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Anisotropic Pseudodifferential Operators of Type 1, 1 (*).

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Abstract. – *The author studies boundedness and microlocal properties for a general class of pseudodifferential operators of type* 1, 1 *in the frame of the anisotropic Sobolev spaces.*

Introduction.

Moving from an essentially formal point of view, we say pseudodifferential operator on $\boldsymbol{R}^n$ the linear map from $\mathcal{S}(\boldsymbol{R}^n)$ to $\mathcal{S}'(\boldsymbol{R}^n)$, defined by

$$a(x, D)\, u(x) = (2\pi)^{-n} \int e^{i\langle x, \eta\rangle} a(x, \eta)\, \widehat{u}(\eta)\, d\eta\,, \tag{0.0.1}$$

where $\widehat{u}$ represents the Fourier transform of $u \in \mathcal{S}(\boldsymbol{R}^n)$ and $a(x, \eta) \in \mathcal{S}'(\boldsymbol{R}^n)$ is the operator symbol.

In the literature of the last 25 years, many general classes of symbols have been introduced and the problem of their boundedness on suitable weighted Sobolev spaces widely studied.

Particularly in his doctoral thesis [3] and in more recent works [4], [5] G. BOURDAUD shows that the pseudodifferential operators in $(\mathrm{Op}\, S^0_{1,1}(\boldsymbol{\Omega})) \cap (\mathrm{Op}\, S^0_{1,1}(\boldsymbol{\Omega}))^*$ are bounded from $H^s_{\mathrm{comp}}(\boldsymbol{\Omega})$ to $H^s_{\mathrm{loc}}(\boldsymbol{\Omega})$, $s \in \boldsymbol{R}$, $\boldsymbol{\Omega}$ open subset of $\boldsymbol{R}^n$.

For better arguing the pathologies of the pseudodifferential operators of type 1, 1 HÖRMANDER [11], 1988, points out that

$$\widehat{a}^*(\xi, \eta) = \overline{\widehat{a}} \circ T(\xi, \eta)\,, \qquad T(\xi, \eta) = (-\xi, \xi + \eta)\,, \tag{0.0.2}$$

where the Fourier transform acts only on the x variable and $a^*(x, D)$ is the adjoint of the operator $a(x, D)$. After observing that the map T changes the horizontal space $\{(\xi, 0);\ \xi \in \boldsymbol{R}^n\}$ into the twisted diagonal $\{(\xi, -\xi);\ \xi \in \boldsymbol{R}^n\}$, HÖRMANDER [11] identi-

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fies the class of all operators in $\mathrm{Op}\, S^0_{1,1}(\boldsymbol{\Omega})$ which are H^s-bounded for every $s \in \boldsymbol{R}$, the set $(\mathrm{Op}\, S^0_{1,1}(\boldsymbol{\Omega})) \cap (\mathrm{Op}\, S^0_{1,1}(\boldsymbol{\Omega}))^*$ and $\mathrm{Op}\, \tilde{S}^0_{1,1}(\boldsymbol{\Omega})$. This last operator class is given by all the pseudodifferential operators in $\mathrm{Op}\, S^0_{1,1}(\boldsymbol{\Omega})$ whose symbol $a(x, \eta)$ «vanishes» in a suitable way in a neighborhood of the twisted diagonal.

In addition, under more restrictive conditions on the symbols, HÖRMANDER [11] gives a suitable symbolic calculus for adjoint and composition.

In a recent work [8], starting from the results described above, I introduce a new selfadjoint operator algebra contained in $\mathrm{Op}\, S^m_{1,1}(\boldsymbol{\Omega})$ where I consider a somewhat more precise symbolic calculus, which leads me to study suitable microlocal properties for Sobolev singularities.

Always suggested by the arguments in HÖRMANDER [11], the aim of this paper is to investigate the properties of the anisotropic pseudodifferential operators of type 1,1, whose symbol is a function in $\mathcal{C}^\infty(\boldsymbol{\Omega} \times \boldsymbol{R}^n)$, which satisfies, for some anisotropic weight function $[\xi]_M$, $M = (M_1, \ldots, M_n) \in \boldsymbol{N}^n$, $M_j \geq 1$:

$$\sup_{x \in K} |\partial^\alpha_\eta \partial^\beta_x a(x, \eta)| \leq c_{\alpha, \beta, K}(1 + [\xi]_M)^{m + \langle M, \beta - \alpha \rangle}, \tag{0.0.3}$$

where K is any compact subset of $\boldsymbol{\Omega}$, α, β are multi-indeces in $\boldsymbol{Z}^n_+$ and $\langle M, \alpha \rangle = M_1 \alpha_1 + \ldots + M_n \alpha_n$.

More precisely using some properties of the anisotropic Littlewood-Paley decomposition, introduced in § 1, in § 2 I consider some restrictive conditions on the symbols, which are necessary and sufficient for the continuity of the respective operators, in the frame of the anisotropic Sobolev spaces. In § 3 I set up a suitable symbolic calculus which leads me to show in § 4 a microlocal property for anisotropic Sobolev singularities. In the second part of this last section, in the more general framework of the inhomogeneous pseudodifferential calculus, I perform a counter-example, which shows as the previous result is in some sense optimal.

Following closely the arguments in HÖRMANDER [11, § 7], in the last part of § 3 I state an inequality of «sharp Gårding» type. This result generalizes in some way the classic «sharp Gårding» inequality, see [10], and its anisotropic version as stated in SEGÀLA [16]; it may be useful, I hope, in future studies on the propagation of singularities.

At the end let me notice that boundedness and microlocal properties for anisotropic *paradifferential* operators, which are widely studied, in the more general frame of quasi-homogeneous Triebel spaces in the works of YAMAZAKI [18], [19], are someway confirmed in the present paper.

1. – Anisotropic Littlewood-Paley decomposition.

Let $M = (M_1, \ldots, M_n)$ be an n-tuple of integer positive numbers, each one greater or equal than one, and define, for $\xi \in \boldsymbol{R}^n$, $[\xi] = [\xi]_M$ the unique positive root of the equation $t^{-2M_1}\xi_1^2 + \ldots + t^{-2M_n}\xi_n^2 = 1$, $[0] = 0$.

The quasi-homogeneous (anisotropic) weight function $[\xi]$ satisfies the following properties, which are widely proved in literature, see for example LASCAR [13].

With standard vectorial notation:

(1.0.1) $$[\xi+\eta] \leqslant [\xi]+[\eta];$$

(1.0.2) $$\min\{|\xi|, |\xi|^{1/M_0}\} \leqslant [\xi] \leqslant |\xi|, \qquad \text{where } M_0=\max_i M_i;$$

(1.0.3) $$[t^M \xi]=[(t^{M_1}\xi_1, \ldots, t^{M_n}\xi_n)]=t[\xi], \qquad \text{for } t>0;$$

(1.0.4) $$[-\xi]=[\xi];$$

(1.0.5) $$[\xi-\eta] \geqslant |[\xi]-[\eta]|;$$

(1.0.6) $$[\xi] \in \mathcal{C}^\infty(\boldsymbol{R}^n \backslash 0) \qquad \text{and} \qquad \partial_\xi^\alpha[\xi]=\chi_\alpha([\xi_0])[\xi]^{1-\langle M, \alpha\rangle},$$

where $\xi_0=[\xi]^{-M}\xi=([\xi]^{-1})^M\xi \in S^{n-1}$ and χ_α, is a continuous function of $T \in \boldsymbol{R}_+$.

Generally speaking we shall say that a function $\Lambda(\theta)$ is M-homogeneous of degree $\lambda \in \boldsymbol{Z}$ if $\Lambda(t^M\theta)=t^\lambda \Lambda(\theta)$, for every $t>0$. We can easily see that the derivatives of order $\alpha \in \boldsymbol{Z}_+^n$ of $\Lambda(\theta)$ are M-homogeneous of degree $\lambda-\langle M, \alpha\rangle$ and moreover $\Lambda(\theta) \leqslant c[\theta]^\lambda$, for some constant $c>0$.

Lastly we shall say that $W \subset \boldsymbol{R}^n$ is M-conic if for every $t>0$, $t^M\xi \in W$ when $\xi \in W$.

Let us consider now $\varphi \in \mathcal{C}^\infty(\boldsymbol{R})$, $0 \leqslant \varphi(t) \leqslant 1$, such that, for some $K \geqslant 1$, $\operatorname{supp}\varphi \subset \{t \in \boldsymbol{R};\ |t|<K\}$, $\varphi(t)=1$ when $|t|<1/2K$, and define, for $p \geqslant 0$: $\psi_p(\xi)=\varphi_{p+1}(\xi)-\varphi_p(\xi)$, where $\varphi_p(\xi)=\varphi([\xi]/2^p)$.

Since

(1.1.1) $$\operatorname{supp}\psi_p \subset C_p=\left\{\frac{1}{K}2^{p-1}<[\xi]<K2^{p+1}\right\},$$

we can easily verify that, for every $q \in \boldsymbol{N}$: $\varphi_q(\xi)+\sum_{p>q}\psi_p(\xi)=1$ and, setting $\psi_{-1}(\xi)=\varphi_0(\xi)$, $\sum_{p=-1}^{\infty}\psi_p(\xi)=1$. If we define now, for every $u \in \mathcal{S}'(\boldsymbol{R}^n)$, $u_p(x)=\psi_p(D)u=(2\pi)^{-n}\int e^{i\langle x, \xi\rangle}\psi_p(\xi)\widehat{u}(\xi)\,d\xi$, we can introduce the anisotropic Littlewood-Paley decomposition:

(1.1.2) $$u=\sum_{p=-1}^{\infty}u_p.$$

In the following we will usually assume $K=1$.

Let us say now that $u \in \mathcal{S}'(\boldsymbol{R}^n)$ belongs to the Sobolev space $H_M^s=H_M^s(\boldsymbol{R}^n)$, $s \in \boldsymbol{R}$, if the norm

(1.1.3) $$\|u\|_{M,s}=\|(1+[D]^2)^{s/2}u\|_{L_2}$$

is bounded.

The properties of the spaces H_M^s are similar to those of the usual Sobolev spaces; particularly, if we define in the standard way $H_{M,\mathrm{loc}}^s(\boldsymbol{\Omega})$, we can show that: $\bigcup_{s\in\boldsymbol{R}}H_{M,\mathrm{loc}}^s(\boldsymbol{\Omega})=\mathcal{D}_F'(\boldsymbol{\Omega})$, $\bigcap_{s\in\boldsymbol{R}}H_{M,\mathrm{loc}}^s(\boldsymbol{\Omega})=\mathcal{C}^\infty(\boldsymbol{\Omega})$, also in the topological sense.

PROPERTY 1.1. – *For every* $s \in \boldsymbol{R}$, $u \in \mathcal{S}'(\boldsymbol{R}^n)$, *we have* $u \in H^s_M$ *if and only if*:

$$\|u_p\|_{L_2} \leq c_p 2^{-sp}, \qquad \text{where } \{c_p\} \in l^2; \tag{1.1.4}$$

moreover, for suitable positive constants c, C *greater than zero*:

$$c\|u\|^2_{M,s} \leq \sum_{p=-1}^{\infty} \|u_p\|^2_{L_2} 2^{sp} \leq C\|u\|^2_{M,s}. \tag{1.1.5}$$

We can also verify that $u = \sum_{p=-1}^{\infty} v_p \in H^s_M$ when, for every $p = -1, \ldots$, we have: $v_p \in \mathcal{C}^\infty$, supp $\hat{v}_p \subset C_p$ and (1.1.4) is verified.

For the proof of the previous Property and more details on the quasi-homogeneous Littlewood-Paley decomposition, the reader can see for example YAMAZAKI [18], where H^s_M is considered as a special case of Triebel space, and GARELLO [9], where more general weight functions are introduced.

Let us now denote briefly $S^m_M(\boldsymbol{\Omega})$, instead of $S^m_{1,1,M}(\boldsymbol{\Omega})$, the set of symbols $a(x, \eta) \in \mathcal{C}^\infty(\boldsymbol{\Omega} \times \boldsymbol{R}^n)$, which satisfy (0.0.3) and Op $S^m_M(\boldsymbol{\Omega})$ the respective class of pseudodifferential operators defined by (0.0.1).

For every $m \in \boldsymbol{R}$, $S^m_M(\boldsymbol{\Omega})$ are all Fréchet spaces, with topology defined by taking the best constants $c_{\alpha,\beta,K}$ in (0.0.3) as semi-norms.

Without any restriction, in what follows, we always consider compactly supported pseudodifferential operators, so that they extend to linear operators from $\mathcal{D}'(\boldsymbol{\Omega})$ to itself and the Fourier transform $\mathcal{F}$ can act on the x variable of the respective symbols.

We can now give a first application of the Littlewood-Paley decomposition.

LEMMA 1.2. – *Let us consider* $a(x, \eta) \in S^m_M(\boldsymbol{\Omega})$ *such that* $a(x, \eta) = 0$ *when* $[\eta] < 1/2$ *and define, for* $p \geq -1$:

$$\hat{b}_p(\xi, \eta) = \psi_p([\eta]^{-M}\xi)\, \hat{a}(\xi, \eta). \tag{1.2.1}$$

Then:

$$a(x, \eta) = \sum_{p=-1}^{\infty} b_p(x, \eta), \tag{1.2.2}$$

where $2^{pN} b_p(x, \eta)$ *is bounded, for each* $N \in \boldsymbol{N}$, *in the topology of* $S^m_M(\boldsymbol{\Omega})$ *and the series converges in the same space.*

PROOF. – Let us notice that, for the M-homogeneity of $[\xi]$, $\psi_p(\xi) = \psi_0((2^p)^{-M}\xi)$, $p \geq 0$. If we set now $t = 2^p[\eta]$ and $\Psi \in \mathcal{S}(\boldsymbol{R}^n)$ the inverse Fourier transform of ψ_0, we have:

$$\begin{aligned} b_p(x, \eta) &= \mathcal{F}^{-1}_\xi[\psi_p([\eta]^{-M}\xi)\, \hat{a}(\xi, \eta)](x, \eta) = \\ &= \left[\int e^{\langle \cdot, \xi\rangle} \psi_0(t^{-M}\xi)\, d\xi * a(\cdot, \eta)\right](x, \eta) = t^{|M|}\, \Psi(t^M \cdot) * a(\cdot, \eta)(x, \eta). \end{aligned} \tag{1.2.3}$$

By using now the Taylor expansion of $a(\cdot\,,\eta)$, centered at x, for a suitable $N_0>0$, we have:

$$
(1.2.4)\qquad b_p(x,\eta)=\sum_{|\alpha|<N_0}(\alpha!)^{-1}t^{|M|}\,\Psi(t^M\cdot)*\partial_x^\alpha a(x,\eta)(\cdot-x)^\alpha+
$$

$$
+\sum_{|\alpha|=N_0}t^{|M|}\,\Psi(t^M\cdot)*\int_0^1(1-T)\,\partial_x^\alpha a(x+T(\cdot-x),\eta)(\cdot-x)^\alpha\,dT\,.
$$

Since for $\alpha\in\boldsymbol{Z}_+^n$:

$$
(1.2.5)\qquad \int t^{|M|}\,\Psi(t^My)\,y^\alpha\,dy=\int\Psi(y)(t^{-M}y)^\alpha\,dy=
$$

$$
=t^{-\langle M,\alpha\rangle}\int\Psi(y)\,y^\alpha\,dy=t^{-\langle M,\alpha\rangle}(-D)^\alpha\psi_0(0)=0\,,
$$

each term in the sum in the first line of (1.2.4) identically vanishes.

In order to estimate the remainder notice that, for every $\alpha\in\boldsymbol{Z}_+^n$

$$
(1.2.6)\qquad t^{\langle M,\alpha\rangle}\int t^{|M|}\,|\Psi(t^My)|\,|y^\alpha|\,dy=\int|\Psi(y)|\,|y|^{|\alpha|}\,dy\leqslant c_{|\alpha|}\,.
$$

Since $|\partial_x^\alpha a(x+Ty,\eta)|\leqslant c_\alpha(1+[\eta])^{m+\langle M,\alpha\rangle}$ and η is far from the origin, for every integer $N>0$ we may estimate:

$$
(1.2.7)\qquad |b_p(x,\eta)|\leqslant\sum_{|\alpha|=N}c_\alpha 2^{-p\langle M,\alpha\rangle}(1+[\eta])^m\leqslant C_N2^{-pN}(1+[\eta])^m\,.
$$

In order to evaluate the derivatives let us notice that differentiation with respect to the x variables only involves the symbol $a(x,\eta)$. By regards to the derivatives with respect to the η variables we can observe that

$$
(1.2.8)\qquad \partial_\eta^\gamma(G(t))=\sum_q G^{(q)}(t)\,\partial^{\beta_1}t\ldots\partial^{\beta_q}t\,;
$$

with $1\leqslant q\leqslant|\gamma|$ and $\sum_{j=1}^q\beta_j=\gamma$. Then

$$
(1.2.9)\qquad \partial_\eta^\gamma(G(t))=[\xi]^{-\langle M,\gamma\rangle}\sum_q\chi_q([\xi_0])\,t^qG^{(q)}(t)\,,
$$

with χ_q defined by (1.0.6) and $\xi_0\in S^{n-1}$.

Let us observe now that:

$$
(1.2.10)\qquad \left(t\frac{d}{dt}\right)^h t^{|M|}\,\Psi(t^My)=t^{|M|}\sum_{k\leqslant h}c_k\left(t\frac{d}{dt}\right)^k\Psi(t^My)\,;
$$

thus

$$(1.2.11)\qquad [\xi]^{\langle M,\gamma\rangle}\partial_\eta^\gamma(t^{|M|}\Psi(t^M y)) = \sum_q \chi_q([\xi_0])\, t^{|M|} \sum_{h\leq q} c_h \left(t\frac{d}{dt}\right)^h \Psi(t^M y) =$$

$$= \sum_k c_k \chi_k([\xi_0])\, t^{|M|}\, \Psi_k(t^M y);$$

with $\Psi_k(t^M y) = (t(d/dt))^k \Psi(t^M y)$.

Now

$$(1.2.12)\qquad \partial_\eta^\alpha b_p(x,\eta) = \sum_{|\gamma|\leq|\alpha|}\binom{\alpha}{\gamma}\int \partial_\eta^\gamma[t^{|M|}\Psi(t^M y)]\,\partial_\eta^{\alpha-\gamma} a(x-y,\eta)\,dy\,.$$

Let us observe that:

$$(1.2.13)\qquad \widehat{\Psi_k(t^M y)}(\xi) = t^{-|M|}\widehat{\Psi}_k(t^{-M}\xi) = t^{-|M|}\left(t\frac{d}{dt}\right)\widehat{\Psi}(t^{-M}\xi)\,.$$

Setting now $\zeta = t^{-M}\xi$ and $\bar{\zeta} = (-M_1\zeta_1, \ldots, -M_n\zeta_n)$ we have:

$$(1.2.14)\qquad \widehat{\Psi}_k(\zeta) = \left(t\frac{\partial}{\partial t}\right)^k \widehat{\Psi}(t^{-M}\xi) = \left(t\frac{\partial\zeta}{\partial t}\frac{\partial}{\partial\zeta}\right)^k\widehat{\Psi}(\zeta) = \left\langle\bar{\zeta}, \frac{\partial}{\partial\zeta}\right\rangle^k \widehat{\Psi}(\zeta),$$

since $t(\partial\zeta_j/\partial t) = -tM_j t^{-M_j-1}\xi_j = -M_j\zeta_j$. Finally supp $\widehat{\Psi}_k \subset \psi_0$, then arguing as in the proof of (1.2.7), we obtain:

$$(1.2.15)\qquad |\partial_\eta^\alpha b_p(x,\eta)| \leq c_N 2^{-pN}(1+[\eta])^{m-\langle M,\alpha\rangle},$$

which conclude the proof.

Following closely the proof in HÖRMANDER [11, Proposition 2.2] and keeeping in mind (1.0.2) and (1.0.6), we can state:

LEMMA 1.3. – *Let us suppose that $a(x,\eta)$ has bounded derivatives of each order which are rapidly decreasing as η tends to infinitive, then $a(x,D)$ is continuous from $H^{-s}_{M,\mathrm{loc}}(\boldsymbol{\Omega})$ to $L_{2,\mathrm{loc}}(\boldsymbol{\Omega})$ for each positive s. Moreover its norm in $\mathcal{L}(H_M^{-s}, L_2)$ is bounded by*

$$(1.3.1)\qquad C_{s,t}\sum_{|\alpha|\leq n+1}\sup_x\int|\partial_x^\beta\partial_\eta^\alpha a(x,\eta)|(1+[\eta])^s\,d\eta\,.$$

If $a^*(x,\eta)$ is the symbol of the adjoint operator of $a(x,D)$, for arbitrary positive integers L, N:

$$(1.3.2)\qquad \sup_{x,\eta}(1+[\eta])^L \left| a^*(x,\eta) - \sum_{\alpha<N}(\alpha!)^{-1} D_\eta^\alpha \partial_x^\alpha \overline{a(x,\eta)} \right| \leq$$

$$\leq C_{L,M} \sum_{|\alpha|+|\beta|+|\gamma|\leq 2N+L+2n+1,\ |\beta|\geq N} \sup_{x,\eta} |\eta^\gamma \partial_x^\beta \partial_\eta^\alpha a(x,\eta)|\,.$$

Setting now $c(x,D)=b(x,D)\,a(x,D)$, with $b(x,\eta)$ satisfying the same properties of $a(x,\eta)$, we have:

$$(1.3.3)\qquad \sup_{x,\eta}(1+[\eta])^L \left| c(x,\eta) - \sum_{|\alpha|<N}(\alpha!)^{-1} \partial_\eta^\alpha b(x,\eta)\, D_x^\alpha a(x,\eta) \right| \leq$$

$$\leq C_{N,L} \sum_{|\alpha|+|\beta|+|\gamma|\leq 2N+L+2n+1,\ |\beta|\geq N} \sup_{x,\eta} |\eta^\gamma \partial_\eta^\alpha b(x,\eta)\, D_x^\alpha a(x,\eta)\,.$$

THEOREM 1.4. – *Let $a(x,\eta)$ be in $C^N(\boldsymbol{R}^{2n})$ and assume that*

$$(1.4.1)\qquad |\partial_{x,\eta}^\alpha a(x,\eta)| \leq 1 \quad \text{when } |\alpha| = N\,.$$

If the norm H of $a(x,D)$ as operator in L_2 is less than one, then:

$$(1.4.2)\qquad |\partial_{x,\eta}^\alpha a(x,\eta)| \leq c_N H^{(N-|\alpha|)/(N+n)}, \qquad |\alpha|<N\,.$$

For the proof see HÖRMANDER [11, Theorem A.1]

REMARK. – (1.4.1) applies to $b(x,\eta)=(AB)^{-M_0N}a((A/B)^M x,(B/A)^M\eta)$ where A, B are chosen greater or equal than one, $M_0=\max_j M_j$, $a(x,\eta)\in C^N(\boldsymbol{R}^{2n})$ and satisfies:

$$(1.4.3)\qquad |\partial_\eta^\alpha \partial_x^\beta a(x,\eta)| \leq A^{\langle M,\alpha\rangle} B^{\langle M,\beta\rangle}, \qquad |\alpha+\beta| = N\,.$$

Moreover, for every $p(x,\eta)\in \mathcal{S}'(\boldsymbol{R}^{2n})$ and $T>0$, we have:

$$(1.4.4)\qquad p(T^{-M}x, T^M D)\,u(x) = [p(\cdot\,,D)\,u(T^M\cdot)](T^{-M}\cdot)\,;$$

then, if $p(x,D)$ is bounded as operator in L_2,

$$(1.4.5)\qquad \|p(T^{-M}x, T^M D)\,u\|_{L_2} \leq \|p(x,D)\|_{\mathcal{L}(L_2)} \|u\|_{L_2}\,.$$

If we suppose now that the norm H of $a(x,D)$ as operator in L_2 is bounded by $(AB)^{M_0N}$, using (1.4.2) we can conclude:

$$(1.4.6)\qquad |\partial_\eta^\alpha \partial_x^\beta a(x,\eta)| \leq (A/B)^{\langle M,\alpha-\beta\rangle}(AB)^{M_0N((|\alpha|+|\beta|+n)/(N+n))} H^{(N-|\alpha|-|\beta|)/(N+n)},$$

where for great N the right hand side reads $(AB)^{M_0(\langle M,\alpha+\beta\rangle+n)}(A/B)^{\langle M,\alpha-\beta\rangle}H$.

At the end of this section let us quote a technical lemma which is essentially shown in HÖRMANDER [11, proof of Lemma 3.2].

LEMMA 1.5. – *Let us set, for arbitrary* $s \in \boldsymbol{R}$ *and* $\xi \in \boldsymbol{R}^n$:

$$F(\xi) = (1 + Q(\xi)^2)^s \sum_p (2^{2p} + Q(\xi)^2)^{-s}, \tag{1.5.1}$$

where the sum is extended to $p \geq 0$ *such that* $(Q(\xi))/(2(2A+1)) < 2^p < 2B(Q(\xi)+1)$, $A \geq 1$, $B \geq 1$. *We can then verify, for every* $\xi \in \boldsymbol{R}^n$:

$$F(\xi) \leq 8^{|s|}(\log_2(8A+4) + (1-(8B)^{-2s})/s). \tag{1.5.1}$$

2. – Continuity.

LEMMA 2.1. – *Let* $a(x, \eta)$ *be in* $S_M^m(\boldsymbol{\Omega})$ *and satisfy*:

$$\hat{a}(\xi, \eta) = 0 \quad \text{when} \quad [\xi] > A([\eta]+1) \quad \text{or} \quad [\xi+\eta]+1 < [\eta]/B, \tag{2.1.1}$$

then $a(x, D)$ *is continuous from* $H^{s+m}_{M,\,\mathrm{loc}}(\boldsymbol{\Omega})$ *to* $H^s_{M,\,\mathrm{loc}}(\boldsymbol{\Omega})$, *for every* $s \in \boldsymbol{R}$.

PROOF. – Since $a(x, D)\,u = a(x, D)(1 + [D]^2)^{-m/2}(1 + [D]^2)^{m/2}$ let us suppose that $m = 0$. By means of the Littlewood-Paley decomposition we can write

$$u = \sum_{p=-1}^{\infty} u_p \quad \text{and} \quad a(x, D)\,u = \sum_{p=-1}^{\infty} a(x, D)\,u_p = \sum_{p=-1}^{\infty} h_p. \tag{2.1.2}$$

Since

$$\hat{h}_p(\xi) = (2\pi)^{-n} \int \hat{a}(\xi - \eta, \eta)\,\psi_p(\eta)\,\hat{u}(\eta)\,d\eta, \tag{2.1.3}$$

using the hypotesis (2.1.1) we can assure that $\hat{a}(\xi-\eta, \eta)$ identically vanishes if $[\xi-\eta] > A([\eta]+1)$ or $[\xi]+1 < [\eta]/B$. Recalling that $2^{p+1} < [\eta] < 2^{p+1}$ in the support of ψ_p, using also (1.0.5) we can conclude:

$$\operatorname{supp} \hat{h}_p(\xi) \subset \left\{\xi \in \boldsymbol{R}^n;\ \frac{2^{p-1}-B}{B} < [\xi] < (2A+1)\,2^{p+1}\right\}. \tag{2.1.4}$$

Thus for a suitable constant $N > 0$:

$$\|\psi_q(D)\,a(x, D)\,u\|^2_{L_2} = \left\| \sum_{-N<j<N} \psi_q(D)\,h_{q+j} \right\|^2_{L_2} \leq 2N \sum_{-N<j<N} \|h_{q+j}\|^2_{L_2}. \tag{2.1.5}$$

By observing now that $h_p = \sum_{-1 \leq k \leq 1} a(x, D)\,\psi_{p+k}(D)\,u_p$, we can easily conclude that the symbol $b_p(x, \eta) = \sum_{-1 \leq k \leq 1} a((2^p)^{-M}x, (2^p)^M \eta)\,\psi_{p+k}((2^p)^M \eta)$ has bounded support by respect to η. Using then (1.3.1) and restoring the former variables we obtain

$$\|h_q\|_{L_2} \leq c_s \|u_q\|_{L_2}, \tag{2.1.6}$$

which, in view of Property 1.1, ends the proof.

REMARK 2.2. – In order to better investigate the norm of $a(x, D)$ as bounded linear operator, let us first notice that, in view of (1.3.1), $b_p(x, D)$ defined above is continuous from H^s_M to L_2, with norm bounded by $c_s(a)$, for every $s \in \boldsymbol{R}$. Then by a change of variable we obtain:

$$\|(2^{2p} + [D]^2)^{s/2} h_p\|_{L_2} \leq c_s(a) 2^{ps} \|u_p\|_{L_2}. \tag{2.2.1}$$

Moreover by the Cauchy Schwarz inequality we have:

$$(1 + [\xi]^2)^s \left| \sum_{= -1}^{\infty} \widehat{h}_p(\xi) \right|^2 \leq (1 + [\xi]^2)^s \sum_{p=-1}^{\infty} (2^{2p} + [\xi]^2)^{-s} \sum_{p=-1}^{\infty} (2^{2p} + [\xi]^2)^s \, |\widehat{h}_p(\xi)|^2,$$

where the first factor in the right-hand side is bounded by (1.5.1). We can then conclude, using also Property 1.1 and the remark which follows, that the norm of $a(x, D)$ in $\mathcal{L}(H^{s+m}_M, H^s_M)$ is bounded by

$$c_s(a) 40^{|s|/2} (\log_2(8A + 4) + (1 - (8B)^{-2s})/s)^{1/2}, \tag{2.2.2}$$

where $c_s(a)$ is a seminorm in $S^m_M(\boldsymbol{\Omega})$.

THEOREM 2.3. – *Let $a(x, \eta)$ be in $S^m_M(\boldsymbol{\Omega})$ and satisfy*

$$\widehat{a}(\xi, \eta) = 0 \quad \textit{when} \quad [\xi + \eta] + 1 < [\eta]/B, \quad B \geq 1, \tag{2.3.1}$$

then $a(x, D)$ is bounded from $H^{s+m}_{M, \mathrm{loc}}(\boldsymbol{\Omega})$ to $H^s_{M, \mathrm{loc}}(\boldsymbol{\Omega})$ for every $s \in \boldsymbol{R}$. Moreover its operator norm in $\mathcal{L}(H^{s+m}_M, H^s_M)$ is bounded by

$$c_s(a)((1 - (8B)^{-2s})/s)^{1/2}, \tag{2.3.2}$$

where $c_s(a)$ is a seminorm in $S^m_M(\boldsymbol{\Omega})$ such that $c_s(a) 40^{-|s|/2}$ is an increasing function of s.

PROOF. – Let us decompose:

$$a(x, D) = a(x, D)\, \varphi_0(D) + a(x, D)(1 - \varphi_0(D)), \tag{2.3.3}$$

where the first term in the right-hand side is in $\mathcal{L}(H^t_M, H^s_M)$, for every $t, s \in \boldsymbol{R}$; we can then suppose that $a(x, \eta)$ identically vanishes for $[\eta] < 1/2$ and apply Property 1.2; then $a(x, \eta) = \sum_{p=-1}^{\infty} b_p(x, \eta)$, with $2^{pN} b_p(x, \eta)$ bounded in $S^m_M(\boldsymbol{\Omega})$. The hypotesis (2.3.1) is inherited by any term in the sum and moreover each of them identically vanishes when $[\xi] > 2^{p+1}[\eta]$. Applying now term by term Lemma 2.1 and Remark 2.2 we can conclude that the operator norm of $a(x, D)$ in $\mathcal{L}(H^{s+m}_M, H^s_M)$ is bounded by

$$C_s(a)(1 + (1 - 8B)^{-2s}/s)^{1/2} + \sum_{p=-1}^{\infty} ((2p)^{1/2} + ((1 - (8B)^{-2s})/s)^{1/2}) 2^{-p}, \tag{2.3.4}$$

then by

$$c'_s(a)(1 + (1 - (8B)^{-2s})/s)^{1/2}. \tag{2.3.5}$$

Changing the constant c_s again we can drop the first term since the convexity of the exponential function assures that $(1-(8B)^{-2s}/2s)$ is greater than

$$\begin{cases} (1-8^{-2s})/2s \geq 8^{-2s}\log 8\,, & s>0\,, \\ \log(8B) \geq \log 8\,, & s \leq 0\,. \end{cases} \tag{2.3.6}$$

LEMMA 2.4. – *For $s \in \boldsymbol{R}$, $s<0$, let $a(x,D) \in \operatorname{Op} S^0_M(\boldsymbol{\Omega})$ be bounded in $H^s_{M,\mathrm{loc}}(\boldsymbol{\Omega})$. If we set now*

$$\hat{c}_B(\xi,\eta) = \sum_{2^p>B} \varphi_0((B/2^p)^M(\xi+\eta))\,\psi_p(\eta)\,\hat{a}(\xi,\eta)\,, \qquad B \geq 1\,, \tag{2.4.1}$$

we obtain the following estimates for the symbol $c(x,\eta)$:

$$|\partial^\alpha_\eta \partial^\beta_x c_B(x,\eta)| \leq c_{\alpha,\beta,s} B^{M_0\langle M,\alpha+\beta\rangle + \langle M,\alpha-\beta\rangle + M_0 n + s}(1+[\eta])^{\langle M,\beta-\alpha\rangle}\,. \tag{2.4.2}$$

PROOF. – For $R \geq 1$ let us introduce the symbol $a_R(x,\eta) = a(x,\eta)\,\psi_0(R^{-M}\eta)$. Since in the support of $\psi_0(R^{-M}\eta)$ we have $(1+[\eta]^2)^{1/2} > R/2$, $a(x,D)$ has norm L as operator on H^s_M and s is negative, we can easily conclude:

$$\|a_R(x,D)\,u\|_{s,M} \leq L\left(\frac{R}{2}\right)^s \|u\|_{L_2}. \tag{2.4.3}$$

In view of (1.4.4) and (1.4.5) we have then:

$$\|(1+[R^M D]^2)^{s/2} a_R(R^{-M}x, R^M D)\,u\|_{L_2} \leq \frac{L}{2^s} R^s \|u\|_{L_2}. \tag{2.4.4}$$

Since $\varphi_0(\xi) \leq 1$ and, in $\operatorname{supp}\varphi_0(B^M\xi)$, $[\xi] < B^{-1}$, we have: $(R^{-2}+[\xi]^2)^{-s/2} \leq$ $\leq (R^{-2}+B^{-2})^{-s/2}$, for $s<0$.

Then if we denote by $b_{RB}(x,D)$ the product to the left of $a_R(R^{-M}x, R^M D)$ by $\varphi_0(B^M D)$ and we suppose also $R>B$:

$$\begin{aligned} \|b_{RB}(x,D)\,u\|_{L_2} &\leq (R^{-2}+B^{-2})^{-s/2}\|(R^{-2}+[D]^2)^{s/2} a_R(R^{-M}x, R^M D)\,u\|_{L_2} \leq \\ &\leq (R^{-2}+B^{-2})^{-s/2} R^{-s}\|(1+[R^M D]^2)^{s/2} a_R(R^{-M}x, R^M D)\|_{L_2} \leq L2^{-(3/2)s} B^s \|u\|_{L_2}, \end{aligned} \tag{2.4.5}$$

then $b_{RB}(x,D)$ is L_2 bounded with norm less than $L2^{-(3/2)s}B^s$.

Moreover, if Φ is the inverse Fourier transform of φ_0, we have:

$$\hat{b}_{RB}(\xi,\eta) = R^{|M|}\varphi_0(B^M(\xi+\eta))\,\hat{a}_R(R^M\xi, R^M\eta)\,, \tag{2.4.6}$$

$$b_{RB}(x,\eta) = B^{-|M|}\int e^{-i\langle y,\eta\rangle}\Phi(B^{-M}y)\,a_R(R^{-M}(x-y), R^M\eta)\,dy\,. \tag{2.4.7}$$

The derivatives of $b_{RB}(x,\eta)$ with respect to x are uniformly bounded, but differentiation by η produces factors y. The best estimate we can give is so:

$$|\partial^\alpha_\eta \partial^\beta_x b_{RB}(x,\eta)| \leq c_{\alpha,\beta} B^{\langle M,\alpha\rangle}. \tag{2.4.8}$$

Using then Theorem 1.4 and the remark which follows it, we have:

$$(2.4.9) \qquad |\partial_\eta^\alpha \partial_x^\beta b_{RB}(x,\eta)| \leq c_{\alpha,\beta,s} B^{M_0\langle M,\alpha+\beta\rangle + \langle M,\alpha-\beta\rangle + M_0 n + s},$$

which is obviously significative only for great negative s.

Returning now to the original variables we have:

$$(2.4.10) \qquad R^{-|M|}\hat{b}_{RB}(R^{-M}\xi, R^{-M}\eta) = \varphi_0((B/R)^M(\xi+\eta))\,\hat{a}(R^M\xi,\eta)\,\psi_0(R^{-M}\eta).$$

Then, for $R = 2^p$:

$$(2.4.11) \qquad \hat{c}_B(\xi,\eta) = \sum_{2^p > B} (2^p)^{-|M|}\hat{b}_{2^pB}((2^p)^{-M}\xi,(2^p)^{-M}\eta),$$

$$(2.4.12) \qquad c_B(x,\eta) = \sum_{2^p > B} b_{2^pB}((2^p)^M x,(2^p)^{-M}\eta),$$

since $1 + [\eta] \sim 2^p$ in supp $\hat{b}_{2^pB}((2^p)^{-M}\xi,(2^p)^{-M}\eta)$, we plainly obtain (2.4.2).

Let us consider now the cutoff function $\chi \in \mathcal{C}^\infty(\boldsymbol{R}^{2n})$ such that:

$$(2.5.1) \qquad \operatorname{supp}\chi \subset \{(\xi,\eta)\in\boldsymbol{R}^{2n};\ [\xi]\leq[\eta],\ [\eta]\geq 1\};$$

$$(2.5.2) \qquad \chi(\xi,\eta) = 1 \text{ in } \{(\xi,\eta)\in\boldsymbol{R}^{2n};\ 2[\xi]\leq[\eta],\ [\eta]\geq 2\};$$

$$(2.5.3) \qquad \chi(\xi,\eta) \text{ is } M\text{-homogeneous of degree } 0 \text{ for } [\eta]\geq 2.$$

THEOREM 2.5. – *Let us define for $a(x,\eta)\in S_M^0(\boldsymbol{\Omega})$ and $0<\varepsilon<1$:*

$$(2.5.4) \qquad \hat{a}_{\varepsilon\chi}(\xi,\eta) = \chi(\xi+\eta,\varepsilon^M\eta)\,\hat{a}(\xi,\eta),$$

with $\chi(\xi,\eta)$ defined above; then $a(x,D)$ is bounded from $H^s_{M,\mathrm{loc}}(\boldsymbol{\Omega})$ to itself for every $s\in\boldsymbol{R}$ if and only if the estimate

$$(2.5.5) \qquad |\partial_\eta^\alpha\partial_x^\beta a_{\varepsilon\chi}(x,\eta)| \leq c_{\alpha,\beta,N}\,\varepsilon^N(1+[\eta])^{\langle M,\beta-\alpha\rangle},$$

is satisfied for every $N\in\boldsymbol{N}$.

PROOF. – Let us notice that, for $\varepsilon < 1/4B$,

$$(2.5.6) \qquad \sum_{2^p\geq B}\varphi_0((B/2^p)^M\theta)\,\psi_p(\eta) = 1 \qquad \text{when } \chi(\theta,\varepsilon^M\eta)\neq 0,$$

since in $\operatorname{supp}\chi(\theta,\varepsilon^M\eta)$ we have $\varepsilon[\eta]\geq 1$, that is $[\eta]\geq 4B$; then using the properties of the Littlewood-Paley decomposition we have $\sum_{2^p\geq B}\psi_p(\eta)=1$. Moreover $[\theta]\leq\varepsilon[\eta]$, then in $\operatorname{supp}\psi_p(\eta)$ we have $[(B/2^p)^M\theta]\leq B\varepsilon[\eta]/2^p\leq 2B\varepsilon\leq 1/2$ that implies $\varphi_0((B/2^p)^M\theta)=1$.

We can then show:

$$(2.5.7) \qquad \hat{a}_{\varepsilon\chi}(\xi,\eta) = \chi(\xi+\eta,\varepsilon^M\eta)\,\hat{c}_B(\xi,\eta) \qquad \text{when } B\varepsilon < \frac{1}{4};$$

with $\hat{c}_B(\xi, \eta)$ defined in (2.4.1). Consequentely if we define $\Xi(x, \eta)$ the inverse Fourier transform of $\chi(\xi, \eta)$, we have:

$$(2.5.8) \qquad a_{\varepsilon\chi}(x, \eta) = \mathcal{F}_\xi^{-1}(\chi(\xi + \eta, \varepsilon^M \eta)\, \hat{c}_B(\xi, \eta)) =$$

$$= e^{-i\langle \cdot, \eta\rangle} \Xi(\cdot, \varepsilon^M \eta) * c_B(\cdot, \eta) = \int \Xi(y, \varepsilon^M \eta)\, e^{-i\langle y, \eta\rangle} c_B(x-y, \eta)\, dy .$$

Using now (2.4.2), the derivatives $\partial_\eta^\alpha \partial_x^\beta a_{\varepsilon\chi}(x, \eta)$, $\alpha, \beta \in \boldsymbol{Z}_+^n$, may be estimated by:

$$(2.5.9) \qquad \sum_{j \leq \alpha} c_j \varepsilon^{M_0\langle m, -\alpha-\beta+j\rangle + \langle M, \beta-\alpha+j\rangle - M_0 n - s} (1 + [\eta])^{\langle M, \beta-\alpha+j\rangle} \int |U_j(y, \eta)|\, dy ,$$

where $U_j(y, \eta) = \partial_\eta^j (e^{-i\langle y, \eta\rangle} \Xi(y, \varepsilon^M \eta))$. Now:

$$(2.5.10) \qquad y^k \partial_\eta^{j-k} \Xi(y, \varepsilon^M \eta) = \mathcal{F}_\xi^{-1}((\partial_\xi^k \partial_\eta^{j-k} \chi(\xi, \varepsilon^M \eta)) =$$

$$= \varepsilon^{\langle M, (j-k)\rangle} (\mathcal{F}^{-1} \partial_\xi^k \partial_\zeta^{j-k} \chi(\xi, \zeta)) , \qquad \zeta = \varepsilon^M \eta .$$

Since $\chi(\xi, \zeta)$ is M-homogeneous of degree 0, for large ζ, then the L_1-norm of $\mathcal{F}^{-1}((\partial_\xi^k \partial_\zeta^{j-k} \chi(\xi, \zeta))(y, \zeta)$, with respect to the y variable, is M-homogeneous of degree $-\langle M, j\rangle$, hence it is bounded by $c[\eta]^{-\langle M, j\rangle}$; we can thus conclude:

$$(2.5.11) \qquad \|y^k \partial_\eta^{j-k} \Xi(y, \varepsilon^M \eta)\|_{L_1(y)} \leq \varepsilon^{-\langle M, k\rangle} (1 + [\eta])^{-\langle M, j\rangle} \leq \varepsilon^{-\langle M, j\rangle} (1 + [\eta])^{-\langle M, j\rangle} .$$

Then:

$$(2.5.12) \qquad |\partial_\eta^\alpha \partial_x^\beta a_{\varepsilon\chi}(x, \eta)| \leq c_{\alpha, \beta, s}\, \varepsilon^{-M_0\langle M, \alpha+\beta\rangle - \langle M, \alpha-\beta\rangle - M_0 n - s} (1 + [\eta])^{\langle M, \beta-\alpha\rangle} .$$

In order to prove the if part, let us decompose:

$$(2.5.13) \qquad a(x, D) = a(x, D) - a_{1\chi}(x, D) + \sum_{p=0}^{\infty} a_{2^{-p}\chi}(x, D) - a_{2^{-p-1}\chi}(x, D) .$$

From (2.5.2) we have:

$$(2.5.14) \quad \chi(\xi + \eta, 2^{-p}\eta) - \chi(\xi + \eta, 2^{-p-1}\eta) = 0 , \qquad \text{when } [\xi] + \eta \leq 2^{-p-2}[\eta] .$$

We can then apply Theorem 2.3 to every term in the right-hand side of of (2.5.13); using also (2.5.5) we can end that each one has norm in $\mathcal{L}(H^s)$ bounded by $C_s 2^{-pN}$, for every $N \in \boldsymbol{N}$, which assures that $a(x, D) \in \mathcal{L}(H^s)$.

REMARK. – (2.5.5) reads as a necessary and sufficient condition in order to have continuity for *every* $s \in \boldsymbol{R}$. On the other hand (2.5.12) shows that, for fixed suitably large negative order s, a great gap exists between such necessary and sufficient conditions. For deeply arguing on this arguments the reader can see HÖRMANDER [11, Theorem 3.6], [12].

REMARK. – In the following we will always denote by $\tilde{S}_M^m(\boldsymbol{\Omega})$ the set of all the symbols in $S_M^m(\boldsymbol{\Omega})$, $m \in \boldsymbol{R}$, which satisfy (2.5.4), that is which maps continuously $H_{M, \mathrm{loc}}^s(\boldsymbol{\Omega})$ into $H_{M, \mathrm{loc}}^{s-m}(\boldsymbol{\Omega})$, for every $s \in \boldsymbol{R}$.

3. – Symbolic calculus.

LEMMA 3.1. – *Let us consider $a(x, \eta) \in S_M^m(\boldsymbol{\Omega})$ such that, for some $N \in \boldsymbol{N}$, $r \in \boldsymbol{R}_+$, $\partial_x^\beta a(x, \eta) \in S^{m+\langle M, \beta\rangle - r}(\boldsymbol{\Omega})$, when $|\beta| = N$, and moreover $\hat{a}(\xi, \eta)$ satisfies* (2.3.1). *Then:*

$$\hat{a}^*(\xi, \eta) = 0 \quad \text{when } [\xi + \eta] > B([\eta] + 1), \tag{3.1.1}$$

and we have the following estimates:

$$\left| \partial_\eta^\alpha \partial_x^\beta \left(a^*(x, \eta) - \sum_{|\lambda| < N} (\lambda!)^{-1} \partial_\eta^\lambda D_x^\lambda \overline{a(x, \eta)} \right) \right| \leqslant \tag{3.1.2}$$

$$\leqslant c_{\alpha, \beta}(a)\, B(B^{m+\langle M, \beta - \alpha\rangle - r} + 1)(1 + [\eta])^{m + \langle M, \beta - \alpha\rangle - r}.$$

where $c_{\alpha,\beta}(a)$ are sum of seminorms of $\partial_x^\gamma a(x, \eta)$ in $S_M^{m+\langle M, \gamma\rangle - r}(\boldsymbol{\Omega})$, $|\gamma| = N$.

PROOF. – Let us decompose $a(x, \eta) = \sum_{p=-1}^{\infty} a_p(x, \eta)$ and notice that, for $p > 0$ the symbol

$$A_p(x, \eta) = a_p((2^p)^{-M} x, (2^p)^M \eta) 2^{-mp} = a((2^p)^{-M} x, (2^p)^M \eta)\, \psi_0(\eta) 2^{-mp} \tag{3.1.3}$$

identically vanishes for $[\eta] > 1$. We can then estimate:

$$|\partial_\eta^\alpha \partial_x^\beta A_p(x, \eta)| \leqslant c_{\alpha, \beta}(a) 2^{-rp} \quad \text{for } |\beta| \geqslant N; \tag{3.1.4}$$

where $c_{\alpha,\beta}(a)$ are semi-norms of $a(x, \eta)$ in $S_M^{m+\langle M, \gamma\rangle - r}(\boldsymbol{\Omega})$, $|\gamma| = N$. Let us denote now $A_p^*(x, D)$ and $a_p^*(x, D)$ the adjoints of the operators $A_p(x, D)$ and $a_p(x, D)$ respectively. Applying then (1.3.2) we obtain, for L positive integer:

$$(1 + [\eta])^L \left| A_p^*(x, \eta) - \sum_{|\alpha| < N} (\alpha!)^{-1} \partial_\eta^\alpha D_x^\alpha \overline{A_p(x, \eta)} \right| \leqslant c_{L, N} 2^{-rp}. \tag{3.1.5}$$

Since $a_p^*(x, \eta) = 2^{mp} A_p^*((2^p)^M x, (2^p)^{-M} \eta)$ we have:

$$\left| a_p^*(x, \eta) - \sum_{|\alpha| < N} (\alpha!)^{-1} \partial_\eta^\alpha D_x^\alpha \overline{a_p(x, \eta)} \right| \leqslant c_{L, N} 2^{(m-r)p} (1 + [(2^p)^{-M} \eta])^{-L}. \tag{3.1.6}$$

For $\hat{a}^*(\xi, \eta) = \overline{\hat{a}(-\xi, \xi + \eta)}$, (3.1.1) immediately follows. Moreover, since we have $2^{p-1} \leqslant [\eta] \leqslant 2^{p+1}$ in supp $\hat{a}_p(\xi, \eta)$,

$$2^{p-1} \leqslant [\eta] + \xi \leqslant 2^{p+1} \quad \text{in supp } \hat{a}_p^*(\xi, \eta); \tag{3.1.7}$$

$$B([\eta] + 1) \geqslant 2^{p-1} \quad \text{in supp } \hat{a}_p^*(\xi, \eta). \tag{3.1.8}$$

Then, for $[\eta] \leqslant 1$, the series

$$B(x, \eta) = \sum_{p=0}^{\infty} \left(a_p^*(x, \eta) - \sum_{|\alpha| < N} (\alpha!)^{-1} \partial_\eta^\alpha \partial_x^\alpha \overline{a_p(x, \eta)} \right) \tag{3.1.9}$$

has at most $\log_2(8B)$ terms different from 0 and applying (3.1.6), with $L = 0$, each term may be estimated by $B^{m-r}+1$. For $[\eta]>1$ let us consider in (3.1.6) the smallest constant $L\geqslant 0$ such that $L+m-r\geqslant 1$. Observing that now the terms different from 0 are at most $\log_2 2B([\eta]+1)$, we obtain:

$$|B(x,\eta)|\leqslant c_L\sum_{p=0}^{\infty}2^{(m-r+L)p}[\eta]^{-L}\leqslant \tag{3.1.10}$$

$$\leqslant 2c_L(2B([\eta]+1))^{m-r+L}[\eta]^{-L}\leqslant c_L' B^{m-r+L}(1+[\eta])^{m-r},$$

which shows (3.1.2) when $\alpha=\beta=0$. The general statement follows immediately if, arguing as in HÖRMANDER [11, Lemma 4.1], we observe that:

$$(\partial_\eta^\alpha D_x^\beta a)^*(x,\eta)=\partial_\eta^\alpha D_x^\beta a^*(x,\eta). \tag{3.1.11}$$

LEMMA 3.2. – *For $m, r\in\boldsymbol{R}$, $r\geqslant 0$, let us consider $a(x,\eta)\in S_M^m(\boldsymbol{\Omega})$ such that $\partial_x^\beta a(x,\eta)\in S_M^{m+\langle M,\beta\rangle-r}(\boldsymbol{\Omega})$ when $|\beta|=N$, N positive integer, moreover let $\hat{a}(\xi,\eta)$ satisfy* (2.1.1).

If we set now $c(x,D)=b(x,D)\,a(x,D)$ with $b(x,\eta)\in S_M^\mu(\boldsymbol{\Omega})$ we have:

$$\left|\partial_\eta^\alpha\partial_x^\beta\left(c(x,\eta)-\sum_{|\lambda|<N}(\lambda!)^{-1}\partial_\eta^\lambda b(x,\eta)\,D_x^\lambda a(x,\eta)\right)\right|\leqslant \tag{3.2.1}$$

$$\leqslant c_{\alpha,\beta}(AB)^{K(\alpha,\beta,N,n,r,\mu)}(1+[\eta])^{m+\mu+\langle M,\beta-\alpha\rangle-r}.$$

PROOF. – First let us recall that:

$$c(x,\eta)=(2\pi)^{-n}\int e^{i\langle x,\theta\rangle}b(x,\theta+\eta)\,\hat{a}(\theta,\eta)\,d\theta\,. \tag{3.2.2}$$

Differentiation with respect to η_j acts simmetrically on $a(x,\eta)$, $b(x,\eta)$, lowering the degree, (in $[\eta]$) by M_j. Since $\theta_j\hat{a}(\theta,\eta)$ is the Fourier transform of $D_{x_j}a(x,\eta)$, also the derivatives with respect to x act on $a(x,\eta)$, $b(x,\eta)$ raising the degree by M_j. It sufficies then to consider $\alpha=\beta=0$. Since (2.1.1) holds, when θ belongs to $\operatorname{supp}\hat{a}(\cdot,\eta)$ we have:

$$[\theta]\leqslant A(1+[\eta]),\qquad [\eta]\leqslant B(1+[\theta+\eta]); \tag{3.2.3}$$

$$[\theta+\eta]\leqslant A(1+[\eta])+[\eta]. \tag{3.2.4}$$

Let us suppose first that $[\eta]\leqslant 2B$. Then $[\theta+\eta]\leqslant 2AB+A+2B$. Thus, introducing a factor $\varphi_0((10AB)^{-M}(\theta+\eta))$ in (3.2.2), $c(x,\eta)$ does not change. Now $[\eta]$ is bounded in the support of the symbols

$$\begin{cases} A(x,\eta)=a(x,\eta)\,\varphi_0((4B)^{-M}\eta) \quad \text{and}\\ B(x,\eta)=b(x,\eta)\,\varphi_0((10AB)^{-M}\eta)\,. \end{cases} \tag{3.2.5}$$

We can then apply (1.3.3) with $L=N=0$, pointing out that, when $[\eta]<2B$, the symbol of $B(x,D)\,A(x,D)$ is equal to that of $b(x,D)\,a(x,D)$, the lemma is so proved in this case.

Let now $[\eta]$ be greater than $2B$ and set $T=[\eta]$; from (3.2.3) we easily obtain $T\leqslant 2B[\theta+\eta]$, $[\theta+\eta]\leqslant(2A+1)T$. We can then change $a(x,\eta)$, $b(x,\eta)$ into $a_T(x,\eta)$, $b_T(x,\eta)$, defined by:

$$\begin{cases} a_T(x,\zeta)=a(x,\zeta)\,\psi_0(T^{-M}\zeta), \\ b_T(x,\zeta)=b(x,\zeta)\left[\varphi_0((2T(2A+1))^{-M}\zeta)-\varphi_0\left(\left(\dfrac{T}{4B}\right)^{-M}\zeta\right)\right], \end{cases} \tag{3.2.6}$$

without changing $c(x,\eta)$ when $[\eta]=T$.

All the derivatives of the functions

$$\begin{aligned} \partial_x^\gamma A_T(x,\zeta)&=\partial_x^\gamma(a_T(T^{-M}x,T^M\zeta))\,T^{-m+r}= \\ &=\psi_0(\zeta)\,a(T^{-M}x,T^M\zeta)\,T^{-m+r},\qquad |\gamma|=N; \end{aligned} \tag{3.2.7}$$

$$\begin{aligned} B_T(x,\zeta)&=b_T(T^{-M}x,T^M\zeta)\,T^{-\mu}= \\ &=b(T^{-M}x,T^M\zeta)[\varphi_0((4A+2)^M\zeta)-\varphi_0((4B)^M\zeta)]\,T^{-\mu}; \end{aligned} \tag{3.2.8}$$

are bounded by a power of AB and a seminorm of $\partial_x^\gamma a(x,\eta)$ in $S_M^{m+\langle M,\gamma\rangle-r}(\boldsymbol{\Omega})$ and $b(x,\eta)$ in $S_M^\mu(\boldsymbol{\Omega})$. Moreover $[\zeta]\leqslant 4A+2$ in their supports.

Since $c(T^{-M}x,T^M\eta)$ is equal to the symbol of $T^{m+\mu-r}B_T(x,D)\,A_T(x,D)$, for $[\eta]=1$, we obtain the result applying (1.3.3) with $L=0$ and a suitable N.

REMARK. – The hypotesis (2.1.1) may be weakened dropping down the first assumption, by means of Lemma 1.2; we argue essentially as in the first part of the proof of Theorem 2.3.

DEFINITION 3.3. – *For $m,r\in\boldsymbol{R}$, $r>0$, we define $S_{M,r}^m(\boldsymbol{\Omega})$ the set of all the symbols $a(x,\eta)\in\tilde{S}_M^m(\boldsymbol{\Omega})$ such that, for N suitable positive integer:*

$$\partial_x^\beta a(x,\eta)\in\tilde{S}_M^{m+\langle M,\beta\rangle-r}(\boldsymbol{\Omega})\qquad \textit{when}\ |\beta|=N. \tag{3.3.1}$$

THEOREM 3.4. – *Let $a(x,\eta)\in S_{M,r}^m(\boldsymbol{\Omega})$, $b(x,\eta)\in S_M^\mu(\boldsymbol{\Omega})$, $m,\mu\in\boldsymbol{R}$; then if we set $c(x,D)=b(x,D)\,a(x,D)$, we have, for a suitable large integer N:*

$$\varrho(x,\eta)=a^*(x,\eta)-\sum_{|\alpha|<N}(\alpha!)^{-1}\partial_\eta^\alpha D_x^\alpha\overline{a(x,\eta)}\in S_M^{m-r}(\boldsymbol{\Omega}); \tag{3.4.1}$$

$$R(x,\eta)=c(x,\eta)-\sum_{|\alpha|<N}(\alpha!)^{-1}\partial_\eta^\alpha b(x,\eta)\,D_x^\alpha a(x,\eta)\in S_M^{m+\mu-r}(\boldsymbol{\Omega}). \tag{3.4.2}$$

PROOF. – For $0<\varepsilon<1$ let $a_{\varepsilon\chi}(x,\eta)$ be defined as in (2.5.4) and consider:

$$\varrho_{\varepsilon\chi}(x,\eta)=(a_{\varepsilon\chi}(x,\eta))^*-\sum_{|\alpha|<N}(\alpha!)^{-1}\partial_\eta^\alpha D_x^\alpha\overline{a_{\varepsilon\chi}(x,\eta)}. \tag{3.4.3}$$

Decomposing now $a(x, \eta)$ and $\varrho(x, \eta)$ as in (2.5.13) we immediately obtain:

$$(3.4.4)\qquad a^*(x, \eta) = (a(x, \eta) - a_{1\chi}(x, \eta))^* + \sum_{p=0}^{\infty} (a_{2^{-p}\chi}(x, \eta) - a_{2^{-p-1}\chi}(x, \eta))^* ;$$

$$(3.4.5)\qquad \varrho(x, \eta) = \varrho(x, \eta) - \varrho_{1\chi}(x, \eta) + \sum_{p=0}^{\infty} \varrho_{2^{-p}\chi}(x, \eta) - \varrho_{2^{-p-1}\chi}(x, \eta).$$

Since $\partial_x^\gamma a_{\varepsilon\chi}(x, \eta)$ follows from (2.5.4) starting from $\partial_x^\gamma a(x, \eta)$, we can show, for suitable N and $|\gamma| = N$

$$(3.4.6)\qquad |\partial_\eta^\alpha \partial_x^\beta (\partial_x^\gamma a_{\varepsilon\chi}(x, \eta))| \leq C_{\alpha,\beta,K}\, \varepsilon^J (1 + [\eta])^{m + \langle M, \gamma+\beta-\alpha\rangle - r}, \qquad \text{for every } J > 0 .$$

Arguing as in the last part of the proof of Theorem 2.5, applying now Lemma 3.1 and (3.4.3), we can state that each summand in the right-hand side of (3.4.5) is bounded in $S_M^{m-r}(\boldsymbol{\Omega})$ by $c_N 2^{-pN}$; then $\varrho(x, \eta) \in S_M^{m-r}(\boldsymbol{\Omega})$.

In order to show (3.4.2) we argue as above using now Lemma 3.2, and the proof is so concluded.

Let $a(x, \eta) \in S_M^m(\boldsymbol{\Omega})$ satisfy only the hypotesis (2.3.1); applying then (3.1.3) we can say that $(1 + [\eta])^L |A_p^*(x, \eta)| \leq C_L$. We may then argue as in the last part of the proof of Lemma 3.1; we obtain so (3.1.1) and moreover:

$$(3.5.1)\qquad |\partial_\eta^\alpha \partial_x^\beta a^*(x, \eta)| \leq c_{\alpha,\beta}(a)\, B(B^{m+\langle M, \beta-\alpha\rangle} + 1)(1 + [\eta])^{m+\langle M, \beta-\alpha\rangle},$$

with $c_{\alpha,\beta}(a)$ seminorms in $S_M^m(\boldsymbol{\Omega})$.

Again, for $a(x, \eta) \in S_M^m(\boldsymbol{\Omega})$ which satisfies (2.1.1) and $b(x, \eta) \in S_M^\mu(\boldsymbol{\Omega})$, let us refer to the proof of Lemma 3.2. When we apply (1.3.3) we clearly loose the gain of order $-r$ in the estimates of the derivatives of order $|\beta| \geq N$ with respect to the x variable; however we can still verify that:

$$(3.5.2)\qquad |\partial_\eta^\alpha \partial_x^\beta c(x, \eta)| \leq c_{\alpha,\beta} (AB)^{k(\alpha,\beta,m,\mu)} (1 + [\eta])^{m+\langle M, \beta-\alpha\rangle},$$

with $c_{\alpha,\beta}$ product of seminorms of $a(x, \eta) \in S_M^m(\boldsymbol{\Omega})$ and $b(x, \eta) \in S_M^\mu(\boldsymbol{\Omega})$.

Arguing now as in the proof of Theorem 3.4, with particular attention to (3.4.4) we can assure that, for $a(x, \eta) \in \widetilde{S}_M^m(\boldsymbol{\Omega})$ and $b(x, \eta) \in S_M^\mu(\boldsymbol{\Omega})$, we have:

$$(3.5.3)\qquad a^*(x, \eta) \in S_M^m(\boldsymbol{\Omega}) \qquad \text{and} \qquad c(x, \eta) \in S_M^{m+\mu}(\boldsymbol{\Omega}).$$

Let us consider now $a(x, \eta) \in S_M^m(\boldsymbol{\Omega})$ such that $a^*(x, D) = (a(x, D))^*$ belongs to Op $S_M^m(\boldsymbol{\Omega})$. We can then decompose the symbol of the adjoint operator: $a^*(x, D) = B_1(x, D) + B_2(x, D)$ where the first term in the right-hand side is in $\mathcal{L}(H_M^t, H_M^s)$ for every $t, s \in \boldsymbol{R}$ and $B_2(x, \eta)$ identically vanishes when $[\eta] < 1/2$.

We can set now $B_2(x, \eta) = \sum_{p \geq -1}^{\infty} b_p(x, \eta)$, where in the notation of the anisotropic Littlewood-Paley decomposition, $\widehat{b}_p(\xi, \eta) = \varphi_p([\eta]^{-M}\xi)\, \widehat{B}_2(\xi, \eta)$.

In view of Lemma 1.2, $2^{pN} b_p(x, \eta)$ is bounded in $S_M^m(\boldsymbol{\Omega})$, for any $N > 0$. Since $[\xi] \geq 2^{p-1}[\eta]$, we argue, for $p \geq 2$, $[\xi] + \eta \geq [\eta]$ in supp $\widehat{b}_p^*(x, \eta)$, which, in view of Lemma 3.1, implies that, for every $N \in \boldsymbol{N}$, $2^{pN} a_p(x, \eta) = 2^{pN} b^*(x, \eta)$ is bounded in $S_M^m(\boldsymbol{\Omega})$ by an estimate like (3.1.2), for every $B \geq 1$. Moreover $[\xi] \leq [\xi + \eta]$ in supp $\widehat{a}_p(\xi, \eta)$, then, when $\chi(\xi+\eta, \varepsilon^M\eta)\neq 0$, we have $[\xi+\eta]\leqslant\varepsilon[\eta]\leqslant\varepsilon([\xi+\eta]+[\eta])$, hence $[\xi+\eta]\leqslant \leqslant\varepsilon[\xi]/1-\varepsilon$. So $\chi(\xi+\eta, \varepsilon^M\eta)\, a_p(\xi,\eta)=0$ if $2^{p+1}\varepsilon/1-\varepsilon\leqslant 1$; then:

$$\hat{a}_{\varepsilon\chi}(\xi+\eta)=\sum_{2^p>(1-\varepsilon)/2\varepsilon}\chi(\xi+\eta, \varepsilon^M\eta)\,\hat{a}_p(\xi,\eta)+\chi(\xi+\eta,\varepsilon^M\eta)\,\widehat{B}_1(\xi,\eta), \tag{3.5.4}$$

where at any rate $p\geqslant -1$. Arguing now as in the proof of (2.5.12), we obtain, for every $N>0$:

$$|\partial_\beta^\alpha\partial_x^\beta\hat{a}_{\varepsilon\chi}(x,\eta)|\leqslant c_{\alpha,\beta,N}\,\varepsilon^N(1+[\eta])^{m+\langle M,\beta-\alpha\rangle},\qquad 0<\varepsilon<1, \tag{3.5.5}$$

which, jointly with Theorem 2.5, shows:

THEOREM 3.5. – *For $a(x,\eta)\in S_M^m(\boldsymbol{\Omega})$, $m\in\boldsymbol{R}$, the following statements are equivalent:*

$$a(x,D)\in\mathcal{L}(H_M^s, H_M^{s-m}),\qquad \text{for every } s\in\boldsymbol{R}; \tag{3.5.6}$$

$$a(x,\eta)\in\tilde{S}_M^m(\boldsymbol{\Omega}); \tag{3.5.7}$$

$$(a(x,D))^*\in \operatorname{Op} S_M^m(\boldsymbol{\Omega}). \tag{3.5.8}$$

Let us observe again that for $a(x,\eta)\in\tilde{S}_M^m(\boldsymbol{\Omega})$, $b(x,\eta)\in\tilde{S}_M^\mu(\boldsymbol{\Omega})$, by means of (3.5.3) and the previous Theorem, since $(b(x,D)\,a(x,D))^*=a(x,D)^*\,b(x,D)^*$ is in $\operatorname{Op} S_M^{m+\mu}(\boldsymbol{\Omega})$ we obtain:

$$a(x,D)b(x,D)\in\operatorname{Op}\tilde{S}_M^{m+\mu}(\boldsymbol{\Omega}). \tag{3.5.9}$$

We can then conclude that the operator space:

$$\operatorname{Op}\tilde{S}_M^0(\boldsymbol{\Omega})=\operatorname{Op}S_M^0(\boldsymbol{\Omega})\cap(\operatorname{Op}S_M^0(\boldsymbol{\Omega}))^*$$

is a selfadjoint operator algebra.

Applying now all the arguments in GARELLO [8, § 1] we can state the following

THEOREM 3.6. – *For every $r>0$, the operator space $\operatorname{Op}S_{M,r}^0(\boldsymbol{\Omega})$ is a selfadjoint operator algebra; moreover the remainder $\varrho(x,\eta)$ and the expansion in the right-hand side of* (3.4.1) *are respectively in the symbol spaces $\tilde{S}_M^{-r}(\boldsymbol{\Omega})$ and $S_{M,r}^0(\boldsymbol{\Omega})$.*

Following closely HÖRMANDER [11, Theorem 7.1] we shall prove an inequality of «sharp Gårding» type, which may be useful in the application of operators in $\operatorname{Op}S_{M,r}^m(\boldsymbol{\Omega})$ to the study of propagation of singularities. The theorem we will prove now includes both the classical «sharp Gårding» inequality, as stated in HÖRMANDER [10, Theorem 18.1.4] and its extension to anisotropic pseudodifferential operators, showed in SEGÀLA [16, Theorem 4.8].

THEOREM 3.7. – *For $m\in\boldsymbol{R}$, $0<r<2$, $M_0=\max\limits_j M_j$, let us consider $a(x,\eta)\in \in\tilde{S}_M^m(\boldsymbol{\Omega})$, such that $\partial_x^\beta a(x,\eta)\in\tilde{S}_{1,1}^{m+\langle M,\beta\rangle-2M_0r}(\boldsymbol{\Omega})$ when $|\beta|=2$. If $\Re a(x,\eta)\geqslant 0$ then* *for some* $C > 0$:

$$\Re(a(x, D)\, u,\, u) \geq -C\|u\|^2_{(m-r)/2}. \tag{3.7.1}$$

PROOF. – Following the proof in [11, Theorem 7.1], we obtain rightly from the hypotesis,

$$\partial_x^\beta a(x, \eta) \in \tilde{S}_M^{m+\langle M, \beta\rangle - M_0 r|\beta|}(\boldsymbol{\Omega}) \subset \tilde{S}_M^{m+(1-r)\langle M, \beta\rangle}(\boldsymbol{\Omega}), \qquad |\beta| \leq 2, \tag{3.7.2}$$

and we can set, for $p = 0, 1, \ldots$:

$$b_p(x, \eta) = \iint \psi(q_p^M(x-y), q_p^{-M}(\eta - \theta)) a_p(y, \theta)\, dy\, d\theta = \tag{3.7.3}$$

$$= \iint \psi(q_p^M y, q_p^{-M}\eta)\, a_p(x-y, \eta-\theta)\, dy\, d\theta,$$

where $q_p = 2^{p(1-r/2)}$ and $\psi(\xi - \eta) \in \mathcal{S}(\boldsymbol{R}^{2n})$ is an even function such that:

$$\iint \psi(x, \eta)\, dx\, d\eta = 1, \qquad (\psi(x, D)\, u, u) \geq 0, \qquad u \in \mathcal{S}(\boldsymbol{R}^n). \tag{3.7.4}$$

Since the operator $\psi(q_p^M(x-y), q_p^{-M}(D-\theta))$ is positive and $\Re a_p(y, \theta) \geq 0$, it follows that $b_p(x, D) + b_p(x, D)^* \geq 0$. Let us split now $a_p(x, \eta) = b_p(x, \eta) + c_p(x, \eta)$, the proof will be completed if we show that $c(x, \eta) = \sum_0^\infty c_p(x, \eta) \in \tilde{S}_M^{m-r}(\boldsymbol{\Omega})$.

At first let us show that, for any $N > 0$,

$$|c_p(x, \eta)| \leq C_N(2^p + [\eta])^{-N}, \qquad \text{when } [\eta] < 2^{p-2} \text{ or } [\eta] > 2^{p+2}. \tag{3.7.5}$$

Infact in the support of the first integrand in (3.7.3) we have $2^{p-1} \leq [\theta] \leq 2^{p+1}$. If $[\eta] < 2^{p-2}$, then $[\theta - \eta] > 2^{p-2}$, hence $2^p + [\eta] \leq 5[\theta - \eta]$; on the other hand if $[\eta] > 2^{p+2}$ then $[\theta - \eta] \geq [\eta]/2$, hence $2^p + [\eta] \leq 3[\theta - \eta]$. In both cases we can conclude:

$$(2^p + [\eta])^{r/2} \leq \frac{(2^p + [\eta])}{2^{p(1-r/2)}} \leq 5\, \frac{[\eta - \theta]}{q^p} \leq 5[q_p^{-M}(\eta - \theta)], \tag{3.7.6}$$

for $r/2 < 1$. Since $\psi \in \mathcal{S}(\boldsymbol{R}^{2n})$ and moreover $a_p(x, \eta) \equiv 0$ when the hypotesis in (3.7.5) are satisfied, this last is proved.

By means of the Taylor expansion and (3.7.2) we have

$$\left| a_p(x-y, \eta-\theta) - \sum_{|\alpha+\beta|<2} \frac{1}{\alpha!\beta!}\, \partial_\eta^\alpha \partial_x^\beta a(x, \eta)(-y)^\beta(-\theta)^\alpha \right| \leq \tag{3.7.7}$$

$$\leq c \sum_{|\alpha+\beta|=2} 2^{p(m - r\langle M, \beta\rangle + \langle M, \beta - \alpha\rangle)}\, |y^\beta \theta^\alpha| \leq$$

$$\leq 2^{p(m-r)} \sum_{|\alpha+\beta|=2} q_p^{\langle M, \beta-\alpha\rangle} |y^\beta \theta^\alpha| = 2^{p(m-r)} \sum_{|\alpha+\beta|=2} |(q_p^M y)^\beta (q_p^{-M}\theta)^\alpha|,$$

since $r - r\langle M, \beta\rangle + \langle M, \beta - \alpha\rangle - (1 - r/2)\langle M, \beta - \alpha\rangle = r(2 - \langle M, \beta + \alpha\rangle)/2 \leq 0$. ψ is even, hence the first order terms drop down from the second integral in (3.7.3); since ψ is also rapidly decreasing, we have:

$$|c_p(x,\eta)| \leq 2^{p(m-r)}, \qquad \text{when } 2^{p-1} \leq [\eta] \leq 2^{p+1}, \tag{3.7.8}$$

which jointly with (3.7.5) shows us that $c(x,\eta) \in S_M^{m-r}(\boldsymbol{\Omega})$, for $\partial_\eta^\alpha \partial_x^\beta b_p(x,\eta)$ follows by differentiation of $a_p(x,\eta)$. Following now the arguments in [11, Theorem 7.1] we obtain also $c^*(x,D) \in \operatorname{Op} S_M^{m-r}(\boldsymbol{\Omega})$, thus the proof is concluded.

4. – Microlocal properties.

DEFINITION 4.1. – *For $u \in \mathcal{D}'(\boldsymbol{\Omega})$, $s \in \boldsymbol{R}$ we shall say that $(x_0, \xi^0) \in T^*(\boldsymbol{\Omega})\backslash 0 = \Omega \times (\boldsymbol{R}^n \backslash \{0\})$ does not belong to $\mathrm{WF}_{s,M}u$ if and only if we can find $g \in \mathcal{C}_0^\infty(\boldsymbol{\Omega})$, $g(x_0) \neq 0$ and Γ_{M,ξ^0} open M-conic neighborhood of ξ^0 such that*

$$\int_{\Gamma_{M,\xi^0}} (1+[\xi])^{2s} |\widehat{gu}(\xi)|^2 d\xi \leq c < \infty . \tag{4.1.1}$$

For every $u \in \mathcal{D}'(\boldsymbol{\Omega})$, the anisotropic Sobolev wave front set $\mathrm{WF}_{s,M}u$ is a closed M-conic subset of $T^*(\boldsymbol{\Omega})\backslash 0$ (M-conic with respect to the ξ variable).

Since an open M-conic subset of $\boldsymbol{R}^n$ may be identified by means of its intersection with the unit sphere, using well known arguments, see for example TRÈVES [17], we can say that $(x_0, \xi^0) \notin \mathrm{WF}_{s,M}u$ if and only if $g(D)\varphi u \in H^s_{M,\mathrm{loc}}(\boldsymbol{\Omega})$ for some $\varphi \in \mathcal{C}_0^\infty(\boldsymbol{\Omega})$, $g \in \mathcal{C}^\infty(\boldsymbol{R}^n)$, such that $\varphi(x_0) \neq 0$, $g(\xi)$ is M-homogeneous of degree 0 for large ξ, $\operatorname{supp} g \subset \Gamma_{M,\xi_0}$ and $g(x) = 1$ in $\Gamma'_{M,\xi_0} \subset \Gamma_{M,\xi_0}$.

Moreover the projection on $\boldsymbol{\Omega}$ of $\mathrm{WF}_{s,M}(u)$ is exactly $\text{sing-supp}_{s,M} u$, that is the set of all the points $x \in \boldsymbol{\Omega}$ such that $u \notin H^s_{M,\mathrm{loc}}(V_{x_0})$ for any open neighborhood V_{x_0} of x_0.

Since $a(x,D) \in \operatorname{Op} \tilde{S}_M^m(\boldsymbol{\Omega})$ maps continuously $H^s_{M,\mathrm{loc}}(\boldsymbol{\Omega})$ into $H^{s-m}_{M,\mathrm{loc}}(\boldsymbol{\Omega})$ we can show in standard way, see for example TRÈVES [17], the pseudolocal property:

$$\operatorname{sing\,supp}_{M,s-m} a(x,D)\,u \subset \operatorname{sing\,supp}_{M,s} u , \tag{4.1.2}$$

for every $u \in \mathcal{D}'(\boldsymbol{\Omega})$.

Since the composition of two operators, respectively in $\operatorname{Op} \tilde{S}_M(\boldsymbol{\Omega})m$, $\operatorname{Op} \tilde{S}_M^\mu(\boldsymbol{\Omega})$, is in $\operatorname{Op} \tilde{S}_M^{m+\mu}(\boldsymbol{\Omega})$ and moreover $S^\mu_{M,(1,0)}(\boldsymbol{\Omega}) \subset \tilde{S}_M^\mu(\boldsymbol{\Omega})$, we can state the following property:

$$a(x,\eta)\,\chi(\eta) \in \tilde{S}_M^{m+\mu}(\boldsymbol{\Omega}), \tag{4.1.3}$$

for every $a(x,\eta) \in \tilde{S}_M^m(\boldsymbol{\Omega})$ and $\chi(\eta) \in S^\mu_{M,(1,0)}(\boldsymbol{\Omega})$, $m, \mu \in \boldsymbol{R}$.

THEOREM 4.2 (Microlocal property). – *Let σ, s, m, r belong to $\boldsymbol{R}$, r strictly positive. Then for $u \in H_M^\sigma$ and $a(x,\eta)$ in $S^m_{M,r}(\boldsymbol{\Omega})$ we have:*

$$\mathrm{WF}_{M,s-m}\,a(x,D)\,u \subset \mathrm{WF}_{M,s}\,u \qquad \text{when } s \leq \sigma + r . \tag{4.2.1}$$

PROOF. – For $x_0 \in$ sing-supp$_{M,s}u$, let us consider $(x_0, \xi^0) \notin \mathrm{WF}_{M,s}u$. We can then find $\varphi \in \mathcal{C}_0^\infty(\boldsymbol{\Omega})$, $\varphi(x) = 1$ in V_{x_0} open neighborhood of x_0 and $g(\xi)$ smooth, M-homogeneous of degree 0 for large ξ, supported in Γ_M and identically 1 in Γ'_M, $\Gamma'_M \subset \Gamma_M$ M-conic neighborhood of ξ_0, in such a way that $g(D)\,\varphi u \in H^s_{M,\mathrm{loc}}(\boldsymbol{\Omega})$.

We can easily see that $a(x, D)(1-\varphi)\,u \in \mathcal{C}^\infty(V_{x_0})$, then the restriction to V_{x_0} of $\mathrm{WF}_{M,s}v$ and $\mathrm{WF}_{M,s}\varphi v$ coincides. Since $a(x, D)\,g(D)\,\varphi u$ is in $H^{s-m}_{M,\mathrm{loc}}(\boldsymbol{\Omega})$ we have only to consider $a(x, D)(1-g(D))\,\varphi u$. Let us consider then the operator $c(x, D) = \chi(D)\,a(x, D)(1-g(D))$, with $\chi(\eta) \in \mathcal{C}^\infty(\boldsymbol{R}^n)$ real valued, M-homogeneous of degree 0 for large ξ, supported in Γ'_M and identically equal to 1 in $\Gamma''_M \subset \Gamma'_M$.

Since the operator $a(x, D)(1-g(D))$ has symbol $b(x, \eta) = a(x, \eta)(1-g(x))$ in $S^m_{M,r}(\boldsymbol{\Omega})$, we can write:

$$c(x, \eta) = \sum_{|\alpha| < N} (\alpha!)^{-1}\,\partial^\alpha_\eta \chi(\eta)\,D^\alpha_x b(x, \eta) + R(x, \eta)\,, \tag{4.2.2}$$

where the sum in the right-hand side identically vanishes and $R(x, \eta) \in S^{m-r}_M(\boldsymbol{\Omega})$.

Let us observe now that $R^*(x, D) = c^*(x, D) = b^*(x, D)\,\chi(D)$; we can then write by means of (3.4.1):

$$R^*(x, \eta) = b^*(x, \eta)\,\chi(\eta) = \sum_{|\alpha| < N} (\alpha!)^{-1}\,\partial^\alpha_\eta D^\alpha_x \overline{b(x, \eta)}\,\chi(\eta) + \varrho(x, \eta)\,\chi(\eta)\,. \tag{4.2.3}$$

Also in this case the sum in the right-hand side identically vanishes; then we have $R^*(x, \eta) = \varrho(x, \eta)\,\chi(\eta) \in S^{m-r}_M(\boldsymbol{\Omega})$. This is enough for concluding:

$$\chi(D)\,a(x, D)\,\varphi u \in H^{\inf(s-m,\,\sigma-m+r)}(V_{x_0})\,. \tag{4.2.4}$$

In view of the pseudolocal property, letting x_0 range all over sing-supp$_{M,s}u$, we have concluded the proof.

In order to construct an example of pseudodifferential operator in $\mathrm{Op}\,\tilde{S}^m_M(\boldsymbol{\Omega})$ which does not satisfy the microlocal property, let us argue in the more general framework of inhomogeneous pseudodifferential calculus. Namely let us introduce the basic weight vector $\psi(\xi) = (\psi_1(\xi), \ldots, \psi_n(\xi))$, whose components $\psi_j(\xi)$ are in $\mathcal{C}^\infty(\boldsymbol{R}^n)$, $j = 1, \ldots, n$ and satisfy, for C, c positive constants:

$$c(1+|\xi|)^c \leq \psi_j(\xi) \leq C(1+|\xi|)^C, \tag{4.3.1}$$

$$c \leq \psi_j(\xi+\theta)\,\psi_j(\xi)^{-1} \leq C\,, \qquad \text{when } \sum_{k=1}^n |\theta_k|\,\psi_k(\xi)^{-1} \leq c\,. \tag{4.3.2}$$

For more details on the basic weight vectors see [1], [14], [15].

Consider now $\varphi \in \mathcal{C}^\infty(\boldsymbol{R})$ as in § 1, we can then introduce the functions: $\chi^j_t = \varphi^j_{t+1} - \varphi^j_t$, $\chi^j_{-1} = \varphi^j_0$, where for $j = 1, \ldots, n$:

$$\varphi^j_t(\xi) = \varphi\left(\frac{\psi_j(\xi)}{2^t}\right) \in \mathcal{C}^\infty(\boldsymbol{R}^n)\,, \qquad t = 0, 1, 2, \ldots\,. \tag{4.3.3}$$

Since $\operatorname{supp}\varphi^j_t \subset \{\xi \in \boldsymbol{R}^n;\ \psi_j(\xi) < 2^t\}$ and $\varphi^j_t(\xi) = 1$ when $\psi_j(\xi) < 2^{t-1}$, we can easily see that:

$$\operatorname{supp}\chi_t^j \subset C_t^j = \{\xi \in \boldsymbol{R}^n;\ 2^{t-1} < \psi_j(\xi) < 2^{t+1}\}, \qquad j = 1, \ldots, n. \tag{4.3.4}$$

Moreover we obtain $C_t^j \cap C_u^j = \emptyset$, $j = 1, \ldots, n$, when $t, u \in \boldsymbol{N}$, $|t-u| > N$, for suitable constant $N \in \boldsymbol{N}$. It then follows:

$$\sum_{t=-1}^{\infty} \chi_t^j(\xi) = \varphi(\psi_j(\xi)) + \sum_{t=0}^{\infty} \chi_t^j(\xi) = 1, \qquad j = 1, \ldots, n. \tag{4.3.5}$$

For $P = (p_1, \ldots, p_n) \in \tilde{\boldsymbol{N}}^n = (\boldsymbol{N} \cup \{-1\})^n$, let us introduce the vector $X_P(\xi) = (\chi_{p_1}^1(\xi), \ldots, \chi_{p_n}^n(\xi))$. We can then define, for $u \in \mathcal{S}'(\boldsymbol{R}^n)$:

$$u_P(x) = X_P(D)\, u = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} \chi_{p_1}^1(\xi) \ldots \chi_{p_n}^n(\xi)\, \hat{u}(\xi)\, d\xi. \tag{4.3.6}$$

The symbol $\sigma(X_P(D))(\xi) = \chi_{p_1}^1(\xi) \ldots \chi_{p_n}^n(\xi)$ has support in $\bigcap_{j=1}^{n} C_{p_j}^j$ and in view of (4.3.5):

$$\sum_{P \in \tilde{\boldsymbol{N}}^n} X_P(\xi) = (1, \ldots, 1), \qquad \sum_{P \in \tilde{\boldsymbol{N}}^n} \sigma((X_P(D))(\xi) = 1. \tag{4.3.7}$$

We can then decompose $u \in \mathcal{S}'(\boldsymbol{R}^n)$ as follows:

$$u = \sum_{P \in \tilde{\boldsymbol{N}}^n} u_P = \sum_{P \in \tilde{\boldsymbol{N}}^n} X_P(D)\, u. \tag{4.3.8}$$

We shall say inhomogeneous Littlewood-Paley partition of unity the vector family $\{X_P(\xi)\}_{P \in \tilde{\boldsymbol{N}}^n}$, and inhomogeneous Littlewood-Paley decomposition the corresponding one defined by (4.3.8).

Arguing now as in § 1 we can verify that, for $u \in \mathcal{S}'(\boldsymbol{R}^n)$ and $\nu \in \boldsymbol{R}^n$:

$$u \in H_\psi^\nu(\boldsymbol{R}^n) \Leftrightarrow \sum_{P \in \tilde{\boldsymbol{N}}^n} \|u_p\|_{L_2}^2 2^{2\langle P, \nu\rangle} < +\infty. \tag{4.3.9}$$

Where, with standard vectorial notation:

$$\|u\|_{\psi, \nu} = \|\psi(D)^\nu(u)\|_{L_2}, \tag{4.3.10}$$

and H_ψ^ν is the respective Sobolev space, defined in standard way.

Let us introduce the symbol class $S_{1,1,\psi}^0(\boldsymbol{R}^n)$ of all $a(x, \xi) \in \mathcal{C}^\infty(\boldsymbol{R}^{2n})$, whose elements $a(x, \eta)$ satisfy, for every K compact subset of $\boldsymbol{R}^n$ and α, β multiindices:

$$|D_\xi^\alpha \partial_x^\beta a(x, \eta)| \leq C_{\alpha, \beta, K}\, \psi(\eta)^{-\alpha+\beta}. \tag{4.4.1}$$

Consider, for every $Y \in \boldsymbol{R}^n$, $\varepsilon > 0$:

$$Y_{\varepsilon\psi} = \{\xi \in \boldsymbol{R}^n;\ |\xi_j - \xi_j^0| < \varepsilon\psi_j(\xi^0),\ \text{for some } \xi^0 \in Y\},$$

For $x_0 \in \boldsymbol{R}^n$, $u \in \mathcal{D}'(\boldsymbol{R}^n)$, $\nu \in \boldsymbol{R}^n$ we can then define the Sobolev ψ-filter of microlocal smoothness:

$$\Sigma^{\nu}_{\psi, x_0} u = \bigcup_{\phi} \Sigma^{\nu}_{\psi} \phi u\,, \qquad \phi \in \mathcal{C}_0^{\infty}(\boldsymbol{R}^n),\ \phi(x_0) \neq 0\,; \tag{4.4.2}$$

where for every $v \in \mathcal{E}'(\boldsymbol{R}^n)$:

$$\Sigma^{\nu}_{\psi} v = \left\{ X \subset \boldsymbol{R}^n;\ \int_{(\boldsymbol{R}^n \setminus X)_{\varepsilon\psi}} \psi(\xi)^{2\nu} |\hat{v}(\xi)|^2 d\xi < +\infty \right\}, \qquad \text{for some } \varepsilon > 0\,. \tag{4.4.3}$$

Both $\Sigma^{\nu}_{\psi, x_0} u$ and $\Sigma^{\nu}_{\psi} v$ are ψ filter in the sense that, for all $X \in \Sigma^{\nu}_{\psi, x_0} u$ (resp. $\Sigma^{\nu}_{\psi} v$), there exists $\varepsilon > 0$ such that: $\boldsymbol{R}^n \setminus (\boldsymbol{R}^n \setminus X)_{\varepsilon\psi} \in \Sigma^{\nu}_{\psi, x_0} u$ (resp. $\Sigma^{\nu}_{\psi} v$).

For more details on the inhomogeneous pseudodifferential operators and the Sobolev ψ-filter of microlocal smoothness, see [7] and the references given there.

Let us consider now the vectors $\omega_k = (2^k, 0, \ldots, 0) \in \boldsymbol{R}^n$, $k \in \boldsymbol{N}$ and the test function $\theta \in \mathcal{C}_0^{\infty}(\boldsymbol{R})$, $\theta(t) \leq 1$, $\operatorname{supp} \theta \subset \{t \in \boldsymbol{R};\ |t| < 1/4\}$, $\theta(t) = 1$ for $|t| < 1/8$. Following RODINO [14] we can so define, for $k \in \boldsymbol{N}$:

$$\zeta_k(\eta) = \theta\left(\frac{\eta_1 - 2^k}{\psi_1(\omega_k)}\right) \prod_{j=2}^{n} \theta\left(\frac{\eta_j}{\psi_j(\omega_k)}\right) \in \mathcal{C}^{\infty}(\boldsymbol{R}^n)\,; \tag{4.4.4}$$

whose supports are clearly contained, for $k \in \boldsymbol{N}$, in

$$I_k = \{\omega_k\}_{\psi/4} = \left\{ \eta \in \boldsymbol{R}^n;\ |\eta_1 - 2^k| < \frac{\psi_1(\omega_k)}{4}\,,\ |\eta_j| < \frac{\psi_j(\omega_k)}{4} \right\}, \qquad j = 2, \ldots, n\,. \tag{4.4.5}$$

In view of (4.3.1), (4.3.2), $\psi_1(\omega_k) < 2^k$ and $\psi_j(\omega_k) > c 2^{\varepsilon k}$, for some $c > 0$, $\varepsilon > 0$. Then $\operatorname{supp} \zeta_k \cap \operatorname{supp} \zeta_h = \emptyset$ for $h \neq k$ and $\zeta_k(\eta) = 1$ when η belongs to the set:

$$\bar{I}_k = \left\{ \eta \in \boldsymbol{R}^n;\ |\eta_1 - 2^k| < c\frac{2^{\varepsilon k}}{8}\,,\ |\eta_j| < c\frac{2^{\varepsilon k}}{8}\,,\ j = 2, \ldots, n \right\}. \tag{4.4.6}$$

Moreover we can find an integer M_k and a positive constant τ independent from k such that:

$$3\psi_n(\omega_k) \leq M_k \leq \tau \psi_n(\omega_k)\,, \qquad \text{for every } k \in \boldsymbol{N}\,. \tag{4.4.7}$$

We define now the symbol

$$a(x, \eta) = \sum_{k=1}^{\infty} e^{iM_k x_n} \zeta_k(\eta)\,. \tag{4.4.8}$$

Observing that $\psi_j^{-1}(\omega_k) < C\psi_j^{-1}(\eta)$ when $\eta \in \operatorname{supp} \zeta_k$, we can say that:

$$|D_\eta^\beta \zeta_k(\eta)| \leq c_\beta \psi(\eta)^{-\beta}, \qquad \eta \in \mathbf{R}^n, \tag{4.4.9}$$

which jointly with (4.4.7) assures that $a(x, \eta) \in S^0_{1,1,\psi}(\mathbf{R}^n)$.

For every $u \in \mathcal{E}'(\mathbf{R}^n)$ we obtain:

$$\widehat{a(x, D)\, u}(\xi) = \sum_{k=1}^{\infty} (\zeta_k \widehat{u})(\xi', \xi_n - M_k), \qquad \xi' = (\xi_1, \ldots, \xi_{n-1}). \tag{4.4.10}$$

We can see now that, for $\overline{\omega}_k = (2^k, 0, \ldots, 0, M_k) \in \mathbf{R}^n$, $\operatorname{supp} \zeta_k(\xi', \xi_n - M_k)$ is contained in

$$J_k = \{\overline{\omega}_k\}_{\psi/4} = \left\{ \xi \in \mathbf{R}^n; \quad \begin{array}{l} |\xi_1 - 2^k| < \dfrac{\psi_1(\omega_k)}{4} \\ |\xi_n - M_k| < \dfrac{\psi_n(\omega_k)}{4} \\ |\xi_j| < \dfrac{\psi_j(\omega_k)}{4}, \quad j = 2, \ldots, n-1 \end{array} \right\} \tag{4.4.11}$$

which satisfies

$$J_k \cap J_h = \emptyset, \qquad \text{when } h \neq k. \tag{4.4.12}$$

For every $k \in \mathbf{N}$ there exists $P = (p_1, \ldots, p_n) \in \widetilde{\mathbf{N}}^n$ such that ω_k belongs to $\operatorname{supp} X_P$, that is $\sigma(X_P(D))(\omega_k) \neq 0$, where X_P is an element of the Littlewood-Paley partition of unity as defined in (4.3.6). In view of (4.4.7) and (4.3.2), $\overline{\omega}_k$ belongs in its turn to $\operatorname{supp} X_P$ and moreover, using also (4.4.5) and (4.4.11), we can show that there exist some positive constants c, C such that, for $j = 1, \ldots, n$:

$$c2^{p_j - 1} < \psi_j(\xi) < C2^{p_j + 1}, \qquad \text{for every } \xi \in I_k \text{ or } \xi \in J_k, \tag{4.4.13}$$

that is both I_k and J_k are contained in $\bigcup_{|p_j - q_j| < N} \operatorname{supp} X_Q$, for suitable $N \in \mathbf{N}$.

Then, for $P \in \widetilde{\mathbf{N}}^n$, notice that $I_P = \bigcup_{\omega_k \in \operatorname{supp} X_P} I_k \subset \bigcup_{|p_j - q_j| < N} \operatorname{supp} X_Q$ and let us estimate for $u \in \mathcal{E}'(\mathbf{R}^n)$:

$$\begin{aligned} \|(a(x, D)\, u)_P\|_{L_2}^2 &\leq \sum_{\omega_k \in \operatorname{supp} X_P} \int_{J_k} (\chi^1_{p_1}(\xi) \ldots \chi^n_{p_n}(\xi))^2 |\zeta_k \widehat{u}(\xi', \xi_n - M_k)^2| \, d\xi \leq \\ &\leq \sum_{\omega_k \in \operatorname{supp} X_P} \int_{I_k} |\widehat{u}(\xi)|^2 d\xi \leq \int_{I_P} \widehat{u}^2(\xi)\, d\xi \leq \operatorname{cost} \|u_P\|_{L_2}^2, \end{aligned} \tag{4.4.14}$$

which in view of (4.3.9) shows that the operator $a(x, D)$ defined by (4.4.8) maps continuosly $H^v_{\psi, \mathrm{comp}}(\mathbf{R}^n)$ into $H^v_{\psi, \mathrm{loc}}(\mathbf{R}^n)$.

Let $\tau(x)$ belong to $\mathcal{C}_0^\infty(\boldsymbol{R}^n)$ and introduce, for $T>0$, the function:

$$u(x)=\tau(x)\sum_{k=1}^{\infty}2^{-k/T}e^{i2^kx_1}\in L^2_{\mathrm{loc}}(\boldsymbol{R}^n). \tag{4.4.15}$$

Observe now that for every $\varphi\in\mathcal{C}_0^\infty(\boldsymbol{R}^n)$, $\nu\in\boldsymbol{R}^n$ which satisfy $\sum_{i=1}^n\nu_i\geq 0$ and $\xi''=(\xi_2,\dots,\xi_n)$:

$$\int\psi^{2\nu}(\xi_1+2^k,\xi'')\,|\widehat{\varphi\tau}(\xi)|^2d\xi\geq \tag{4.4.16}$$

$$\geq\int_{\xi_1\geq 0}|\widehat{\varphi\tau}(\xi)|^2(1+|\xi|+2^k)^{2c(\nu_1+\dots+\nu_n)}d\xi\geq\mathrm{const.}\,2^{2kc(\nu_1+\dots+\nu_n)}.$$

Let us set for $\varphi\in\mathcal{C}_0^\infty(\boldsymbol{R}^n)$:

$$\Phi(\xi)=\sum_{\substack{h\neq k\\ h,k\in\boldsymbol{N}}}2^{-(h+k)/T}\widehat{\varphi\tau}(\xi_1-2^k,\xi'')\,\widehat{\varphi\tau}(\xi_1-2^h,\xi''). \tag{4.4.17}$$

Since $|2^h-2^k|\geq 2^k/2$, $h,k\in\boldsymbol{N}$, $h\neq k$, when $\xi_1\geq 0$ we can estimate for every $H>0$ and suitable $t\in\boldsymbol{N}$, $0\leq\lambda<1$, $c_H>0$:

$$\sum_{\substack{h\neq k\\ h,k\in\boldsymbol{N}}}2^{-(h+k)/T}\,|\widehat{\varphi\tau}(2^t-2^k+2\lambda,\xi'')|\,|\widehat{\varphi\tau}(2^t-2^h+2\lambda,\xi'')|\leq \tag{4.4.18}$$

$$\leq c_H|\xi''|^{-H}\,|(2^t-2\lambda,\xi'')|^{-H}\left(\sum_{\substack{k\neq h\\ k,h\in\boldsymbol{N}}}2^{-(h+k)/T}\right)\leq c_H|\xi|^{-H}.$$

Moreover $\Phi(\xi)$ is clearly rapidly decreasing when $\xi_1<0$. Since

$$\widehat{\varphi u}(\xi)^2=\sum_{k=1}^{\infty}2^{-2(k/T)}\widehat{\varphi\tau}(\xi_1-2^k,\xi'')^2+\Phi(\xi), \tag{4.4.19}$$

by means of (4.4.16), (4.4.17), (4.4.18) we can realize that for any $\nu\in\boldsymbol{R}^n$ such that $\nu_1+\dots+\nu_n>1/Tc$, $\operatorname{sing\,supp}_{\nu,\psi}u=\operatorname{supp}\tau$, where $\operatorname{sing\,supp}_{\nu,\psi}u$ is the complement in $\boldsymbol{R}^n$ of the set of points which admit some neighborhood U such that $u\in H^\nu_{\psi\mathrm{loc}}(U)$. Let us introduce now for every $x_0\in\boldsymbol{R}^n$ and $u\in\mathcal{D}'(\boldsymbol{R}^n)$ the ψ filter of microlocal smoothness $\Sigma_{\psi,x_0}u$ defined by means of (4.4.2) and (4.4.3), where $\widehat{v}(\xi)$ is rapidly decreasing in $(\boldsymbol{R}^n\backslash X)_{\varepsilon\psi}$. For every $\mu\in\boldsymbol{R}^n$, $f\in\mathcal{D}'(\boldsymbol{R}^n)$, we can easily verify that $\operatorname{sing\,supp}_{\mu,\psi}f\subset\operatorname{sing\,supp}f$ and $\Sigma_{\psi,x_0}f\subset\Sigma^\mu_{\psi,x_0}f$. Now, following the last part of the proof in RODINO [14, Theorem 3.9] and observing that T may grow all over $\boldsymbol{R}_+$, we can state:

THEOREM 4.4. – *There exists $a(x,\eta)\in S^0_{1,1,\psi}(\boldsymbol{R}^n)$ which satisfy:*

i) *$a(x,D)$ maps $H^\mu_{\psi,\mathrm{comp}}$ into $H^\mu_{\psi,\mathrm{loc}}$, $\mu\in\boldsymbol{R}^n$;*

ii) *for any $\nu\in\boldsymbol{R}^n$ such that $\sum_{j=1}^n\nu_j>0$ there exists $Y\in\Sigma^\nu_{\psi,x_0}u$ which is not in $\Sigma^\nu_{\psi,x_0}a(x,D)\,u$, for some $u\in L_2(\boldsymbol{R}^n)$.*

REMARK. – If we set $\widetilde{\psi}(\xi) = ([\xi]^{M_1}, \dots, [\xi]^{M_n})$, $\mu = (m/M_1, \dots, m/M_n)$, $\nu = (s/M_1, \dots, s/M_n)$, the symbol class $S_M^m(\boldsymbol{\Omega})$ and the anisotropic Sobolev space H_M^s can be identified respectively as $S_{1,1,\widetilde{\psi}}^{\mu}(\boldsymbol{\Omega})$ and H_{ψ}^{ν}; see for example RODINO [14], [15].

Let us consider now, for $s \in \boldsymbol{R}$, $x \in \boldsymbol{\Omega}$, $u \in \mathcal{D}'(\boldsymbol{\Omega})$:

$$\Sigma_{M,x}^s u = \bigcap_k X_k, \qquad X_k \in \Sigma_{\widetilde{\psi}(\xi),x}^s u, \tag{4.4.20}$$

then $\mathrm{WF}_{s,M} u$ and the closed conic set:

$$\{(x, \xi) \in \boldsymbol{\Omega} \times \boldsymbol{R}^n \backslash 0;\ \xi \in \Sigma_{M,x}^s u\}; \tag{4.4.21}$$

differ only for a bounded part, see [7].

Let us consider now for $r > 0$

$$b(x, \eta) = a(x, \eta)(1 + [\eta])^{-r} \in \widetilde{S}_M^{-r}(\boldsymbol{R}^n) \subset S_{M,r}^0(\boldsymbol{R}^n), \tag{4.4.22}$$

with $a(x, \eta)$ defined by (4.4.8), where $\psi(\xi) = \widetilde{\psi}(\xi)$. If we take $u \in L_2$ as in (4.4.15), we obtain $b(x, D)u \in H_M^r(\boldsymbol{R}^n)$; then $\mathrm{WF}_{s,M} b(x, D)\, u$ is empty when $s \leq r$. On the other hand Theorem 4.4 shows that $\mathrm{WF}_{s,M} u$ does not contain $\mathrm{WF}_{s-m,M} b(x, D)\, u$ when $s > r$. We can then easily end that $\mathrm{WF}_{s,M} u$, $u \in H_{M,\mathrm{loc}}^{\sigma}(\boldsymbol{\Omega})$, is preserved under the action of $b(x, D) \in S_{1,1,r}^m(\boldsymbol{\Omega})$ if and only if s belongs to the interval $(\sigma, \sigma + r]$.

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