not satisfy Ambrosetti–Rabinowitz condition.

Finally we mention that the results that follow remain valid, if we use a more general differential operator of the form

$$-\mathrm{div} a\,(z, \nabla u(z)) \quad \forall u \in W_n^{1,p(z)}(\Omega),$$

where

$$a(z,\zeta) = h\,(z, \|\zeta\|)\,\zeta \quad \forall (z,\zeta) \in \overline{\Omega} \times \mathbb{R}^N,$$

with $h(z,t) > 0$ for all $z \in \overline{\Omega}$, all $t > 0$ and

(i) $a \in C^{0,\alpha}\left(\overline{\Omega} \times \mathbb{R}^N; \mathbb{R}^N\right) \cap C^1\left(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}); \mathbb{R}^N\right)$, $0 < \alpha \leqslant 1$;

(ii) there exists $\widehat{c}_1 > 0$, such that

$$\left\|\nabla_{\xi} a(z,\xi)\right\| \leqslant \widehat{c}_1 \|\xi\|^{p(z)-2}$$

for all $(z,\xi) \in \overline{\Omega} \times (\mathbb{R}^N \setminus \{0\})$;

(iii) there exists $\widehat{c}_0 > 0$, such that

$$\left(\nabla_{\xi} a(z,\xi)y, y\right)_{\mathbb{R}^N} \geqslant \widehat{c}_0 \|\xi\|^{p(z)-2} \|y\|^2$$

for all $(z,\xi) \in \overline{\Omega} \times (\mathbb{R}^N \setminus \{0\})$ and all $y \in \mathbb{R}^N$;

(iv) if the potential $G(z,\xi)$ is determined by $\nabla_\xi G(z,\xi) = a(z,\xi)$ with $(z,\xi) \in \overline{\Omega} \times \mathbb{R}^N$ and $G(z,0) = 0$ for all $z \in \overline{\Omega}$, then

$$p_{max} G(z,\xi) - (a(z,\xi),\xi)_{\mathbb{R}^N} \geqslant \eta(z)$$

for almost all $z \in \Omega$, all $\xi \in \mathbb{R}^N$ with $\eta \in L^1(\Omega)$ (see Zhang [37]).

Clearly the $p(z)$-Laplacian is a particular case of such an operator. However, for simplicity in the exposition, we have decided to present everything in terms of the $p(z)$-Laplacian.

3 Two auxiliary results

Let $g\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be the Carathéodory function, such that

$$|g(z,\zeta)| \leqslant \widehat{a}(z) + \widehat{c}|\zeta|^{r(z)-1} \quad \text{for a.a. } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}, \tag{3.1}$$

with $r \in C(\overline{\Omega})$ being such that $(p^* - r)^- > 0$ and with $\widehat{a} \in L^\infty(\Omega), \widehat{c} > 0$. Also, without any loss of generality, we may assume that $(r-p)^- > 0$. We consider the following nonlinear Neumann problem

$$\begin{cases} -\Delta_{p(z)} u(z) = g(z, u(z)) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.2}$$

Any regularity result up to the boundary for the weak solutions of (3.2) (see Lieberman [29] ($p = constant$) and Fan [14] (p being variable)), requires that the weak solution belongs also in $L^\infty(\Omega)$. In the Dirichlet case, this can be deduced from Theorem 7.1 of Ladyzhenskaya–Uraltseva [28] (problems with standard growth conditions) and Theorem 4.1 of Fan–Zhao [17] (problems with nonstandard growth conditions). However, in the Neumann case, the aforementioned theorems cannot be used since they require that $u|_{\partial\Omega}$ is bounded (u being the weak solution). So, we need to show that a weak solution u of (3.2) belongs in $L^\infty(\Omega)$. We do this using a suitable variation of the Moser iteration technique.

Proposition 3.1 *If $p \in C^1(\overline{\Omega})$ satisfies hypothesis H_0, $g\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying the subcritical growth condition (3.1) and $u \in W_n^{1,p(z)}(\Omega)$ is a nontrivial weak solution of (3.2), then $u \in L^\infty(\Omega)$ and $\|u\|_\infty < M_0 = M_0\left(\|\widehat{a}\|_\infty, \widehat{c}, N, p_{max}, \|u\|_{\widehat{p}^*}\right)$.*

Proof Since $u = u^+ - u^-$ and $u^\pm \in W_n^{1,p(z)}(\Omega)$, we may assume without any loss of generality that $u \geqslant 0$.

Let

$$p_0 = \widehat{p}^* = \frac{N p_{min}}{N - p_{min}} \leqslant p^*(z) = \frac{N p(z)}{N - p(z)}$$

(recall that $p_{max} < N$; see hypothesis H_0) and recursively, define

$$p_{n+1} = \widehat{p}^* + \frac{\widehat{p}^*}{p_{max}}(p_n - r_{max}) \quad \forall n \geqslant 0.$$

Evidently the sequence $\{p_n\}_{n \geqslant 0} \subseteq \mathbb{R}_+$ is increasing. We set

$$\theta_n = p_n - r_{max} > 0 \quad \forall n \geqslant 0.$$

We have

$$A(u) = N_g(u),$$

where

$$N_g(y)(\cdot) = g\left(\cdot, y(\cdot)\right) \quad \forall y \in W_n^{1,p(z)}(\Omega). \tag{3.3}$$

For every integer $k \geqslant 1$, we set

$$u_k = \min\{u, k\} \in W_n^{1,p(z)}(\Omega) \cap L^\infty(\Omega).$$

On (3.3) we act with $u_k^{\theta_n+1} \in W_n^{1,p(z)}(\Omega)$ and we obtain

$$\left\langle A(u), u_k^{\theta_n+1} \right\rangle = \int_\Omega g(z,u) u_k^{\theta_n+1}\, dz. \tag{3.4}$$

From the definition of the map A, we have

$$\begin{aligned}
\left\langle A(u), u_k^{\theta_n+1} \right\rangle &= \int_\Omega \|\nabla u\|^{p(z)-2} \left(\nabla u,\ \nabla u_k^{\theta_n+1}\right)_{\mathbb{R}^N} dz \\
&= (\theta_n+1) \int_\Omega u_k^{\theta_n} \|\nabla u\|^{p(z)-2} \left(\nabla u,\ \nabla u_k\right)_{\mathbb{R}^N} dz \\
&= (\theta_n+1) \int_\Omega u_k^{\theta_n} \|\nabla u_k\|^{p(z)}\, dz.
\end{aligned} \tag{3.5}$$

Also, we have

$$\begin{aligned}
\nabla u_k^{\frac{\theta_n+p(z)}{p(z)}} &= \nabla u_k^{\frac{\theta_n}{p(z)}+1} \\
&= \left(\frac{\theta_n}{p(z)}+1\right) u_k^{\frac{\theta_n}{p(z)}} \nabla u_k + u_k^{\frac{\theta_n}{p(z)}+1} \left(-\frac{\theta_n}{p(z)^2}\right) (\ln u_k) \nabla p(z),
\end{aligned}$$

so

$$\left\| \nabla u_k^{\frac{\theta_n+p(z)}{p(z)}} \right\|^{p(z)} \leqslant \left(\frac{\theta_n}{p(z)}+1\right)^{p_{max}} u_k^{\theta_n} \|\nabla u_k\|^{p(z)} + c_2 u_k^{(\theta_n+p(z))} |\ln u_k|^{p(z)} \tag{3.6}$$

for some $c_2 = c_2(\theta_n) > 0$ (see hypothesis H_0). Note that

$$\lim_{\zeta \to 0^+} \zeta^{(\theta_n+p(z))} |\ln \zeta|^{p(z)} = 0.$$

Also, recall that for every $\varepsilon > 0$, we have

$$\lim_{\zeta \to +\infty} \frac{\ln \zeta}{\zeta^\varepsilon} = 0.$$

Therefore, for any $\varepsilon \in (0, r_{max} - p_{max})$, we can find $c_3 = c_3(\varepsilon) > 0$, such that

$$c_2 u_k^{(\theta_n+p(z))} |\ln u_k|^{p(z)} \leqslant c_3 \left(1 + u_k^{\theta_n+p(z)+\varepsilon}\right).$$

If we use this estimate in (3.6), we obtain

$$\begin{aligned}
\left\| \nabla u_k^{\frac{\theta_n+p(z)}{p(z)}} \right\|^{p(z)} &\leqslant (\theta_n+1)^{p_{max}} u_k^{\theta_n} \|\nabla u_k\|^{p(z)} + c_3 \left(1 + u_k^{\theta_n+p(z)+\varepsilon}\right) \\
&\leqslant (\theta_n+1)^{p_{max}} u_k^{\theta_n} \|\nabla u_k\|^{p(z)} + c_4 \left(1 + u_k^{p_n}\right),
\end{aligned}$$

for some $c_4 > 0$ (since $\theta_n + p(z) + \varepsilon < p_n(z)$ for all $z \in \overline{\Omega}$), so, using also (3.4) and (3.5), we have

$$\begin{aligned}\int_\Omega \left\| \nabla u_k^{\frac{\theta_n+p(z)}{p(z)}} \right\|^{p(z)} dz \\ &\leqslant c_5 \left(1 + \|u_k\|_{p_n}^{p_n}\right) + (\theta_n+1)^{p_{max}} \int_\Omega u_k^{\theta_n} \|\nabla u_k\|^{p(z)} dz \\ &\leqslant c_5 \left(1 + \|u_k\|_{p_n}^{p_n}\right) + (\theta_n+1)^{p_{max}-1} \int_\Omega g(z,u) u_k^{\theta_n+1} dz, \end{aligned} \tag{3.7}$$

for some $c_5 > 0$. From the growth condition on $g(z,\cdot)$ (see (3.1)), we have

$$\begin{aligned}\int_\Omega g(z,u) u_k^{\theta_n+1} dz &\leqslant \int_\Omega \left(\widehat{a}(z) u_k^{\theta_n+1} + \widehat{c} u_k^{\theta_n + r(z)}\right) dz \\ &\leqslant c_6 \left(\|u_k\|_{\theta_n+1}^{\theta_n+1} + \|u_k\|_{\theta_n+r_{max}}^{\theta_n+r_{max}}\right) \\ &\leqslant c_7 \left(1 + \|u_k\|_{p_n}^{p_n}\right), \end{aligned} \tag{3.8}$$

for some $c_6, c_7 > 0$ (since $\theta_n + 1 < \theta_n + r_{max} = p_n$). Using (3.8) in (3.7), we obtain

$$\int_\Omega \left\| \nabla u_k^{\frac{\theta_n+p(z)}{p(z)}} \right\|^{p(z)} dz + \int_\Omega \left| u_k^{\frac{\theta_n+p(z)}{p(z)}} \right|^{p(z)} dz \leqslant c_8 \left(1 + \|u_k\|_{p_n}^{p_n}\right),$$

for some $c_8 = c_8(\theta_n) > 0$, so

$$\left\| u_k^{\frac{\theta_n+p(z)}{p(z)}} \right\|^{p_{max}} \wedge \left\| u_k^{\frac{\theta_n+p(z)}{p(z)}} \right\|^{p_{min}} \leqslant c_8 \left(1 + \|u_k\|_{p_n}^{p_n}\right) \tag{3.9}$$

(see Proposition 2.3(c) and (d)). Because $\theta_n p_{max} \geqslant \theta_n p(z)$ for all $z \in \overline{\Omega}$, we have

$$\frac{\theta_n + p(z)}{p(z)} \geqslant \frac{\theta_n + p_{max}}{p_{max}}.$$

Also, by definition $p_{n+1} = \widehat{p}^* + \frac{\widehat{p}^*}{p_{max}} \theta_n$, hence

$$\frac{\theta_n + p_{max}}{p_{max}} = \frac{p_{n+1}}{\widehat{p}^*}.$$

Therefore,

$$u_k(z)^{\frac{\theta_n+p(z)}{p(z)}} \geqslant \chi_{\{u_k \geqslant 1\}} u_k^{\frac{\theta_n+p_{max}}{p_{max}}} = \chi_{\{u_k \geqslant 1\}} u_k^{\frac{p_{n+1}}{\widehat{p}^*}} \quad \text{for a.a. } z \in \Omega \tag{3.10}$$

(recall that $u_k \geqslant 0$). Note that $u_k^{\frac{p_{n+1}}{\widehat{p}^*}} \in L^{\widehat{p}^*}(\Omega)$ and from the Sobolev embedding theorem for variable exponent (see Proposition 2.1), we have that the embedding $W_n^{1,p(z)}(\Omega) \subseteq L^{\widehat{p}^*}(\Omega)$ is continuous. Since $u_k^{\frac{\theta_n+p(z)}{p(z)}} \in W_n^{1,p(z)}(\Omega)$, we have

$$c_9 \left\| u_k^{\frac{\theta_n+p(z)}{p(z)}} \right\|_{\widehat{p}^*}^{p_{max}} \leqslant \left\| u_k^{\frac{\theta_n+p(z)}{p(z)}} \right\|^{p_{max}},$$

for some $c_9 > 0$, so

$$\|u_k\|_{p_{n+1}}^{\frac{p_{n+1}}{\widehat{p}^*} p_{max}} \wedge \|u_k\|_{p_{n+1}}^{\frac{p_{n+1}}{\widehat{p}^*} p_{min}} \leqslant c_{10}\left(1 + \|u_k\|_{p_n}^{p_n}\right), \tag{3.11}$$

for some $c_{10} = c_{10}(\theta_n) > 0$. Letting $k \to +\infty$ and using the monotone convergence theorem, we obtain

$$\|u\|_{p_{n+1}}^{\frac{p_{n+1}}{\widehat{p}^*} p_{max}} \wedge \|u\|_{p_{n+1}}^{\frac{p_{n+1}}{\widehat{p}^*} p_{min}} \leqslant c_{10}\left(1 + \|u\|_{p_n}^{p_n}\right). \tag{3.12}$$

Since $p_0 = \widehat{p}^*$ and the embedding $W_n^{1,p(z)}(\Omega) \subseteq L^{\widehat{p}^*}(\Omega)$ is continuous (see Proposition 2.1), from (3.12), it follows that

$$u \in L^{p_n}(\Omega) \quad \forall n \geqslant 0. \tag{3.13}$$

Note that $p_n \longrightarrow +\infty$ as $n \to +\infty$. To see this, suppose that the increasing sequence $\{p_n\}_{n\geqslant 0} \subseteq [\widehat{p}^*, +\infty)$ is bounded. Then we have $p_n \longrightarrow \widehat{p} \geqslant \widehat{p}^*$ as $n \to +\infty$. By definition, we have

$$p_{n+1} = \widehat{p}^* + \frac{\widehat{p}^*}{p_{max}}(p_n - r_{max}) \quad \forall n \geqslant 0,$$

with $p_0 = \widehat{p}^*$, so

$$\widehat{p} = \widehat{p}^* + \frac{\widehat{p}^*}{p_{max}}(\widehat{p} - r_{max}),$$

thus

$$\widehat{p}\left(\frac{\widehat{p}^*}{p_{max}} - 1\right) = \widehat{p}^*\left(\frac{r_{max}}{p_{max}} - 1\right),$$

and so

$$\widehat{p}\left(\widehat{p}^* - p_{max}\right) = \widehat{p}^*\left(r_{max} - p_{max}\right).$$

Since $p_{max} \leqslant r_{max} < \widehat{p}^* \leqslant \widehat{p}$, we have

$$\widehat{p}^*\left(\widehat{p}^* - p_{max}\right) \leqslant \widehat{p}^*\left(r_{max} - p_{max}\right),$$

so

$$\widehat{p}^* \leqslant r_{max},$$

a contradiction.

But recall that for any measurable function $u\colon \Omega \longrightarrow \mathbb{R}$, the set

$$S_u = \left\{p \geqslant 1 : \|u\|_p < +\infty\right\}$$

is an interval. Hence $S_u = [1, +\infty)$ (see (3.13)) and so

$$u \in L^s(\Omega) \quad \forall s \geqslant 1. \tag{3.14}$$

Now let $\sigma_0 = \widehat{p}^*$ and recursively define

$$\sigma_{n+1} = (\sigma_n + p_{max} - 1)\,\frac{\widehat{p}^*}{p_{max}} \quad \forall n \geqslant 0.$$

We have that the sequence $\{\sigma_n\}_{n\geqslant 0} \subseteq [\widehat{p}^*, +\infty)$ is increasing and $\sigma_n \longrightarrow +\infty$ as $n \to +\infty$. Moreover as $\sigma_n \geqslant \widehat{p}^*$

$$(\sigma_n)' = \frac{\sigma_n}{\sigma_{n-1}} \leqslant (\widehat{p}^*)' = \frac{\widehat{p}^*}{\widehat{p}^* - 1}.$$

Using (3.14), we have

$$\int\limits_\Omega g(z,u)u^{\frac{\sigma_n}{\widehat{p}^*}}\,dz \leqslant \int\limits_\Omega \left(c_{11}(1+u^{r_{max}-1})\right) u^{\frac{\sigma_n}{\widehat{p}^*}}\,dz \leqslant c_{12}\|u\|_{\sigma_n}^{\frac{\sigma_n}{\widehat{p}^*}},$$

for some $c_{11}, c_{12} > 0$.

Repeating the estimation conducted in the first part of the proof with $\theta_n = \frac{\sigma_n}{\widehat{p}^*} - 1 \geqslant 0$ for all $n \geqslant 0$, we obtain

$$\|u\|_{\sigma_{n+1}}^{\sigma_{n+1}} \leqslant c_{13}\sigma_{n+1}^p \|u\|_{\sigma_n}^{\sigma_n}, \tag{3.15}$$

for some $c_{13} > 0$.

Since $\sigma_{n+1} > \sigma_n$ for all $n \geqslant 0$ and $\sigma_n \longrightarrow +\infty$, from (3.15), it follows that

$$\|u\|_{\sigma_{n+1}} \leqslant M_0 \quad \forall n \geqslant 0,$$

for some $M_0 = M_0\left(\|\widehat{a}\|_\infty, \widehat{c}, N, p_{max}, \|u\|_{\widehat{p}^*}\right)$, so

$$\|u\|_\infty \leqslant M_0$$

(since $\sigma_n \longrightarrow +\infty$) $\square$

Another auxiliary result which we will need in the study of problem (1.1), is the next one which relates local C_n^1-minimizers and local W_n^1-minimizers. This result too is of independent interest. For constant exponent Dirichlet Sobolev spaces, the result was obtained by Brezis–Nirenberg [7] (for $p = 2$), García Azorero–Manfredi–Peral Alonso [21] (for $p > 1$) and Guo–Zhang [25] (for $p \geqslant 2$). For variable exponent Dirichlet Sobolev spaces, the result is due to Fan [15], while for the constant exponent Neumann Sobolev spaces (i.e., for $W_n^{1,p}(\Omega)$, $1 < p < +\infty$), the result can be found in Motreanu–Motreanu–Papageorgiou [33]. Here, we extend their result to the case of the variable exponent Neumann Sobolev spaces. Moreover, our proof is simpler than those of [21,25,33], since it avoids the complicated estimates that characterize the other proofs.

So, again $p(\cdot)$ satisfies H_0, $p_{max} < \widehat{p}^* = \frac{Np_{min}}{N-p_{min}}$ and $g\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is the Carathéodory function of problem (3.2). We set

$$G(z,\zeta) = \int\limits_0^\zeta g(z,s)\,ds$$

and consider the C^1-functional $\psi\colon W_n^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\psi(u) = \int\limits_\Omega \frac{1}{p(z)}\|\nabla u\|^{p(z)}\,dz - \int\limits_\Omega G(z,u)\,dz \quad \forall u \in W_n^{1,p(z)}(\Omega).$$

We start with the simple observation concerning an equivalent norm on $W_n^{1,p(z)}(\Omega)$.

Lemma 3.2 $|u| = \|\nabla u\|_{p(z)} + \|u\|_{q(z)}$ *with* $q \in C(\overline{\Omega})$, $(p^* - q)^- > 0$ *is an equivalent norm on* $W_n^{1,p(z)}(\Omega)$.

Proof By virtue of Proposition 2.1(b), we can find $c_{14} > 0$, such that

$$\|u\|_{q(z)} \leqslant c_{14}\|u\| \quad \forall u \in W_n^{1,p(z)}(\Omega),$$

so

$$|u| \leqslant (1 + c_{14})\|u\| \quad \forall u \in W_n^{1,p(z)}(\Omega). \tag{3.16}$$

On the other hand, if $u_n \xrightarrow{|\cdot|} u$ in $W_n^{1,p(z)}(\Omega)$, then since $p_{min} \leqslant p(z)$, $q_{min} \leqslant q(z)$ for all $z \in \overline{\Omega}$, we have

$$\nabla u_n \longrightarrow \nabla u \quad \text{in } L^{p_{min}}(\Omega; \mathbb{R}^N)$$

and

$$u_n \longrightarrow u \quad \text{in } L^{q_{min}}(\Omega)$$

(see Kováčik–Rákosnik [27, Theorem 2.8]). Recall that

$$u \longmapsto \|\nabla u\|_{p_{min}} + \|u\|_{q_{min}}$$

is an equivalent norm on $W_n^{1,p_{min}}(\Omega)$ (as $q^- < \widehat{p}^*$, see e.g., Gasiński–Papageorgiou [22, Theorem 2.5.24(b), p. 227]). So, we have

$$u_n \longrightarrow u \quad \text{in } W_n^{1,p_{min}}(\Omega)$$

and thus

$$u_n \longrightarrow u \quad \text{in } L^{\theta}(\Omega)$$

for all $\theta < \widehat{p}^*$ (Sobolev embedding theorem).

In particular since $p_{max} < \widehat{p}^*$, we have

$$u_n \longrightarrow u \quad \text{in } L^{p_{max}}(\Omega)$$

and so

$$u_n \longrightarrow u \quad \text{in } L^{p(z)}(\Omega).$$

We also have

$$\nabla u_n \longrightarrow \nabla u \quad \text{in } L^{p(z)}(\Omega; \mathbb{R}^N),$$

hence we infer that

$$u_n \longrightarrow u \quad \text{in } W_n^{1,p(z)}(\Omega).$$

This fact and (3.16) imply that $\|\cdot\|$ and $|\cdot|$ are equivalent norms in $W_n^{1,p(z)}(\Omega)$. ◻

Proposition 3.3 *If $u_0 \in W_n^{1,p(z)}(\Omega)$ is a local $C_n^1(\overline{\Omega})$-minimizer of ψ, i.e., there exists $r_0 > 0$, such that*

$$\psi(u_0) \leqslant \psi(u_0 + h) \quad \forall h \in C_n^1(\overline{\Omega}),\ \|h\|_{C_n^1(\overline{\Omega})} \leqslant r_0,$$

then $u_0 \in C_n^1(\overline{\Omega})$ and it is a local $W_n^{1,p(z)}(\Omega)$-minimizer of ψ, i.e., there exists $r_1 > 0$, such that

$$\psi(u_0) \leqslant \psi(u_0 + h) \quad \forall h \in W_n^{1,p(z)}(\Omega),\ \|h\| \leqslant r_1.$$

Proof Let $h \in C_n^1(\overline{\Omega})$ and let $\lambda > 0$ be small. Then by hypothesis, we have

$$\psi(u_0) \leqslant \psi(u_0 + \lambda h),$$

so

$$0 \leqslant \langle \psi'(u_0), h \rangle \quad \forall h \in C_n^1(\overline{\Omega}). \tag{3.17}$$

But $C_n^1(\overline{\Omega})$ is dense in $W_n^{1,p(z)}(\Omega)$. So, from (3.17), we have

$$0 \leqslant \langle \psi'(u_0), h \rangle \quad \forall h \in W_n^{1,p(z)}(\Omega),$$

thus

$$\psi'(u_0) = 0$$

and

$$A(u_0) = N_g(u_0)$$

so

$$\begin{cases} -\Delta_{p(z)} u(z) = g(z, u(z)) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.18}$$

From Proposition 3.1, we have that $u_0 \in L^\infty(\Omega)$ and then invoking Theorem 1.3 of Fan [14], we infer that

$$u_0 \in C_n^{1,\alpha}(\overline{\Omega}) \subseteq C_n^1(\overline{\Omega})$$

for some $\alpha \in (0, 1)$.

Next we show that u_0 is a local $W_n^{1,p(z)}(\Omega)$-minimizer of ψ. We argue indirectly. So, suppose that u_0 is not a local $W_n^{1,p(z)}(\Omega)$-minimizer of ψ. Exploiting the compactness of the embedding $W_n^{1,p(z)}(\Omega) \subseteq L^{r(z)}(\Omega)$ (see Proposition 2.1 and recall that by hypothesis $(p^* - r)^- > 0$), we can easily check that ψ is sequentially weakly lower semicontinuous. For $\varepsilon > 0$, let

$$\overline{B}_\varepsilon^{r(z)} = \left\{ u \in W_n^{1,p(z)}(\Omega) : \|u\|_{r(z)} \leqslant \varepsilon \right\}.$$

We will show that we can find $h_\varepsilon \in \overline{B}_\varepsilon^{r(z)}$, such that

$$\psi(u_0 + h_\varepsilon) = \inf \left\{ \psi(u_0 + h) : h \in \overline{B}_\varepsilon^{r(z)} \right\} = m_\varepsilon < \psi(u_0).$$

To this end, let $\{h_n\}_{n\geqslant 1} \subseteq \overline{B}_\varepsilon^{r(z)}$ be a minimizing sequence. It is clear then that the sequence $\{\nabla h_n\}_{n\geqslant 1} \subseteq L^{p(z)}(\Omega; \mathbb{R}^N)$ is bounded. Invoking Lemma 3.2, we have that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p(z)}(\Omega)$ is bounded. So, we assume that

$$h_n \longrightarrow h_\varepsilon \quad \text{weakly in } W_n^{1,p(z)}(\Omega), \tag{3.19}$$

$$h_n \longrightarrow h_\varepsilon \quad \text{in } L^{r(z)}(\Omega) \tag{3.20}$$

(see Proposition 2.1). From (3.19), it follows that

$$\psi(u_0 + h_\varepsilon) \leqslant \liminf_{n\to+\infty} \psi(u_0 + h_n) = m_\varepsilon \quad \text{and} \quad h_\varepsilon \in \overline{B}_\varepsilon^{r(z)},$$

so

$$\psi(u_0 + h_\varepsilon) = m_\varepsilon.$$

Invoking the Lagrange multiplier rule (see e.g., Gasiński–Papageorgiou [22, p. 700]), we can find $\lambda_\varepsilon \leqslant 0$, such that

$$\psi'(u_0 + h_\varepsilon) = A(u_0 + h_\varepsilon) - N_g(u_0 + h_\varepsilon) = \lambda_\varepsilon |h_\varepsilon|^{r(z)-2} h_\varepsilon,$$

so

$$\begin{cases} -\Delta_{p(z)}(u_0 + h_\varepsilon)(z) = g\left(z, (u_0 + h_\varepsilon)(z)\right) + \lambda_\varepsilon \left|h_\varepsilon(z)\right|^{r(z)-2} h_\varepsilon(z) \text{ in } \Omega, \\ \dfrac{\partial h_\varepsilon}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.21}$$

From (3.18) and (3.21), it follows that

$$\begin{aligned} &-\operatorname{div}\left(\|\nabla(u_0 + h_\varepsilon)(z)\|^{p(z)-2} \nabla(u_0 + h_\varepsilon)(z) - \|\nabla u_0(z)\|^{p(z)-2} \nabla u_0(z)\right) \\ &\quad = g\left(z, (u_0 + h_\varepsilon)(z)\right) - g\left(z, u_0(z)\right) + \lambda_\varepsilon \left|h_\varepsilon(z)\right|^{r(z)-2} h_\varepsilon(z) \quad \text{in } \Omega. \end{aligned} \tag{3.22}$$

We consider two distinct cases.

Case 1: $\lambda_\varepsilon \in [-1, 0]$ for all $\varepsilon \in (0, 1]$.

Let $y_\varepsilon = u_0 + h_\varepsilon$ and let us set

$$V_\varepsilon(z, \xi) = \|\xi\|^{p(z)-2}\xi - \|\nabla u_0(z)\|^{p(z)-2} \nabla u_0(z).$$

Form (3.22), we have that

$$\begin{aligned} &-\operatorname{div} V_\varepsilon\left(z, \nabla y_\varepsilon(z)\right) \\ &\quad = g\left(z, y_\varepsilon(z)\right) - g\left(z, u_0(z)\right) + \lambda_\varepsilon \left|(y_\varepsilon - u_0)(z)\right|^{p(z)-2} (y_\varepsilon - u_0)(z) \quad \text{in } \Omega. \end{aligned}$$

By virtue of Theorem 1.3 of Fan [14], we can find $\beta \in (0, 1)$ and $M_1 > 0$, such that

$$y_\varepsilon \in C_n^{1,\beta}(\overline{\Omega}) \quad \text{and} \quad \|y_\varepsilon\|_{C_n^{1,\beta}(\overline{\Omega})} \leqslant M_1 \quad \forall \varepsilon \in (0, 1]. \tag{3.23}$$

Case 2. $\lambda_{\varepsilon_n} < -1$ along a sequence $\varepsilon_n \searrow 0$.

In this case, we set

$$\widehat{V}_{\varepsilon_n}(z, \xi) = \frac{1}{|\lambda_{\varepsilon_n}|} \left| \|\nabla u_0(z) + \xi\|^{p(z)-2} \left(\nabla u_0(z) + \xi\right) - \|\nabla u_0(z)\|^{p(z)-2} \nabla u_0(z) \right|.$$

Form (3.22), we have

$$\begin{aligned} &-\operatorname{div} \widehat{V}_{\varepsilon_n}\left(z, \nabla h_{\varepsilon_n}(z)\right) \\ &\quad = \frac{1}{|\lambda_{\varepsilon_n}|}\left(g\left(z, (u_0 + h_{\varepsilon_n})(z)\right) - g\left(z, u_0(z)\right) - \left|h_{\varepsilon_n}(z)\right|^{r(z)-2} h_{\varepsilon_n}(z)\right) \quad \text{in } \Omega. \end{aligned}$$

Once again, via Theorem 1.3 of Fan [14], we produce $\beta \in (0, 1)$ and $M_1 > 0$, such that

$$h_{\varepsilon_n} \in C_n^{1,\beta}(\overline{\Omega}) \quad \text{and} \quad \|h_\varepsilon\|_{C_n^{1,\beta}(\overline{\Omega})} \leqslant M_1 \quad \forall n \geqslant 1. \tag{3.24}$$

From (3.23) and (3.24) and recalling that the embedding $C_n^{1,\beta}(\overline{\Omega}) \subseteq C_n^1(\overline{\Omega})$ is compact, we have

$$u_0 + h_{\varepsilon_n} \longrightarrow u_0 \quad \text{in } C_n^1(\overline{\Omega})$$

(recall that $h_{\varepsilon_n} \longrightarrow 0$ in $L^{r(z)}(\Omega)$), so

$$\psi(u_0) \leqslant \psi(u_0 + h_{\varepsilon_n}) \quad \forall n \geqslant n_0 \geqslant 1,$$

a contradiction to the choice of the sequence $\{h_{\varepsilon_n}\}_{n\geqslant 1}$. This prove the proposition. $\square$

4 Three nontrivial smooth solutions

In this section, using a combination of variational and Morse theoretic arguments, together with the results from Sect. 3, we establish the existence of three nontrivial smooth solutions for problem (1.1) under hypotheses H_0 and H_1.

So, for $\lambda > 0$, we introduce the following truncations-perturbations of the reaction $f(z, \zeta)$:

$$f_+^\lambda(z,\zeta) = \begin{cases} 0 & \text{if } \zeta \leqslant 0, \\ f(z,\zeta) + \lambda \zeta^{p(z)-1} & \text{if } \zeta > 0, \end{cases} \tag{4.1}$$

$$f_-^\lambda(z,\zeta) = \begin{cases} f(z,\zeta) + \lambda |\zeta|^{p(z)-2}\zeta & \text{if } \zeta < 0, \\ 0 & \text{if } \zeta \geqslant 0. \end{cases} \tag{4.2}$$

Both are Carathéodory functions. We set

$$F_\pm^\lambda(z,\zeta) = \int_0^\zeta f_\pm^\lambda(z,s)\, ds$$

and consider the C^1-functionals $\varphi_\pm^\lambda \colon W_n^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\varphi_\pm^\lambda(u) = \int_\Omega \frac{1}{p(z)} \|\nabla u\|^{p(z)}\, dz + \lambda \int_\Omega \frac{1}{p(z)} |u|^{p(z)}\, dz - \int_\Omega F_\pm^\lambda(z,u)\, dz \quad \forall u \in W_n^{1,p(z)}(\Omega).$$

Also, we consider energy (Euler) functional $\varphi \colon W_n^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ for problem (1.1), defined by

$$\varphi(u) = \int_\Omega \frac{1}{p(z)} \|\nabla u\|^{p(z)}\, dz - \int_\Omega F(z,u)\, dz \quad \forall u \in W_n^{1,p(z)}(\Omega).$$

Proposition 4.1 *If hypotheses H_0 and H_1 hold, then the functionals φ and $\varphi_\pm^\lambda$ satisfy the Cerami condition.*

Proof First we check that φ satisfies the Cerami condition. So, let $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p(z)}(\Omega)$ be a sequence, such that

$$\left|\varphi(u_n)\right| \leqslant M_2 \quad \forall n \geqslant 1, \tag{4.3}$$

for some $M_2 > 0$ and

$$(1 + \|u_n\|)\varphi'(u_n) \longrightarrow 0 \quad \text{in } W_n^{1,p(z)}(\Omega)^*. \tag{4.4}$$

From (4.4), we have

$$\left|\left\langle A(u_n), h\right\rangle - \int\limits_{\Omega} f(z, u_n) h\, dz\right| \leqslant \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \forall h \in W_n^{1,p(z)}(\Omega), \tag{4.5}$$

with $\varepsilon_n \searrow 0$. In (4.5), we choose $h = u_n \in W_n^{1,p(z)}(\Omega)$. Then

$$-\int\limits_{\Omega} \|\nabla u_n\|^{p(z)}\, dz + \int\limits_{\Omega} f(z, u_n) u_n\, dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{4.6}$$

On the other hand from (4.3), we have

$$\int\limits_{\Omega} \frac{p_{max}}{p(z)} \|\nabla u_n\|^{p(z)}\, dz - \int\limits_{\Omega} p_{max} F(z, u_n)\, dz \leqslant \ p_{max} M_2 \quad \forall n \geqslant 1,$$

so

$$\int\limits_{\Omega} \|\nabla u_n\|^{p(z)}\, dz - \int\limits_{\Omega} p_{max} F(z, u_n)\, dz \leqslant \ p_{max} M_2 \quad \forall n \geqslant 1 \tag{4.7}$$

(since $p(z) \leqslant p_{max}$ for all $z \in \Omega$). We add (4.6) and (4.7) and obtain

$$\int\limits_{\Omega} (f(z, u_n) u_n - p_{max} F(z, u_n))\, dz \leqslant M_3 \quad \forall n \geqslant 1, \tag{4.8}$$

for some $M_3 > 0$. By virtue of hypotheses $H_1(i)$ and (ii), we can find $\beta_1 \in (0, \beta_0)$ and $c_{15} > 0$, such that

$$\beta_1 |\zeta|^{\tau(z)} - c_{15} \leqslant f(z, \zeta)\zeta - p_{max} F(z, \zeta) \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{4.9}$$

We use (4.9) in (4.8) and obtain

$$\beta_1 \int\limits_{\Omega} |u_n|^{\tau(z)}\, dz \leqslant M_4 \quad \forall n \geqslant 1, \tag{4.10}$$

for some $M_4 > 0$, so

$$\text{the sequence } \{u_n\}_{n \geqslant 1} \subseteq L^{\tau(z)}(\Omega) \text{ is bounded} \tag{4.11}$$

(see Proposition 2.2(c) and (d)).

Let $\theta_0 \in (r_{max}, \widehat{p}^*)$ (see hypothesis $H_1(i)$). Also, it is clear from hypothesis $H_1(ii)$, that we can always assume without any loss of generality that $\tau_{min} < r_{max} < \theta_0$. So, we can find $t \in (0, 1)$, such that

$$\frac{1}{r_{max}} = \frac{1-t}{\tau_{min}} + \frac{t}{\theta_0}.$$

Invoking the interpolation inequality (see e.g., Gasiński–Papageorgiou [22, p. 905]), we have

$$\|u_n\|_{r_{max}} \leqslant \|u_n\|_{\tau_{min}}^{1-t} \|u_n\|_{\theta_0}^{t} \quad \forall n \geqslant 1,$$

so

$$\|u_n\|_{r_{max}}^{r_{max}} \leqslant \|u_n\|_{\tau_{min}}^{(1-t) r_{max}} \|u_n\|_{\theta_0}^{t r_{max}} \quad \forall n \geqslant 1,$$

thus

$$\|u_n\|_{r_{max}}^{r_{max}} \leqslant M_5 \|u_n\|_{\theta_0}^{tr_{max}} \quad \forall n \geqslant 1, \tag{4.12}$$

for some $M_5 > 0$ (see (4.10)). By virtue of hypothesis $H_1(i)$, we have

$$f(z,\zeta)\zeta \leqslant c_{16}\left(|\zeta| + |\zeta|^{r_{max}}\right) \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}, \quad \forall n \geqslant 1, \tag{4.13}$$

for some $c_{16} > 0$. In (4.5) we choose $h = u_n \in W_n^{1,p(z)}(\overline{\Omega})$. Then we have

$$\begin{aligned}\int_\Omega \|\nabla u_n\|^{p(z)}\, dz &\leqslant \int_\Omega f(z,u_n)u_n\, dz + c_{17} \\ &\leqslant c_{18}\left(1 + \|u_n\| + \|u_n\|^{tr_{max}}\right) \quad \forall n \geqslant 1,\end{aligned}$$

for some $c_{17}, c_{18} > 0$ (see (4.12)) and (4.13) and recall that $\theta_0 < \widehat{p}^*$). Thus

$$\int_\Omega \|\nabla u_n\|^{p(z)}\, dz + \int_\Omega |u_n|^{\tau(z)}\, dz \leqslant c_{19}\left(1 + \|u_n\| + \|u_n\|^{tr_{max}}\right) \quad \forall n \geqslant 1,$$

for some $c_{19} > 0$ (see (4.10)) and so

$$\|u_n\|^{p_{min}} \leqslant c_{20}\left(1 + \|u_n\| + \|u_n\|^{tr_{max}}\right) \quad \forall n \geqslant 1, \tag{4.14}$$

for some $c_{20} > 0$ (see Lemma 3.2). Note that

$$tr_{max} = \frac{\theta_0(r_{max} - \tau_{min})}{\theta_0 - \tau_{min}} < p_{min}.$$

So, from (4.14), it follows that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p(z)}(\Omega)$ is bounded. Hence, passing to a subsequence if necessary, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } W_n^{1,p(z)}(\Omega), \tag{4.15}$$

$$u_n \longrightarrow u \quad \text{in } L^{r(z)}(\Omega) \tag{4.16}$$

(recall that $r_{max} < \widehat{p}^*$). In (4.5) we choose $h = u_n - u \in W_n^{1,p(z)}(\Omega)$. Then

$$\left|\langle A(u_n), u_n - u\rangle - \int_\Omega f(z,u_n)(u_n - u)\, dz\right| \leqslant \varepsilon_n',$$

with $\varepsilon_n' \searrow 0$, so, using (4.15) and Proposition 2.1(c), we have

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u\rangle = 0,$$

so, from Proposition 2.5, we have

$$u_n \longrightarrow u \quad \text{in } W_n^{1,p(z)}(\Omega).$$

This proves that φ satisfies the Cerami condition.

Next we show that φ_+^λ satisfies the Cerami condition. So, as before we consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p(z)}(\Omega)$, such that

$$\left|\varphi_+^\lambda(u_n)\right| \leqslant M_6 \quad \forall n \geqslant 1, \tag{4.17}$$

for some $M_6 > 0$ and

$$(1 + \|u_n\|)(\varphi_+^\lambda)'(u_n) \longrightarrow 0 \quad \text{in } W_n^{1,p(z)}(\Omega). \tag{4.18}$$

From (4.18), we have

$$\left|\left\langle A(u_n), h\right\rangle + \lambda \int_\Omega |u_n|^{p(z)-2} u_n h\, dz - \int_\Omega f_+^\lambda(z, u_n) h\, dz\right| \leqslant \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \forall h \in W_n^{1,p(z)}(\Omega), \tag{4.19}$$

with $\varepsilon_n \searrow 0$. In (4.19), we choose $h = -u_n^- \in W_n^{1,p(z)}(\Omega)$. Then

$$\left|\int_\Omega \|\nabla u_n^-\|^{p(z)}\, dz + \int_\Omega (u_n^-)^{p(z)}\, dz\right| \leqslant \varepsilon_n,$$

so

$$u_n^- \longrightarrow 0 \quad \text{in } W_n^{1,p(z)}(\Omega) \tag{4.20}$$

(see Proposition 2.3(e)). Next, in (4.19), we choose $h = u_n^+ \in W_n^{1,p(z)}(\Omega)$. Then

$$-\int_\Omega \|\nabla u_n^+\|^{p(z)}\, dz + \int_\Omega f(z, u_n^+) u_n^+\, dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{4.21}$$

On the other hand, from (4.17) and (4.20), Proposition 2.3 and (4.1), we have

$$\int_\Omega \|\nabla u_n^+\|^{p(z)}\, dz - \int_\Omega p_{max} F(z, u_n^+)\, dz \leqslant M_7 \quad \forall n \geqslant 1 \tag{4.22}$$

for some $M_7 > 0$. Adding (4.21) and (4.22), we obtain

$$\int_\Omega \left(f(z, u_n^+) u_n^+ - p_{max} F(z, u_n^+)\right) dz \leqslant M_8 \quad \forall n \geqslant 1,$$

for some $M_8 > 0$. Then we proceed as in the first part of the proof (see the argument after (4.8)). So, we obtain that the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq L^{\tau(z)}(\Omega)$ is bounded and then as before, via the interpolation inequality, we show that the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_n^{1,p(z)}(\Omega)$ is bounded. Finally, using Proposition 2.5, we conclude that φ_+^λ satisfies the Cerami condition.

Similarly we show that φ_-^λ satisfies the Cerami condition, using this time (4.2). □

Proposition 4.2 *If hypotheses H_0 and H_1 hold, then $u = 0$ is a local minimizer of φ and of $\varphi_\pm^\lambda$.*

Proof We do the proof for φ_+^λ, the proofs for $\varphi, \varphi_-^\lambda$ being similar.

Let $\delta_0 > 0$ be as postulated by hypothesis $H_1(iii)$ and let $u \in C_n^1(\overline{\Omega})$ be such that $\|u\|_{C_n^1(\overline{\Omega})} \leqslant \delta_0$. Then, using hypothesis $H_1(iii)$ and (4.1), we have

$$\begin{aligned}\varphi_+^\lambda(u) &= \int_\Omega \frac{1}{p(z)} \|\nabla u\|^{p(z)}\, dz + \lambda \int_\Omega \frac{1}{p(z)} |u|^{p(z)}\, dz - \int_\Omega F_+^\lambda(z, u)\, dz \\ &\geqslant \int_\Omega \frac{1}{p(z)} \|\nabla u\|^{p(z)}\, dz \geqslant 0,\end{aligned}$$

so

$$u = 0 \text{ is a local } C_n^1(\overline{\Omega})\text{-minimizer of } \varphi_+^\lambda,$$

thus, using Proposition 3.3, we have that

$$u = 0 \text{ is a local } W_n^{1,p(z)}(\Omega)\text{-minimizer of } \varphi_+^\lambda.$$

The proof is similar for φ_-^λ and φ. □

An immediate consequence of the p-superlinearity of $F(z, \cdot)$ (see hypothesis $H_1(ii)$), is the following result.

Proposition 4.3 *If hypotheses H_0 and H_1 hold, then*

$$\varphi_\pm^\lambda(\xi) \longrightarrow -\infty \quad \text{as } \xi \to \pm\infty \text{ for every } u \in W_n^{1,p(x)}(\Omega),\, u \neq 0.$$

As we already mentioned earlier, our method of proof uses also Morse theory, This requires the computation of certain critical groups of φ and $\varphi_\pm^\lambda$. In what follows, we assume without any loss off generality, that the critical sets of these functions are finite (otherwise we already have an infinity of solutions and so we are done).

Proposition 4.4 *If hypotheses H_0 and H_1 hold, then*

$$C_k(\varphi, \infty) = 0 \quad \forall k \geqslant 0.$$

Proof By virtue of hypothesis $H_1(ii)$, for a given $\xi > 0$, we can find $M_9 = M_9(\xi) > 0$, such that

$$F(z, \zeta) \geqslant \frac{\xi}{p_{min}} |\zeta|^{p+} - M_9 \quad \text{for a.a. } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{4.23}$$

Let $u \in \partial B_1 = \left\{ u \in W_n^{1,p(z)}(\Omega) : \|u\| = 1 \right\}$ and $\theta > 0$. Then

$$\begin{aligned}
\varphi(\theta u) &= \int_\Omega \frac{\theta^{p(z)}}{p(z)} \|\nabla u\|^{p(z)}\, dz - \int_\Omega F(z, \theta u)\, dz \\
&\leqslant \theta^{\widetilde{p}} \int_\Omega \frac{1}{p(z)} \|\nabla u\|^{p(z)}\, dz - \int_\Omega F(z, \theta u)\, dz \\
&\leqslant \theta^{\widetilde{p}} \int_\Omega \frac{1}{p(z)} \|\nabla u\|^{p(z)}\, dz - \frac{\theta^{p_{max}} \xi}{p_{min}} \|u\|_{p_{max}}^{p_{max}} + M_9 |\Omega|_N \\
&\leqslant \frac{\theta^{\widetilde{p}}}{p_{min}} \left(c_{21} - \xi \|u\|_{p_{max}}^{p_{max}} \right) + M_9 |\Omega|_N,
\end{aligned} \tag{4.24}$$

for some $c_{21} > 0$, where

$$\widetilde{p} = \begin{cases} p_{max} & \text{if } \ \theta \geqslant 1, \\ p_{min} & \text{if } \ \theta < 1. \end{cases}$$

Since $\xi > 0$ was arbitrary, from (4.24), we infer that

$$\varphi(\theta u) \longrightarrow -\infty \quad \text{as } \theta \to +\infty,\ \text{with } u \in \partial B_1. \tag{4.25}$$

By virtue of (2.2) (see hypothesis $H_1(ii)$), we can find $\beta_1 \in (0, \beta_0)$ and $c_{22} > 0$, such that

$$f(z, \zeta)\zeta - p_{max} F(z, \zeta) \geqslant \beta_1 |\zeta|^{\tau(z)} - c_{22} \quad \text{for a.a. } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{4.26}$$

Then for every $u \in W_n^{1,p(z)}(\Omega)$, we have

$$\int_\Omega (p_{max} F(z,u) - f(z,u)u)\, dz \leqslant \int_\Omega \left(-\beta_1 |u|^{\tau(z)} + c_{22}\right) dz$$
$$= -\beta_1 \int_\Omega |u|^{\tau(z)}\, dz + c_{22}|\Omega|_N. \tag{4.27}$$

Let $c_{23} = c_{22}|\Omega|_N + 1 > 0$ and choose $\eta < -\frac{c_{23}}{p_{max}} < 0$. By virtue of (4.25), we see that for $u \in \partial B_1$ and $\theta \geqslant 0$ large enough, we have

$$\varphi(\theta u) \leqslant \eta,$$

so

$$\int_\Omega \frac{\theta^{p(z)}}{p(z)} \|\nabla u\|^{p(z)}\, dz - \int_\Omega F(z,\theta u)\, dz \leqslant \eta$$

and thus

$$\frac{1}{p_{max}} \left(\int_\Omega \frac{\theta^{p(z)} p_{max}}{p(z)} \|\nabla u\|^{p(z)}\, dz - \int_\Omega p_{max} F(z,\theta u)\, dz \right) \leqslant \eta. \tag{4.28}$$

Since $\varphi(0) = 0$, from (4.25) and (4.28), we infer that there exists $\theta^* > 0$, such that

$$\varphi(\theta^* u) = \eta \quad \text{and} \quad \varphi(\theta u) \leqslant \eta \quad \forall \theta \geqslant \theta^*. \tag{4.29}$$

Using (4.27) and (4.28), we have

$$\begin{aligned}
\frac{d}{dt}\varphi(\theta u) &= \langle \varphi'(\theta u), u \rangle \\
&= \int_\Omega \theta^{p(z)-1} \|\nabla u\|^{p(z)}\, dz - \int_\Omega f(z,\theta u) u\, dz \\
&= \frac{1}{\theta} \left(\int_\Omega \left\|\nabla(\theta u)\right\|^{p(z)} dz - \int_\Omega f(z,\theta u)\theta u\, dz \right) \\
&\leqslant \frac{1}{\theta} \left(\int_\Omega \left\|\nabla(\theta u)\right\|^{p(z)} dz - \int_\Omega p_{max} F(z,\theta u)\, dz + c_{22}|\Omega|_N \right) \\
&\leqslant \frac{1}{\theta} \left(\int_\Omega \frac{p_{max}}{p(z)} \left\|\nabla(\theta u)\right\|^{p(z)} dz - \int_\Omega p_{max} F(z,\theta u)\, dz + c_{22}|\Omega|_N \right) \\
&\leqslant \frac{1}{\theta} \left(p_{max}\eta + c_{22}|\Omega|_N \right) < 0,
\end{aligned}$$

for $\theta \geqslant 1$ large and since $\eta < -\frac{c_{23}}{p_{max}}$. So, there is a unique $\theta^*(u) > 0$, such that

$$\varphi\left(\theta^*(u)u\right) = \eta, \quad u \in \partial B_1$$

(see (4.29)). By virtue of the implicit function theorem, we have $\theta^* \in C(\partial B_1)$. For $u \in W_n^{1,p(z)}(\Omega)\backslash\{0\}$, we set

$$\widehat{\theta}^*(u) = \frac{1}{\|u\|}\theta^*\left(\frac{u}{\|u\|}\right).$$

Then $\widehat{\theta}^* \in C(W_n^{1,p(z)}(\Omega)\backslash\{0\})$ and we have

$$\varphi(\widehat{\theta}^*(u)u) = \eta \quad \forall u \in W_n^{1,p(z)}(\Omega) \setminus \{0\}. \tag{4.30}$$

Note that, if $\varphi(u) = \eta$, then $\widehat{\theta}^*(u) = 1$. We set

$$\widehat{\theta}_0^*(u) = \begin{cases} 1 & \text{if} \quad \varphi(u) \leqslant \eta, \\ \widehat{\theta}^*(u)u & \text{if} \quad \varphi(u) > \eta. \end{cases} \tag{4.31}$$

Evidently $\widehat{\theta}_0^* \in C(W_n^{1,p(z)}(\Omega) \setminus \{0\})$. We consider the homotopy

$$h\colon [0,1] \times (W_n^{1,p(z)}(\Omega) \setminus \{0\}) \longrightarrow W_n^{1,p(z)}(\Omega) \setminus \{0\},$$

defined by

$$h(t,u) = (1-t)u + t\widehat{\theta}_0^*(u)u.$$

Note that

$$h(0,u) = u \quad \forall u \in W_n^{1,p(z)}(\Omega) \setminus \{0\},$$
$$h(1,u) \in \varphi^\eta \quad \forall u \in W_n^{1,p(z)}(\Omega) \setminus \{0\}$$

(see (4.30)) and (4.31) and

$$h(t,\cdot)|_{\varphi^\eta} = id|_{\varphi^\eta} \quad \forall t \in [0,1]$$

(see (4.31)). It follows that φ^η is a strong deformation retract of $W_n^{1,p(z)}(\Omega) \setminus \{0\}$. Therefore

$$\varphi^\eta \text{ and } W_n^{1,p(z)}(\Omega) \setminus \{0\} \text{ are homotopy equivalent.} \tag{4.32}$$

On the other hand, if we consider homotopy

$$h_1\colon [0,1] \times (W_n^{1,p(z)}(\Omega) \setminus \{0\}) \longrightarrow W_n^{1,p(z)}(\Omega) \setminus \{0\},$$

defined by

$$h_1(t,u) = (1-t)u + t\frac{u}{\|u\|},$$

we see that

$$h_1(0,u) = u \quad \forall u \in W_n^{1,p(z)}(\Omega) \setminus \{0\},$$
$$h_1(1,u) \in \partial B_1 \quad \forall u \in W_n^{1,p(z)}(\Omega) \setminus \{0\}$$

and

$$h_1(t,\cdot)|_{\partial B_1} = id|_{\partial B_1} \quad \forall t \in [0,1].$$

Hence ∂B_1 is a strong deformation retract of $W_n^{1,p(z)}(\Omega) \setminus \{0\}$. So, we have that

$$\partial B_1 \text{ and } W_n^{1,p(z)}(\Omega) \setminus \{0\} \text{ are homotopy equivalent.} \tag{4.33}$$

From (4.32) and (4.33), it follows that

$$\varphi^\eta \text{ and } \partial B_1 \text{ are homotopy equivalent,}$$

so

$$H_k\left(W_n^{1,p(z)}(\Omega), \varphi^\eta\right) = H_k\left(W_n^{1,p(z)}(\Omega), \partial B_1\right) \quad \forall k \geqslant 0$$

and thus

$$C_k(\varphi, \infty) = H_k\left(W_n^{1,p(z)}(\Omega), \partial B_1\right) \quad \forall k \geqslant 0 \tag{4.34}$$

(choosing $\eta < \inf \varphi(K^\varphi)$). Because $W_n^{1,p(z)}(\Omega)$ is infinite dimensional, then ∂B_1 is contractible (see e.g., Gasiński–Papageorgiou [22, p. 693]). Hence

$$H_k\left(W_n^{1,p(z)}(\Omega), \partial B_1\right) = 0 \quad \forall k \geqslant 0 \tag{4.35}$$

(see Granas–Dugundji [24, p. 389]) Combining (4.34) and (4.35), we conclude that

$$C_k(\varphi, \infty) = 0 \quad \forall k \geqslant 0.$$

□

A suitable modification of the above proof, leads to a similar result for the functionals $\varphi_\pm^\lambda$.

Proposition 4.5 *If hypotheses H_0 and H_1 hold, then*

$$C_k(\varphi_\pm^\lambda, \infty) = 0 \quad \forall k \geqslant 0.$$

Proof We do the proof for φ_+^λ, the proof for φ_-^λ being similar.

By virtue of hypothesis $H_1(ii)$, for a given $\xi > 0$, we can find $c_{24} = c_{24}(\xi) > 0$, such that

$$F_+^\lambda(z, \zeta) \geqslant \frac{\lambda}{p(z)}(\zeta^+)^{p(z)} + \frac{\xi}{p_{min}}(\zeta^+)^{p_{max}} - c_{24} \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{4.36}$$

Let

$$D_+ = \left\{u \in \partial B_1 : u^+ \neq 0\right\}.$$

Using (4.36), for $u \in D_+$ and $\theta > 0$, we have

$$\begin{aligned} \varphi_+^\lambda(\theta u) &= \int_\Omega \frac{\theta^{p(z)}}{p(z)}\|\nabla u\|^{p(z)}\,dz + \lambda \int_\Omega \frac{\theta^{p(z)}}{p(z)}|u|^{p(z)}\,dz - \int_\Omega F_+^\lambda(z, \theta u)\,dz \\ &\leqslant \theta^{\widetilde{p}}\left(\int_\Omega \frac{1}{p(z)}\|\nabla u\|^{p(z)}\,dz + \lambda \int_\Omega \frac{1}{p(z)}(u^-)^{p(z)}\,dz - \frac{\xi}{p_{min}}\|u^+\|_{p_{max}}^{p_{max}}\right) \\ &\quad + c_{24}|\Omega|_N \\ &\leqslant \theta^{\widetilde{p}}\left(\varrho_p(u) - \frac{\xi}{p_{min}}\|u^+\|_{p_{max}}^{p_{max}}\right) + c_{24}|\Omega|_N, \end{aligned} \tag{4.37}$$

where

$$\widetilde{p} = \begin{cases} p_{max} & \text{if } \theta \geqslant 1, \\ p_{min} & \text{if } \theta < 1 \end{cases}$$

and with ϱ_p being the modular function, defined by

$$\varrho_p(u) = \int_\Omega \left(\|\nabla u\|^{p(z)} + \lambda |u|^{p(z)} \right) dz \quad \forall u \in W_n^{1,p(z)}(\Omega).$$

Since $\xi > 0$ is arbitrary, we choose it large such that

$$\varrho_p(u) < \frac{\xi}{p_{min}} \|u^+\|_{p_{max}}^{p_{max}} \quad \forall u \in D_+,$$

so

$$\varphi_+^\lambda(\theta u) \longrightarrow -\infty \quad \text{as } \theta \to +\infty, u \in D_+ \tag{4.38}$$

(see (4.37)).

Hypothesis $H_1(ii)$ implies that we can find $\beta_1 \in (0, \beta_0)$ and $c_{25} > 0$, such that

$$f(z,\zeta^+)\zeta^+ - p_{max}F(z,\zeta^+) \geqslant \beta_1(\zeta^+)^{\tau(z)} - c_{25} \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{4.39}$$

Therefore for every $u \in W_n^{1,p(z)}(\Omega)$, we have

$$\int_\Omega \left(p_{max}F(z,u^+) - f(z,u^+)u^+ \right) dz \leqslant -\beta_1 \int_\Omega (u^+)^{\tau(z)}\, dz + c_{25}|\Omega|_N \tag{4.40}$$

(see (4.39)). Let $c_{26} = c_{25}|\Omega|_N + 1$ and choose $\eta < -\frac{c_{26}}{p_{max}}$. Then because of (4.38), for all $u \in D_+$ and for $\theta > 0$ large enough, we have

$$\varphi_+^\lambda(\theta u) \leqslant \eta$$

so

$$\int_\Omega \frac{\theta^{p(z)}}{p(z)} \|\nabla u\|^{p(z)}\, dz + \lambda \int_\Omega \frac{\theta^{p(z)}}{p(z)} |u|^{p(z)}\, dz - \int_\Omega F_+^\lambda(z, \theta u)\, dz \leqslant \eta,$$

thus, using (4.1), we have

$$\int_\Omega \frac{\theta^{p(z)}}{p(z)} \|\nabla u\|^{p(z)}\, dz + \lambda \int_\Omega \frac{\theta^{p(z)}}{p(z)} (u^-)^{p(z)}\, dz - \int_\Omega F(z, \theta u^+)\, dz \leqslant \eta$$

and so

$$\frac{1}{p_{max}} \left(\int_\Omega \frac{\theta^{p(z)} p_{max}}{p(z)} \|\nabla u\|^{p(z)}\, dz \right.$$
$$\left. + \lambda \int_\Omega \frac{\theta^{p(z)} p_{max}}{p(z)} (u^-)^{p(z)}\, dz - \int_\Omega p_{max} F(z, \theta u^+)\, dz \right) \leqslant \eta. \tag{4.41}$$

Since $\varphi_+^\lambda(0) = 0$, we can find $\widehat{\theta} > 0$, such that

$$\widehat{\varphi}_+^\lambda(\widehat{\theta} u) = 0 \quad \text{with } u \in D_+$$

(see (4.38)). We have

$$
\begin{aligned}
&\frac{d}{d\theta}\varphi_+^\lambda(\theta u)\\
&=\big\langle(\varphi_+^\lambda)'(\theta u), u\big\rangle=\frac{1}{\theta}\big\langle(\varphi_+^\lambda)'(\theta u), \theta u\big\rangle\\
&=\frac{1}{\theta}\Bigg(\int\limits_\Omega \theta^{p(z)}\|\nabla u\|^{p(z)}\,dz+\lambda\int\limits_\Omega \theta^{p(z)}(u^-)^{p(z)}\,dz-\int\limits_\Omega f(z,\theta u^+)\theta u^+\,dz\Bigg)\\
&\leqslant \frac{1}{\theta}\Bigg(\int\limits_\Omega \frac{\theta^{p(z)}p_{max}}{p(z)}\|\nabla u\|^{p(z)}\,dz+\lambda\int\limits_\Omega \frac{\theta^{p(z)}p_{max}}{p(z)}(u^-)^{p(z)}\,dz\\
&\quad-\int\limits_\Omega f(z,\theta u^+)\theta u^+\,dz\Bigg)\\
&\leqslant \frac{1}{\theta}\Bigg(\int\limits_\Omega \frac{\theta^{p(z)}p_{max}}{p(z)}\|\nabla u\|^{p(z)}\,dz+\lambda\int\limits_\Omega \frac{\theta^{p(z)}p_{max}}{p(z)}(u^-)^{p(z)}\,dz\\
&\quad-\int\limits_\Omega p_{max}F(z,\theta u^+)\,dz+c_{25}|\Omega|_N\Bigg)\\
&\leqslant \frac{1}{\theta}(p_{max}\eta+c_{26})<0
\end{aligned}
$$

(see (4.40), (4.41) and recall that $\eta<-\frac{c_{26}}{p_{max}}$). So, as in the proof of Proposition 4.4, we can find a unique $\theta^+\in C(D_+)$, such that

$$\varphi_+^\lambda\big(\theta^+(u)u\big)=\eta\quad\forall u\in D_+.$$

Let

$$E_+=\Big\{u\in W_n^{1,p(z)}(\Omega): u^+\neq 0\Big\}$$

and set

$$\widehat{\theta}^+(u)=\frac{1}{\|u\|}\theta^+\left(\frac{u}{\|u\|}\right).$$

Then

$$\widehat{\theta}^+\in C(E_+)\quad\text{ and }\quad \varphi_+^\lambda\big(\widehat{\theta}^+(u)u\big)=\eta\quad\forall u\in E_+.$$

Note that, if $\varphi_+^\lambda(u)=\eta$, then $\widehat{\theta}^+(u)=1$. So, if we define $\widehat{\theta}_0^+: E_+\longrightarrow\mathbb{R}$, by

$$\widehat{\theta}_0^+(u)=\begin{cases}1 & \text{if } \varphi_+^\lambda(u)\leqslant\eta,\\ \widehat{\theta}^+(u) & \text{if } \varphi_+^\lambda(u)>\eta,\end{cases}\qquad\forall u\in E_+,\tag{4.42}$$

then $\widehat{\theta}_0^+\in C(E_+)$. Consider the homotopy

$$h_+:[0,1]\times E_+\longrightarrow E_+,$$

defined by

$$h_+(t,u)=(1-t)u+t\widehat{\theta}_0^+(u)u.$$

We have

$$h_+(0,u) = u \quad \forall u \in E_+,$$

$$h_+(1,u) \in (\varphi_+^\lambda)^\eta \quad \forall u \in E_+$$

and

$$h_+(t,\cdot)|_{(\varphi_+^\lambda)^\eta} = id|_{(\varphi_+^\lambda)^\eta} \quad \forall t \in [0,1]$$

(see (4.42)). It follows that $(\varphi_+^\lambda)^\eta$ is a strong deformation retract of E_+. Therefore

$$E_+ \text{ and } (\varphi_+^\lambda)^\eta \text{ are homotopy equivalent.} \tag{4.43}$$

Also consider the homotopy

$$h_+^1\colon [0,1] \times E_+ \longrightarrow E_+,$$

defined by

$$h_+^1(t,u) = (1-t)u + t\frac{u}{\|u\|}.$$

Evidently, we have

$$h_+^1(0,u) = u \quad \forall u \in E_+,$$
$$h_+^1(1,u) \in D_+ \quad \forall u \in E_+$$

and

$$h_+(t,\cdot)|_{D_+} = id|_{D_+} \quad \forall t \in [0,1],$$

so D_+ is a strong deformation retract of E_+. Therefore

$$E_+ \text{ and } D_+ \text{ are homotopy equivalent.} \tag{4.44}$$

Form (4.43) and (4.44), it follows that

$$(\varphi_+^\lambda)^\eta \text{ and } D_+ \text{ are homotopy equivalent,}$$

so

$$H_k\left(W_n^{1,p(z)}(\Omega), (\varphi_+^\lambda)^\eta\right) = H_k\left(W_n^{1,p(z)}(\Omega), D_+\right) \quad \forall k \geqslant 0$$

and thus

$$C_k(\varphi_+^\lambda, \infty) = H_k\left(W_n^{1,p(z)}(\Omega), D_+\right) \quad \forall k \geqslant 0 \tag{4.45}$$

(choosing $\eta < \inf \varphi_+^\lambda(K^{\varphi_+^\lambda})$). Consider the homotopy

$$\widehat{h}_+\colon [0,1] \times D_+ \longrightarrow D_+,$$

defined by

$$\widehat{h}_+(t,u) = \frac{(1-t)u + t\xi}{\|(1-t)u + t\xi\|},$$

with $\xi \in \mathbb{R}$, $\xi > 0$, $\|\xi\| = 1$. Note that $[(1-t)u + t\xi]^+ \neq 0$ and so the homotopy is well defined. We infer that the set D_+ is contractible in itself. Therefore

$$H_k\left(W_n^{1,p(z)}(\Omega), D_+\right) = 0 \quad \forall k \geqslant 0$$

(see Granas–Dugundji [24, p. 389]), so

$$C_k(\varphi_+^\lambda, \infty) = 0 \quad \forall k \geqslant 0$$

(see (4.45)). Similarly we show that

$$C_k(\varphi_-^\lambda, \infty) = 0 \quad \forall k \geqslant 0. \qquad \square$$

Now we are ready for the three solutions theorem.

Theorem 4.6 *If hypotheses H_0 and H_1 hold, then problem (1.1) has at least three nontrivial smooth solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -int\, C_+, \quad \widehat{y} \in C_n^1(\overline{\Omega}) \setminus \{0\}.$$

Proof From Proposition 4.2, we know that $u = 0$ is a local minimizer of φ_+^λ. Reasoning as in the proof of Proposition 29 of Aizicovici–Papageorgiou–Staicu [3], we can find small $\varrho \in (0,1)$, such that

$$0 = \varphi_+^\lambda(0) < \inf\left\{\varphi_+^\lambda(u) : \|u\| = \varrho\right\} = \eta_+^\lambda. \tag{4.46}$$

Then (4.46) together with Propositions 4.1 and 4.3, permit the use of the mountain pass theorem (see Theorem 2.4). So, we obtain $u_0 \in W_n^{1,p(z)}(\Omega)$, such that

$$0 = \varphi_+^\lambda(0) < \eta_+^\lambda \leqslant \varphi_+^\lambda(u_0) \quad \text{and} \quad (\varphi_+^\lambda)'(u_0) = 0. \tag{4.47}$$

From the inequality in (4.47), we infer that $u_0 \neq 0$. From the equality, it follows that

$$A(u_0) + \lambda |u_0|^{p(\cdot)-2} u_0 = N_+^\lambda(u_0), \tag{4.48}$$

where

$$N_+^\lambda(u)(\cdot) = f_+^\lambda(\cdot, u(\cdot)) \quad \forall u \in W_n^{1,p(z)}(\Omega).$$

On (4.48) we act with $-u_0^- \in W_n^{1,p(z)}(\Omega)$ and obtain

$$\int\limits_\Omega \|\nabla u_0^-\|^{p(z)}\, dz + \lambda \int\limits_\Omega |u_0^-|^{p(z)}\, dz = 0$$

(see (4.1)), so $u_0^- = 0$ (see Proposition 2.3) and so

$$u_0 \geqslant 0, \quad u_0 \neq 0.$$

Then using Proposition 3.1 and Theorem 1.3 of Fan [14], we have that $u_0 \in C_+ \setminus \{0\}$ solves problem (1.1). By virtue of hypothesis $H_1(iii)$, we have

$$\Delta_{p(z)} u_0 \leqslant c_0 u_0^{p(z)-1} \quad \text{in } W_n^{1,p(z)}(\Omega)^*,$$

so

$$u_0 \in \operatorname{int} C_+$$

(see Theorem 1.2 of Zhang [37]).

Similarly, working with φ_-^λ and using this time (4.2), we obtain another constant sign smooth solution

$$v_0 \in -\operatorname{int} C_+.$$

Clearly both u_0 and v_0 are critical points of φ (see (4.1) and (4.2)).

Suppose that $\{0, u_0, v_0\}$ are the only critical points of φ.

Claim 1 $C_k(\varphi_+^\lambda, u_0) = C_k(\varphi_-^\lambda, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geqslant 0$.

We do the proof for the pair $\{\varphi_+^\lambda, u_0\}$, the proof for $\{\varphi_-^\lambda, v_0\}$ being similar.

As above, we can check that every critical point u of φ_+^λ satisfies $u \geqslant 0$ and so (4.1) implies that $u \in K^\varphi$. Since by hypothesis $K^\varphi = \{0, u_0, v_0\}$, we infer that

$$K^{\varphi_+^\lambda} = \{0, u_0\}.$$

Let $\eta, \theta \in \mathbb{R}$ be such that

$$\theta < 0 = \varphi_+^\lambda(0) < \eta < \varphi_+^\lambda(u_0)$$

(see (4.47)). We consider the following triple of sets

$$(\varphi_+^\lambda)^\theta \subseteq (\varphi_+^\lambda)^\eta \subseteq W = W_n^{1,p(z)}(\Omega).$$

We introduce the long exact sequence of homological groups corresponding to the above triple of sets

$$\ldots \longrightarrow H_k\left(W, (\varphi_+^\lambda)^\theta\right) \xrightarrow{i_*} H_k\left(W, (\varphi_+^\lambda)^\eta\right) \xrightarrow{\partial_*} H_{k-1}\left((\varphi_+^\lambda)^\eta, (\varphi_+^\lambda)^\theta\right) \longrightarrow \ldots. \tag{4.49}$$

Here i_* is the group homomorphism induced by the inclusion

$$\left(W, (\varphi_+^\lambda)^\theta\right) \xrightarrow{i} \left(W, (\varphi_+^\lambda)^\eta\right)$$

and ∂_* is the boundary homomorphism. Recall that $K^{\varphi_+^\lambda} = \{0, u_0\}$, from the choice of the levels θ and η, we have

$$H_k\left(W, (\varphi_+^\lambda)^\theta\right) = C_k(\varphi_+^\lambda, \infty) = 0 \quad \forall k \geqslant 0 \tag{4.50}$$

(see Proposition 4.5),

$$H_k\left(W, (\varphi_+^\lambda)^\eta\right) = C_k(\varphi_+^\lambda, u_0) \quad \forall k \geqslant 0 \tag{4.51}$$

and

$$H_{k-1}\left((\varphi_+^\lambda)^\eta, (\varphi_+^\lambda)^\theta\right) = C_{k-1}(\varphi_+^\lambda, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0 \tag{4.52}$$

(see Proposition 4.2). From the exactness of the sequence (4.49) and using (4.52), we have

$$H_k\left(W, (\varphi_+^\lambda)^\eta\right) \cong H_{k-1}\left((\varphi_+^\lambda)^\eta, (\varphi_+^\lambda)^\theta\right) = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0,$$

so

$$C_k(\varphi_+^\lambda, u_0) = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0.$$

Similarly we show that

$$C_k(\varphi_-^\lambda, v_0) = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0.$$

Claim 2 $C_k(\varphi, u_0) = C_k(\varphi_+^\lambda, u_0)$ and $C_k(\varphi, v_0) = C_k(\varphi_-^\lambda, v_0)$ for all $k \geqslant 0$.

We do the proof for the triple $\{\varphi, \varphi_+^\lambda, u_0\}$, the proof for $\{\varphi, \varphi_-^\lambda, v_0\}$ being similar.

We consider the homotopy

$$\overline{h}(t,u) = t\varphi_+^\lambda(u) + (1-t)\varphi(u) \quad \forall (t,u) \in [0,1] \times W_n^{1,p(z)}(\Omega).$$

Evidently u_0 is a critical point of $\overline{h}(t,\cdot)$ for all $t \in [0,1]$. We will show that u_0 is isolated uniformly in $t \in [0,1]$. Indeed, if this is not the case, then we can find two sequences $\{t_n\}_{n\geqslant 1} \subseteq [0,1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p(z)}(\Omega)$, such that

$$t_n \longrightarrow t \in [0,1] \quad \text{and} \quad u_n \longrightarrow u_0 \quad \text{in } W_n^{1,p(z)}(\Omega) \tag{4.53}$$

and

$$\overline{h}'_u(t_n, u_n) = 0 \quad \forall n \geqslant 1. \tag{4.54}$$

From (4.53), we have

$$A(u_n) + t_n\lambda|u_n|^{p(\cdot)-2}u_n = t_n N_+^\lambda(u_n) + (1-t_n)N(u_n),$$

where

$$N(u)(\cdot) = f(\cdot, u(\cdot)) \quad \forall u \in W_n^{1,p(z)}(\Omega),$$

so

$$\begin{cases} -\Delta_{p(z)}u(z) = t_n f\left(z, u_n^+(z)\right) + t_n\left(u_n^-(z)\right)^{p(z)-1} + (1-t_n)f\left(z, u_n(z)\right) & \text{in} \quad \Omega, \\ \dfrac{\partial u_n}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Proposition 3.1 implies that we can find $M_{10} > 0$, such that

$$\|u_n\|_\infty \leqslant M_{10} \quad \forall n \geqslant 1.$$

Then using the regularity result of Fan [14], we can find $M_{11} > 0$ and $\eta \in (0,1)$, such that

$$u_n \in C_n^{1,\eta}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_n^{1,\eta}(\overline{\Omega})} \leqslant M_{11} \quad \forall n \geqslant 1. \tag{4.55}$$

From (4.55) and since the embedding $C_n^{1,\eta}(\overline{\Omega}) \subseteq C_n^1(\overline{\Omega})$ is compact, we may also assume that

$$u_n \longrightarrow u_0 \quad \text{in } C_n^1(\overline{\Omega})$$

(see (4.54)). But recall that $u_0 \in \operatorname{int} C_+$. So, it follows that

$$u_n \in C_+ \setminus \{0\} \quad \forall n \geqslant n_0,$$

so $\{u_n\}_{n\geqslant n_0} \subseteq C_+ \setminus \{0\}$ are all distinct solutions of (1.1) (see (4.1)).

This contradicts the assumption that $\{0, u_0, v_0\}$ are the only critical points of φ. So, indeed u_0 is an isolated critical point $\overline{h}(t,\cdot)$ uniformly in $t \in [0,1]$. Moreover, as in Proposition 4.1, we can check that for all $t \in [0,1]$, $\overline{h}(t,\cdot)$ satisfies the Cerami condition. This enables us to exploit the homotopy invariance of the critical groups (see Chang [9, p. 336]) and obtain

$$C_k\left(\overline{h}(0,\cdot), u_0\right) = C_k\left(\overline{h}(1,\cdot), u_0\right) \quad \forall k \geqslant 0,$$

so

$$C_k(\varphi, u_0) = C_k(\varphi_+^\lambda, u_0) \quad \forall k \geqslant 0.$$

Similarly, we show that

$$C_k(\varphi, v_0) = C_k(\varphi_-^\lambda, v_0) \quad \forall k \geqslant 0.$$

This proves Claim 2.

From Claims 1 and 2, it follows that

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0. \tag{4.56}$$

From Proposition 2.1, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0. \tag{4.57}$$

Finally, from Proposition 4.4, we know that

$$C_k(\varphi, \infty) = 0 \quad \forall k \geqslant 0. \tag{4.58}$$

Recall that by hypothesis $\{0, u_0, v_0\}$ are the only critical points of φ. So, from (4.56), (4.57), (4.58) and the Morse relation (2.1) with $t = -1$, we have

$$2(-1)^1 + (-1)^0 = (-1)^1 \neq 0,$$

a contradiction. This means that φ has one more critical point $\widehat{y} \notin \{0, u_0, v_0\}$. Then $\widehat{y} \in C_n^1(\overline{\Omega})$ (see Proposition 3.1 and Fan [14]) and solves (1.1). □

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