Anisotropic Universe in $f(Q,T)$ gravity, a novel study

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$f(Q,T)$ theory of gravity is very recently proposed to incorporate within the action Lagrangian, the trace T of the energy-momentum tensor along with the non-metricity scalar Q. The cosmological application of this theory in a spatially flat isotropic and homogeneous Universe is well-studied. However, our Universe is not isotropic since the Planck era and therefore to study a complete evolution of the Universe we must investigate the $f(Q,T)$ theory in a model with a small anisotropy. This motivated us to presume a locally rotationally symmetric (LRS) Bianchi-I spacetime and derive the motion equations. We analyse the model candidate $f(Q,T) = \alpha Q^{n+1} + \beta T$, and to constrain the parameter n, we employ the statistical Markov chain Monte Carlo (MCMC) method with the Bayesian approach using two independent observational datasets, namely, the Hubble datasets, and Type Ia supernovae (SNe Ia) datasets.

I. Introduction

Several recent Type Ia supernovae (SNe Ia) datasets [1, 2] and Planck Collaboration [3] results have revealed that the Universe is expanding at an accelerated rate. To explain this late-time cosmic acceleration in the realm of general relativity (GR), several ideas have been presented in the literature. The most prominent candidate causing this acceleration is thought to be some yet undetected form of Dark Energy (DE), which cannot be described by the baryonic matter. However, such a cosmic scenario is plagued with several issues [4], promoting the development of alternative gravity models. The first in line and simplest form of such modified gravity theories must be the $f(\mathring{R})$ theories, where the Ricci-scalar $\mathring{R}$ in the original Einstein-Hilbert action was replaced by an arbitrary but viable function $f(\mathring{R})$ [5]. Several curvature based modified gravity theories were proposed and analysed in the literature, for a thorough survey one can see [6, 7] and the references therein. In this research direction, extending the popular $f(\mathring{R})$ gravity, a matter-geometry coupling in the form of $f(\mathring{R}, T)$ theory was first proposed by Harko et al [8] and later frequently investigated in cosmological research [9, 10]. Studies of $f(\mathring{R}, T)$ gravity coupled with a real scalar field was also conducted in the inflationary paradigm [11]. However, the most general technique to couple a scalar field with the gravitational degrees of freedom with second-order Euler-Lagrange field equations is given by the well-known Horndeski Lagrangian, initiated in [12]. This theory can only incorporate the Riemannian curvature tensor and the second derivatives of the scalar field in a severely constrained form to maintain the Euler-Lagrange field equations in second order; even though the theory contains arbitrary functions of the scalar field and its kinetic component. This might be motivated to survive the Ostrogradsky instability, which was present in the higher-order theories of gravity, fourth-order $f(\mathring{R})$ theory being an exception. In particular, it is forbidden for the curvature invariants to arise freely via arbitrary functions. This is due to the fact that the curvature invariants already contain second derivatives of the metric, which, in most cases, would result in higher-order equations [13, 14]. After the recent discovery of GW170817, most of Horndeski's terms are severely constrained by the tensor mode propagation speed [15].

On the other hand, an equivalent formulation of gravity was proposed on a flat spacetime geometry based solely either on the torsion (TEGR) or the non-metricity (STEGR), the first is known as metric teleparallel theories and the

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second as symmetric teleparallel theories. Nester and Yo [16] proposed the latter from an action term $\int Q\sqrt{-g}d^4x$. Due to its dependence on the dark sector, Jimenez et al. later extended it to formulate the $f(Q)$ gravity [17] such that the late-time acceleration could be demonstrated from the additional geometric components. In STEGR, an extension was also proposed by considering a linear combination of all the possible quadratic contractions of the nonmetricity tensor [18, 19]. Scalar fields were also coupled to the nonmetricity scalar Q in [20, 21]. The Hordenski type theory was very recently proposed in both metric and symmetric teleparallel geometries, respectively in [22] and [23] and thorough investigation is still due. Even though most of the terms are the same as in the original Hordenski theory in GR, a much richer phenomenology is noticed in the latter two counterparts.

In the recent past, a tremendous amount of works were carried out in the $f(Q)$ theories [24–43].

Very recently, a matter-geometry coupling in the form of $f(Q, T)$ theories were proposed in which the Lagrangian was represented by a viable function of the non-metricity scalar Q, and the trace T of the energy-momentum tensor [44]. Harko argued that this dependence can be caused by exotic imperfect fluids or quantum phenomena [45]. The STEGR, and specially the current $f(Q, T)$ theory is naturally a second-order theory, unlike the curvature-based original Hordenski theory which was constructed with additional constraints to be of second-order. After the first article, several works were published in this new gravity theories [46–49]. However, all the existing works were carried out in a background of spatially flat homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. Whereas, there are sufficient evidences to support a not so symmetrical looking Universe, atleast in its beginning near the initial singularity [50–52]. Moreover, some of the anisotropic Bianchi models (models I, VII0, V, VIIh, and IX) can be interpreted as the homogeneous limit of linear cosmological perturbations of the FLRW spacetime [53, 54]. The homogeneous and isotropic model naturally cannot provide a complete account of evolution of Universe. One must relax the assumption of FLRW geometry from the very start and investigate the transition from an anisotropic and inhomogeneous state to the observed level of homogeneity and isotropy. A number of recent articles in several modified gravity theories can be cited [55–60] for a diverse field of investigation in the background of anisotropic Bianchi type Universe models. In the present discussion we consider a special type of Bianchi Universe, the locally rotationally symmetric (LRS) Bianchi type-I model to denote the anisotropic state of the Universe, given by the metric in the Cartesian coordinates

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)(dy^2 + dz^2). \tag{1}$$

Here, $A(t)$ and $B(t) = C(t)$ are the metric potentials that are time-dependent scale factors. All of our results can be extended to the Bianchi type-I model without much effort.

In this paper, we analyse the model candidate $f(Q, T) = \alpha Q^{n+1} + \beta T$, and as a special case we briefly discuss the model $f(Q, T) = \alpha Q + \beta T$, in the background of an anisotropic Universe with a well-known physically motivated condition of proportionality between the expansion and shear scalar. The present research focuses on observable evidence from SNe, Cosmic Microwave Background (CMB), and Baryon Acoustic Oscillations (BAO), all of which are shown to be useful in constraining cosmological models. The Hubble parameter $H(z)$ datasets reveal the complicated structure of the expansion of the Universe. The ages of the most massive and slowly developing galaxies provide direct measurements of the $H(z)$ at different redshifts z, culminating in the construction of a new type of standard cosmological probe [69]. In this paper, we present 31 Hubble expansion observations spread using the differential age approach [70]. Scolnic et al. previously posted Pantheon, a massive SNe datasets with 1048 locations across the redshift range $z \in [0.01, 2.3]$ [74]. The Hubble and SNe Ia are used in our analysis to constrain the cosmological model.

The paper is structured as follows: After introduction, in Section II, we provide an outline of $f(Q, T)$ gravity, followed by the motion equations and some results crucial for the current study in Section III. In Section IV, we propose the cosmological model used in the paper, along with computation of certain parameters. Then we proceed to analyze the model $f(Q, T) = \alpha Q^{n+1} + \beta T$ in Subsection IV A, and the brief account of the special linear case in Subsection IV B. In Section V, we use Hubble and SNe Ia datasets to constrain the model parameters. Finally, in Section VI, we summarise our findings.

II.   The mathematical formulation

As an extension of symmetric teleparallel gravity theory, the $f(Q, T)$-gravity theory is also constrained by the curvature free and torsion free conditions, i.e., $R^\rho{}_{\sigma\mu\nu} = 0$ and $T^\rho{}_{\mu\nu} = 0$. The disformation tensor is defined as the difference between the associated connection $\Gamma^\lambda{}_{\mu\nu}$ and the Levi-Civita connection $\mathring{\Gamma}^\lambda{}_{\mu\nu}$

$$L^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \mathring{\Gamma}^\lambda{}_{\mu\nu}. \tag{2}$$

It can also be expressed as

$$L^\lambda{}_{\mu\nu} = \frac{1}{2}(Q^\lambda{}_{\mu\nu} - Q_\mu{}^\lambda{}_\nu - Q_\nu{}^\lambda{}_\mu),$$

where $Q_{\lambda\mu\nu} := \nabla_\lambda g_{\mu\nu}$ is the non-metricity tensor. The non-metricity scalar Q reads as [35]

$$Q = Q_{\lambda\mu\nu}P^{\lambda\mu\nu} = -\frac{1}{2}Q_{\lambda\mu\nu}L^{\lambda\mu\nu} + \frac{1}{4}Q_\lambda Q^\lambda - \frac{1}{2}Q_\lambda \tilde{Q}^\lambda, \tag{3}$$

where

$$P^\lambda{}_{\mu\nu} := \frac{1}{4}\left(-2L^\lambda{}_{\mu\nu} + Q^\lambda g_{\mu\nu} - \tilde{Q}^\lambda g_{\mu\nu} - \frac{1}{2}\delta^\lambda_\mu Q_\nu - \frac{1}{2}\delta^\lambda_\nu Q_\mu\right), \tag{4}$$

is the superpotential tensor, $Q_\lambda = Q_{\lambda\mu}{}^\mu$ and $\tilde{Q}_\lambda = Q_{\nu\lambda}{}^\nu$. The action of $f(Q, T)$-gravity is defined as [44]

$$S = \int \left[\frac{1}{2\kappa}f(Q, T) + \mathcal{L}_M\right] \sqrt{-g}\, d^4 x,$$

where $g := \det[g_{\mu\nu}]$, $\mathcal{L}_M$ is the Lagrangian for matter source and T is the trace of the stress energy tensor $T_{\mu\nu}$, which is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}.$$

The following metric field equation can be obtained after varying the action with respect to the metric

$$\frac{2}{\sqrt{-g}}\nabla_\lambda(\sqrt{-g}f_Q P^\lambda{}_{\mu\nu}) - \frac{1}{2}f g_{\mu\nu} + f_T(T_{\mu\nu} + \Theta_{\mu\nu}) + f_Q(P_{\nu\rho\sigma}Q_\mu{}^{\rho\sigma} - 2P_{\rho\sigma\mu}Q^{\rho\sigma}{}_\nu) = \kappa T_{\mu\nu}, \tag{5}$$

where $f_Q = f_Q(Q, T)$ and $f_T = f_T(Q, T)$ denote the partial derivative of $f = f(Q, T)$ with respect to Q and T respectively, and

$$\Theta_{\mu\nu} = \frac{g_{\alpha\beta}\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}.$$

Noticing that the field equation (5) is differ from that of [44] due to the choice of the sign in defining the non-metricity scalar (12), yet it does not affect the results obtained in general. In the present paper a perfect fluid type spacetime is considered, for which the stress energy tensor is given by

$$T_{\mu\nu} = pg_{\mu\nu} + (p + \rho)u_\mu u_\nu,$$

where ρ, p and u^μ represent the energy density, pressure and four velocity of the fluid respectively. We have chosen here the matter Lagrangian to be $\mathcal{L}_M = p$. In addition, the matter Lagrangian is supposed to rely only on the metric tensors. As a result,

$$\Theta_{\mu\nu} = pg_{\mu\nu} - 2T_{\mu\nu}.$$

III.   Equations of motion in the LRS-BI model

In this section, we consider the LRS-BI spacetime whose line element is given by (1) in Cartesian coordinates. We use the usual flat affine connection in this coincident gauge choice to obtain the motion equations of a test particle. Corresponding to (1), the directional Hubble parameters are defined as

$$H_x(t) = \frac{\dot{A}}{A}, \quad H_y(t) = \frac{\dot{B}}{B}, \quad H_z(t) = \frac{\dot{B}}{B}, \tag{6}$$

and

$$H(t) = \frac{1}{3}\frac{\dot{V}}{V} = \frac{1}{3}\left[\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right], \tag{7}$$

is the average Hubble parameter, where the spatial volume is

$$V(t) = AB^2 = a^3. \tag{8}$$

where a is the mean scale factor of the Universe. The rate of expansion is evaluated by anisotropy parameter

$$\Delta(t) = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = \frac{2}{9H^2}\left(H_x - H_y\right)^2. \tag{9}$$

It follows that

$$H_y^2 + 2H_xH_y = 3H^2\left(1 - \frac{\Delta}{2}\right). \tag{10}$$

The expansion scalar $\theta(t)$ and shear $\sigma(t)$ of the fluid are given by

$$\theta(t) = H_x + 2H_y, \quad \sigma(t) = \frac{|H_x - H_y|}{\sqrt{3}}. \tag{11}$$

The non-metricity scalar is given by

$$Q(t) = -6(2H - H_y)H_y. \tag{12}$$

Using (1) and (5), we obtain the following Friedmann-like equations.

$$(\kappa + f_T(Q,T))\rho + f_T(Q,T)\,p = \frac{f(Q,T)}{2} + 6f_Q(Q,T)(2H - H_y)H_y, \tag{13}$$

$$\kappa p = -\frac{f(Q,T)}{2} - \frac{\partial}{\partial t}\left[2f_Q(Q,T)H_y\right] - 6f_Q(Q,T)H_yH, \tag{14}$$

$$\kappa p = -\frac{f(Q,T)}{2} - \frac{\partial}{\partial t}\left[f_Q(Q,T)(3H - H_y)\right] - 3f_Q(Q,T)(3H - H_y)H. \tag{15}$$

It follows from (13)–(15) that

$$\kappa\rho = \frac{f(Q,T)}{2} + \frac{6f_Q(Q,T)}{\kappa + f_T(Q,T)}\left[\kappa(2H - H_y)H_y + f_T(Q,T)H^2\right] + \frac{2f_T(Q,T)}{\kappa + f_T(Q,T)}\frac{\partial}{\partial t}\left[f_Q(Q,T)H\right], \tag{16}$$

$$\kappa p = -\frac{f(Q,T)}{2} - 2\frac{\partial}{\partial t}\left[f_Q(Q,T)H\right] - 6f_Q(Q,T)H^2. \tag{17}$$

On the other hand, using (14)–(15) we obtain

$$0 = \frac{\partial}{\partial t}\left[f_Q(Q,T)(H - H_y)\right] + 3f_Q(Q,T)(H - H_y)H.$$

Solving this differential equation gives

$$f_Q(Q,T)(H - H_y) = la(t)^{-3}, \tag{18}$$

where l is a constant.

IV.   Exact solutions of LRS-BI model

In this section, we opt for the exact solutions of the above-mentioned system which requires some additional assumption to determine. The well-studied physical condition of proportionality between the expansion scalar $\theta(t)$ and the shear scalar $\sigma(t)$ is used here, which yields

$$A(t) = B(t)^\lambda, \tag{19}$$

where the constant λ accounts for the anisotropic nature of the model, i.e., if λ is equal to one, the model is isotropic. The physical basis for this hypothesis is supported by studies of the velocity redshift relation for extragalactic sources, which indicate that the Hubble expansion of the Universe can attain isotropy if $\frac{\sigma(t)}{\theta(t)}$ is constant. Collins proved the physical importance of this condition in the case of a perfect fluid with a barotropic equation of state (EoS). Additionally, it was showed [75] that in radiation era, a quadratic model $f(Q) = f_0 Q^2$ reproduces the condition $\sigma^2 \propto \theta^2$.

Using the condition (19), we can derive the relationship between the directional Hubble parameters $H_x(t)$ and $H_y(t)$ as follows:

$$H_x(t) = \lambda H_y(t). \tag{20}$$

Also, we obtain

$$H_y(t) = \frac{3H(t)}{\lambda + 2}. \tag{21}$$

Using Eqs. (18) and (21), we get

$$f_Q(Q, T) H(t) = \frac{(\lambda + 2)la(t)^{-3}}{(\lambda - 1)}, \tag{22}$$

In this case, the non-metricity scalar $Q(t)$ in Eq. (12) has the form

$$Q(t) = -\frac{18(2\lambda + 1)}{(\lambda + 2)^2} H(t)^2. \tag{23}$$

Substituting these into (16)–(17) and simplifying we get

$$\kappa\rho = \frac{f(Q, T)}{2} + \frac{6f_Q(Q, T) H(t)^2}{\kappa + f_T(Q, T)} \frac{3\kappa(2\lambda + 1)}{(\lambda + 2)^2}, \tag{24}$$

$$\kappa p = -\frac{f(Q, T)}{2}. \tag{25}$$

From the previous two equations, we see that when studying the case of a Universe filled with pressureless matter, i.e. $p = 0$ in association with condition (19), the function $f(Q, T)$ is not defined. Thus, in the next two sections, we study some forms of the function $f(Q, T)$ with $p \neq 0$.

A.   Cosmological model with $f(Q, T) = \alpha Q^{(n+1)} + \beta T$

As a first case of the cosmological model of $f(Q, T)$ gravity, let us consider the scenario when $f(Q, T)$ has the non-linear form $f(Q, T) = \alpha Q^{(n+1)} + \beta T$, where α, β and n are constants. Then, we have $f_Q = \alpha(n + 1)Q^n$ and $f_T = \beta$. Thus, Eq. (22) becomes

$$\alpha(n + 1) \left(\frac{-18(2\lambda + 1)}{(\lambda + 2)^2} \right)^n H(t)^{2n+1} = \frac{(\lambda + 2)la(t)^{-3}}{(\lambda - 1)}. \tag{26}$$

Solving this equation gives

$$H(t) = \frac{2n+1}{3t + (2n+1)c_n},$$ (27)

where c_n is a constant.

Again, solving the field equations (24) and (25) to find the expressions for pressure and energy density, we obtain

$$\rho(t) = -\frac{\alpha 2^{n-1} 9^{n+1} (\beta(3n+2) + 8\pi(2n+1)) \left(-\frac{H^2(2\lambda+1)}{(\lambda+2)^2}\right)^{n+1}}{(\beta+4\pi)(\beta+8\pi)},$$ (28)

$$p(t) = -\frac{\alpha 2^{n-1} 9^{n+1} (\beta(n+2) + 8\pi) \left(-\frac{H^2(2\lambda+1)}{(\lambda+2)^2}\right)^{n+1}}{(\beta+4\pi)(\beta+8\pi)}.$$ (29)

Therefore, at the first epoch $t \to 0$, we see that energy density and pressure have finite values. Also, these quantities diminish in value as cosmic time t increases and approaches to zero at infinite time.

Now, using Eqs. (20) and (21), the directional Hubble parameters are obtained as

$$H_x(t) = \frac{3\lambda(2n+1)}{(\lambda+2)[3t + (2n+1)c_n]},$$ (30)

and

$$H_y(t) = \frac{3(2n+1)}{(\lambda+2)[3t + (2n+1)c_n]}.$$ (31)

Further, the metric potentials are derived as

$$A(t) = \left(\frac{H_0[3t + (2n+1)c_n]}{2n+1}\right)^{\frac{(2n+1)\lambda}{\lambda+2}},$$ (32)

and

$$B(t) = \left(\frac{H_0[3t + (2n+1)c_n]}{2n+1}\right)^{\frac{2n+1}{\lambda+2}},$$ (33)

where

$$H_0 = (\lambda+2)\left[\frac{l}{(\lambda-1)(n+1)\alpha}\right]^{\frac{1}{2n+1}}\left[\frac{-1}{18(2\lambda+1)}\right]^{\frac{n}{2n+1}}.$$ (34)

The model exhibits no singularity at the beginning epoch $t \to 0$, since the metric potentials have constant values. For time $t \to \infty$, the metric potentials tend to infinity as time passes. Thus, using Eqs. (32) and (33), the LRS Bianchi type-I metric becomes

$$ds^2 = -dt^2 + \left(\frac{H_0[3t + (2n+1)c_n]}{2n+1}\right)^{\frac{(2n+1)\lambda}{\lambda+2}} dx^2 + \left(\frac{H_0[3t + (2n+1)c_n]}{2n+1}\right)^{\frac{2n+1}{\lambda+2}} (dy^2 + dz^2).$$ (35)

By applying (8), (11) and (20), the spatial volume V, expansion scalar $\theta(t)$, and shear scalar $\sigma(t)$ are derived as,

$$V(t) = \left(\frac{H_0[3t + (2n+1)c_n]}{2n+1}\right)^{2n+1},$$ (36)

$$\theta(t) = \frac{3(2n+1)}{3t + (2n+1)c_n}, \tag{37}$$

$$\sigma(t) = \frac{\sqrt{3}(\lambda-1)(2n+1)}{(\lambda+2)[3t + (2n+1)c_n]}. \tag{38}$$

Using Eqs. (37) and (38), we obtain

$$\frac{\sigma(t)^2}{\theta(t)} = \frac{(\lambda-1)^2(2n+1)}{(\lambda+2)^2[3t + (2n+1)c_n]}. \tag{39}$$

Here, It is observed that the isotropy condition, i.e., $\frac{\sigma(t)^2}{\theta(t)} \to 0$ as $t \to \infty$, is fulfilled in this case. Eqs. (36) and (37) show that the spatial volume is finite at $t = 0$ and increases with time from a finite to an infinitely large value, but the expansion scalar is infinite, implying that the Universe begins to evolve with finite volume at $t = 0$. This is compatible with the Big Bang scenario.

Also, from Eq. (27), we obtain

$$\frac{H(t)}{H_0} = \frac{3t_0 + (2n+1)c_n}{3t + (2n+1)c_n}, \tag{40}$$

where H_0 and t_0 represent the current value of Hubble parameter and age of the Universe. To get cosmological findings that enable for direct comparison of cosmological model predictions with astronomical data, we use the redshift parameter z as an independent variable in place of the cosmic time variable t. The redshift parameter for a distant source is inversely related to the scale factor of the Universe at the moment in which the photons were produced from the source. In this case, the equation relates the scale factor $a(t)$ with the redshift parameter z is given by

$$a = \frac{a_0}{(1+z)}. \tag{41}$$

Here, the scale factor is normalized such that its current value is one i.e. $a_0 = a(0) = 1$. Using Eqs. (40) and (41), the expression for Hubble parameter $H(t)$ in terms of the redshift parameter z is derived as

$$H(z) = H_0 (1+z)^{\frac{3}{2n+1}}, \tag{42}$$

where

$$H_0 = (\lambda+2) \left[\frac{l}{(\lambda-1)(n+1)\alpha} \right]^{\frac{1}{2n+1}} \left[\frac{-1}{18(2\lambda+1)} \right]^{\frac{n}{2n+1}}. \tag{43}$$

In addition, we find the power-law expansion as the solution to the field equations in an anisotropic Universe. Power-law cosmology provides an attractive solution to several exceptional problems, including flatness and the horizon problem. In literature, the power-law expansion is well justified. The author of [76] examined cosmic parameters using a power-law, and Hubble and Type Ia supernova datasets. Rani et al. [77] used state-finder analysis to investigate the power law cosmology. Recently, Koussour and Bennai [78] employed the power law to investigate cosmic acceleration in an anisotropic Universe in $f(Q)$ gravity.

The deceleration parameter $q(t)$, which indicates the accelerating/decelerating aspect of the expansion of the Universe, is an essential cosmological variable. The deceleration parameter is expressed by the equation:

$$q = -1 + \frac{d}{dt} \left(\frac{1}{H} \right) = -1 - \frac{\dot{H}}{H^2}. \tag{44}$$

Using Eq. (41), the time operator is given by

$$\frac{d}{dt} = \frac{dz}{dt} \frac{d}{dz} = -(1+z) H(z) \frac{d}{dz}. \tag{45}$$

The deceleration parameter $q(t)$ can be calculated as a function of the redshift parameter z using the formula

$$q(z) = -1 + (1+z)\frac{1}{H(z)}\frac{dH(z)}{dz}. \tag{46}$$

Here, the sign of the $q(z)$ (negative or positive) shows if the Universe accelerates or decelerates. The value of the deceleration parameter for this scenario is

$$q = -1 + \frac{3}{1+2n}. \tag{47}$$

This is a constant as predicted according to the power law type expansion of the cosmological model. It is clear from Eq. (47) that there is a transition phase from deceleration to acceleration at $n = 1$. Fig. 1 depicts the evolution of the deceleration parameter in terms of n for the function $f(Q,T) = \alpha Q^{(n+1)} + \beta T$ and in our model, it is directly dependent on the parameter n. According to Fig. 1, the deceleration parameter is positive at $n < 1$ and negative for $n > 1$. Thus, it shows that the Universe is transitioning from deceleration to acceleration. q decreases as n increases.
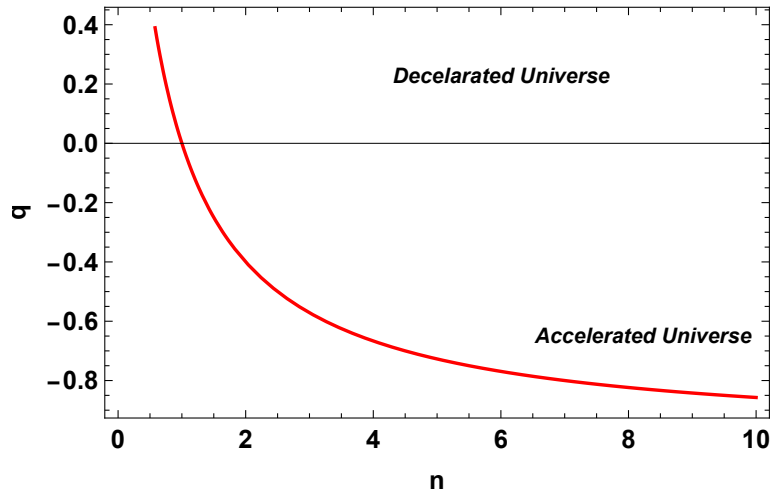

FIG. 1. The plot of deceleration parameter vs. n.

B.   Cosmological model with $f(Q,T) = \alpha Q + \beta T$

We can observe that for $n = 0$, the previous situation is reduced to the linear form of the $f(Q,T)$ function as $f(Q,T) = \alpha Q + \beta T$, where α and β are constants. Then, we have $f_Q = \alpha$ and $f_T = \beta$. So, for $n = 0$, we can solve Eq. (26) and get the Hubble parameter expression as

$$H(t) = \frac{1}{3t + c_0}, \tag{48}$$

where c_0 is the constant i.e. $c_0 = c_n(n = 0)$.

Now, using Eqs. (20) and (21), the directional Hubble parameters are obtained as

$$H_x(t) = \frac{3\lambda}{(\lambda+2)[3t+c_0]}, \tag{49}$$

and

$$H_y(t) = \frac{3}{(\lambda+2)[3t+c_0]}. \tag{50}$$

Using Eq. (39), we obtain

$$\frac{\sigma(t)^2}{\theta(t)} = \frac{(\lambda - 1)^2}{(\lambda + 2)^2 \left[3t + c_0\right]}. \tag{51}$$

Also, it is observed that the isotropy condition, i.e., $\frac{\sigma(t)^2}{\theta(t)} \to 0$ as $t \to \infty$, is fulfilled in this case.

As previously mentioned, all of the cosmological parameters listed above must be expressed in terms of redshift parameter z. Using Eq. (42), the expression of Hubble parameter $H(t)$ for $n = 0$ in terms of the redshift parameter z is derived as

$$H(z) = H_0 (1 + z)^3, \tag{52}$$

where $H_0 = \frac{(\lambda + 2)l}{\alpha(\lambda - 1)}$. It is important to note that Eq. (42) can be interpreted like the famous Hubble's Law which states that the proper distance d between galaxies is proportional to their recessional velocity v as measured by the Doppler effect redshift i.e. $v = H_0 d$. Thus, the value of the Hubble parameter in terms of redshift parameter is extremely important in an astrophysical background. For this scenario, the value of the deceleration parameter is $q = 2$, implying a Universe that is decelerating. Many authors of different modified gravity theories have obtained the same result.

V. Observational constraints

It should be highlighted that a thorough evaluation of the parameter values is important in examining the cosmological features. In this sense, the current section presents observational analyses of the current situation. The statistical method we use helps us to constrain parameters like H_0 and n. Especially, we use the Markov Chain Monte Carlo (MCMC) with the standard Bayesian approach. Furthermore, using the pseudo-chi-squared function χ^2, the best fit values for the parameters are obtained by the probability function,

$$\mathcal{L} \propto e^{-\frac{\chi^2}{2}}, \tag{53}$$

To do this, we now focus on two datasets: Hubble and Type Ia supernova (SNe Ia) data. To begin, we evaluate the parameter space priors, which are $(60.0 < H_0 < 80.0)$ to account for all possible scenarios of the Hubble parameter, $(-10.0 < n < +10.0)$ to get all the scenarios of the expansion of the Universe. Also, take into consideration that our cosmological model must also fit the observational datasets. The next subsections go into further depth on the data sets and statistical analyses.
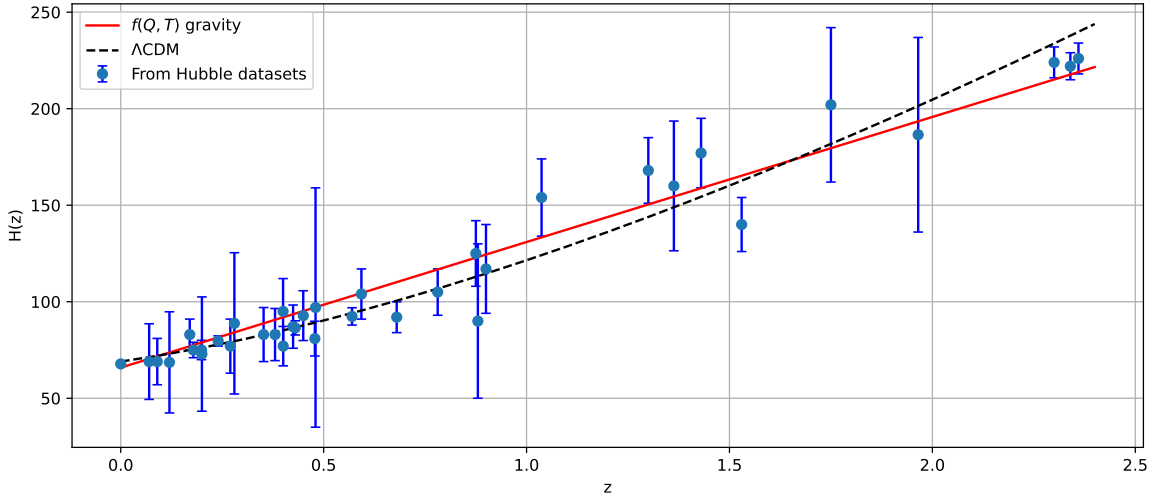
A. Hubble datasets

Here, we employ a standardized collection of 31 measures derived from the differential age technique (DA) in the redshift range $0.07 < z < 2.42$ and are listed in Tab. V A [86, 87]. The DA technique can be used to calculate the rate of expansion of the Universe at redshifts z. The Hubble datasets chi-square (χ^2) is calculated as follows:

$$\chi^2_{Hubble} = \sum_{j=1}^{31} \frac{\left[H_{th}(z_j) - H_{obs}(z_j, p_s)\right]^2}{\sigma(z_j)^2}, \tag{54}$$

where H_{th} and H_{obs} are the theoretical and observed values of the Hubble parameter $H(z)$, and p_s denotes the parameter space of the model to be constrained. In addition, σ^2 denotes the standard error in the observed value of $H(z)$, $p_s = (H_0, n)$ and is the parameter space of the cosmic background.

z	$H(z)$	σ_H	Ref.	z	$H(z)$	σ_H	Ref.
0.070	69	19.6	[79]	0.4783	80	99	[83]
0.90	69	12	[80]	0.480	97	62	[79]
0.120	68.6	26.2	[79]	0.593	104	13	[81]
0.170	83	8	[80]	0.6797	92	8	[81]
0.1791	75	4	[81]	0.7812	105	12	[81]
0.1993	75	5	[81]	0.8754	125	17	[81]
0.200	72.9	29.6	[82]	0.880	90	40	[79]
0.270	77	14	[80]	0.900	117	23	[80]
0.280	88.8	36.6	[82]	1.037	154	20	[81]
0.3519	83	14	[81]	1.300	168	17	[80]
0.3802	83	13.5	[83]	1.363	160	33.6	[85]
0.400	95	17	[80]	1.430	177	18	[80]
0.4004	77	10.2	[83]	1.530	140	14	[80]
0.4247	87.1	11.2	[83]	1.750	202	40	[80]
0.4497	92.8	12.9	[83]	1.965	186.5	50.4	[85]
0.470	89	34	[84]				

TABLE I. $H(z)$ datasets with 31 data points.

FIG. 2. The plot of $H(z)$ vs. redshift parameter z for our $f(Q,T) = \alpha Q^{(n+1)} + \beta T$ model, shown in red, and ΛCDM, shown in black dashed lines, shows an excellent fit to the 31 points of the Hubble datasets.

B. Type Ia supernovae (SNe Ia) datasets

The measurement of SNe Ia is essential to comprehend how the Universe is expanding. The Panoramic Survey Telescope and Rapid Response System (Pan-STARSS1), Sloan Digital Sky Survey (SDSS), Supernova Legacy Survey (SNLS), and Hubble Space Telescope (HST) surveys all collected data on SNe Ia [74]. Here, we employ the Pantheon sample, which consists of 1048 points with distance moduli μ_j in the range $0.01 < z_j < 2.26$ at various redshifts. The

SNe Ia datasets chi-square (χ^2) is calculated as follows:

$$\chi^2_{SNe} = \sum_{j,i=1}^{1048} \Delta\mu_j (C_{SNe}^{-1})_{ji} \Delta\mu_i. \tag{55}$$

Here, C_{SNe} is the covariance matrix, and $\Delta\mu_j = \mu_j^{th}(z_j, p_s) - \mu_j^{obs}$ is the difference between the measured distance modulus value collected from cosmic measurements and its theoretical values estimated from the model with the specified parameter space p_s. The theoretical distance modulus μ^{th} is given as

$$\mu^{th}(z_j) = 25 + 5 log_{10} \left[\frac{d_l}{1Mpc} \right], \tag{56}$$

and the luminosity distance d_l defined as,

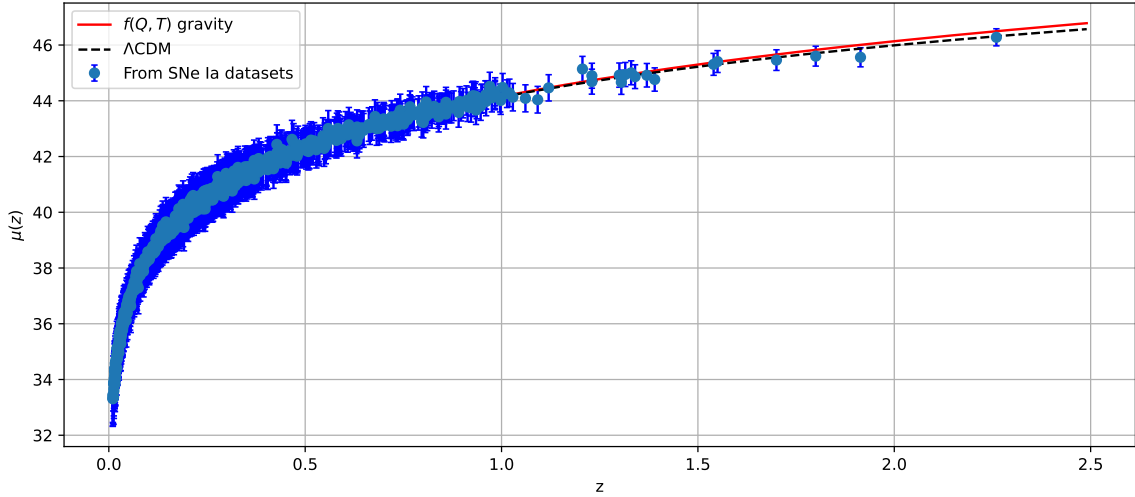$$d_l(z) = (1+z) \int_0^z \frac{c}{H(z, p_s)}. \tag{57}$$

FIG. 3. The plot of $\mu(z)$ vs. redshift parameter z for our $f(Q, T) = \alpha Q^{(n+1)} + \beta T$ model, shown in red, and ΛCDM, shown in black dashed lines, shows an excellent fit to the 1048 points of the SNe Ia datasets.

C. Results

The model parameters for the joint (Hubble+SNe) are constrained using $\chi^2 = \chi^2_{Hubble} + \chi^2_{SNe}$. Tab. II shows the outcomes and results. Figs. 2 and 3 compare our model to the widely accepted ΛCDM model in cosmology i.e. $H(z) = H_0 \sqrt{\Omega_0^m (1+z)^3 + \Omega_0^\Lambda}$; we use $\Omega_0^m = 0.3$, $\Omega_0^\Lambda = 0.7$, and $H_0 = 69 \ km.s^{-1}.Mpc^{-1}$ for the plot. The figures also show the Hubble and SNe Ia experimental findings, with 31 and 1048 data points and errors, respectively, allowing for a direct comparison of the two models. To find out the outcomes of our MCMC study, we employed 100 walkers and 1000 steps for all datasets: Hubble, SNe Ia, and Joint. Also, Fig. 4 shows the likelihood contours for Hubble, SNe Ia, and Joint analysis, and Tab. II shows the numerical findings. From Fig. 4, it is clear that the likelihood functions for all datasets (Hubble, SNe Ia, and Joint) are very well matched to a Gaussian distribution function. In every cosmological model, the Hubble constant H_0 and the deceleration parameter q play a significant role in characterizing the nature of the expansion of the Universe. The first describes the current rate of expansion of

the Universe, whereas the latter describes if the Universe is accelerating ($q < 0$) or decelerating ($q > 0$). We obtain the constraints on these parameters using the most recent Hubble with 31 data points and SNe Ia data with 1048 pantheon sample points. At the $1 - \sigma$ CL, the constraints determined from Hubble datasets are $H_0 = 65.9^{+1.5}_{-1.5}$ and $q = -0.011 \pm 0.01$, whereas the constraints determined from SNe Ia data are $H_0 = 66.8^{+2.6}_{-2.5}$ and $q = -0.261 \pm 0.03$. We also run the joint test with Hubble and SNe Ia datasets, which gives the constraints $H_0 = 65.1^{+1.2}_{-1.2}$ and $q = -0.014 \pm 0.01$. It is worth noting that the values of parameter H_0 correspond to the observations [3]. Also, the deceleration parameter values show that the observational data represent the actual cosmic acceleration within the context of anisotropic $f(Q, T)$ cosmology.
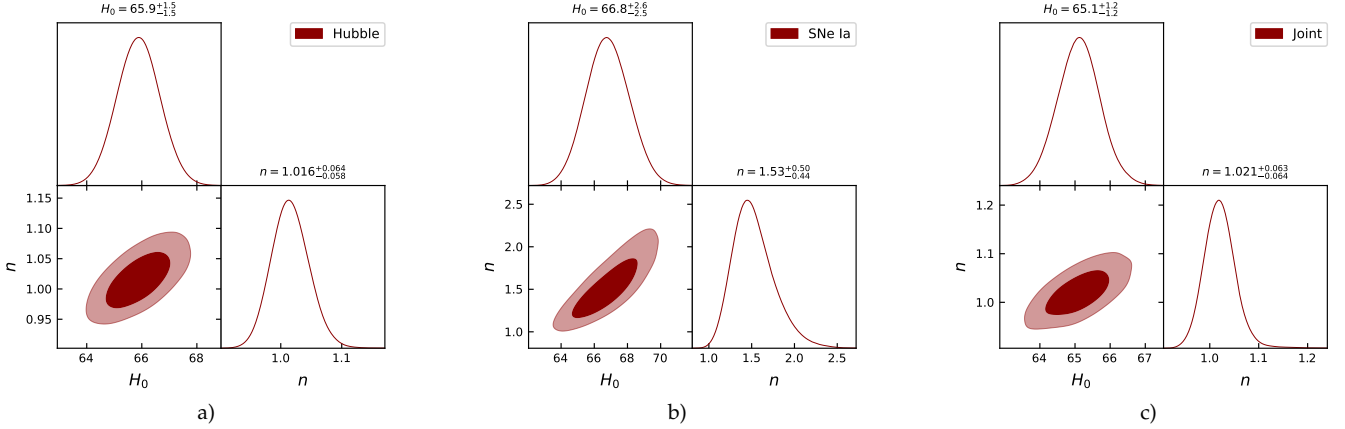

FIG. 4. The marginalized constraints on the parameters H_0 and n are presented using the Hubble (a), SNe Ia (b), and Joint (c) datasets. The dark red shaded areas represent the $1 - \sigma$ confidence level (CL), whereas the light red shaded regions represent the $2 - \sigma$ confidence level. The parameter constraint values are displayed at the $1 - \sigma$ CL.

TABLE II. With a confidence level of 68%, marginalized constrained data of the parameters H_0, n and q for various data samples were obtained.

Dataset	H_0	n	q
Hubble	$65.9^{+1.5}_{-1.5}$	$1.016^{+0.064}_{-0.058}$	-0.011 ± 0.01
SNeIa	$66.8^{+2.6}_{-2.5}$	$1.53^{+0.50}_{-0.44}$	-0.261 ± 0.03
Joint	$65.1^{+1.2}_{-1.2}$	$1.021^{+0.063}_{-0.046}$	-0.014 ± 0.01

VI. Concluding remarks

This study has investigated an anisotropic cosmology in the modified $f(Q, T)$ gravity theory, where Q denotes non-metricity scalar and T is the trace of the energy-momentum tensor. The exact solution of the field equations for an LRS Bianchi type-I spacetime are explored. Because the field equations are highly nonlinear and difficult, we have solved them by assuming that the expansion scalar $\theta(t)$ is proportional to the shear scalar $\sigma(t)$. It has provided $A(t) = B(t)^{\lambda}$, where $A(t)$ and $B(t)$ are the metric potentials and λ is an arbitrary constant that accounts for the anisotropic nature of the model. We have primarily investigated two solutions of modified field equations using two functional forms of $f(Q, T)$.

For the model $f(Q, T) = \alpha Q^{n+1} + \beta T$ the isotropy condition, i.e. $\frac{\sigma(t)^2}{\theta(t)} \to 0$ as $t \to \infty$, has been fulfilled. At $t = 0$, the spatial volume is finite and the expansion scalar is infinite, implying that the Universe began to evolve with a finite volume at $t = 0$. We obtain the scenario of the big bang. The energy density and pressure are finite at the first

epoch. Furthermore, as cosmic time t increases, the value of these quantities decreases and approaches 0 at infinite time. The deceleration parameter $q(t)$ is found to be $q = -1 + \frac{3}{1+2n}$, implying a phase transition from deceleration to acceleration at $n = 1$. From Fig. 1, it is observed that the deceleration parameter is positive (deceleration) at $n < 1$ and negative (acceleration) for $n > 1$. Next, to obtain the constraint value for the parameter n, we employed the statistical Markov chain Monte Carlo (MCMC) method with the Bayesian approach. We also examined the results for two independent observational datasets, Hubble datasets, and Type Ia supernovae (SNe Ia) datasets, which contain SDSS, SNLS, Pan-STARRS1, low-redshift survey, and HST surveys. The best-fit values obtained are $n = 1.016^{+0.064}_{-0.058}$ for the Hubble datasets, $n = 1.53^{+0.50}_{-0.44}$ for the SNe Ia datasets and $n = 1.021^{+0.063}_{-0.046}$ for the Hubble+SNe Ia datasets. Moreover, the deceleration parameter has been constrained, which is important in describing the evolution of the Universe. The best-fit values obtained are $q = -0.011 \pm 0.01$ for the Hubble datasets, $q = -0.261 \pm 0.03$ for the SN Ia datasets and $q = -0.014 \pm 0.01$ for the Hubble+SNe Ia datasets, which indicates an accelerating model of the Universe. Furthermore, using these parameter values, we compared our $f(Q, T) = \alpha Q^{n+1} + \beta T$ model with the most commonly accepted model for the Universe i.e. ΛCDM in Figs. 2 and 3.

Finally, we have obtained the solutions to the field equation by studying the linear case of the function $f(Q, T)$ i.e. $n = 0$. We have observed the same behavior for the cosmological parameters mentioned above, with the exception that the deceleration parameter in this scenario turns out to be a constant $q = 2$, which indicates a decelerating model of the Universe. The paper by Shamir [56] discusses a cosmological model based on $f(R, T)$ gravity in LRS-BI space-time. The author derived the field equations and solved them using an analytical approach. The obtained solutions were then used to investigate the evolution of the scale factor, energy density, and pressure of the universe. When compared to our paper on $f(Q, T)$ gravity, the primary distinction between Shamir's study on LRS-BI cosmology in $f(R, T)$ gravity is that while our paper utilizes observational constraints on the parameters of the $f(Q, T)$ gravity model, especially, we used the MCMC method to fit our $f(Q, T)$ gravity model parameters to Hubble and SNe Ia datasets which allows us to estimate the posterior probability distribution of the model parameters and quantify the uncertainties in our parameter estimates, Shamir does not incorporate any observational data in their analysis of the $f(R, T)$ gravity model. By incorporating observational data in our analysis, our paper provides a more comprehensive study of the $f(Q, T)$ gravity model, as it allows for a more rigorous comparison between the theoretical predictions and observational data. This can lead to more robust constraints on the model parameters, which can help to rule out or support specific modifications to gravity theory. In addition, Kennedy et al. [89] have reconstructed Horndeski's theories from phenomenological modified gravity and dark energy models on cosmological scales. Their approach is complementary to ours, as they aim to reconstruct the theoretical framework of Horndeski gravity from observed data, rather than proposing a specific gravity model. However, it is interesting to compare their results with ours, as both approaches aim to explain the observed acceleration of the universe without introducing dark energy.

Thus, these findings can encourage us to investigate the anisotropic nature of $f(Q, T)$ theory further, since it follows the observational data. Furthermore, it would be interesting to study the acceleration scenario of the Universe using certain parameterizations of the equation of state parameters. We intend to investigate this scenario in the future.

Data Availability

All generated data are included in this manuscript.

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