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Anisotropies in Luminosity Distance

Norimasa Sugiura,[1,*] Naoshi Sugiyama[1,**] and Misao Sasaki[2,***]

[1] *Department of Physics, Kyoto University, Kyoto 606-8502, Japan*
[2] *Department of Earth and Space Science, Oasaka University Toyonaka 560-0043, Japan*

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Anisotropies in the luminosity distance-redshift ($d_{\rm L}$-z) relation caused by the large-scale structure (LSS) of the universe are studied. We solve the Raychaudhuri equation in Friedmann-Robertson-Walker models, taking account of the LSS by the linear perturbation method. Numerical calculations to evaluate the amplitude of the anisotropies are carried out in flat models with the cosmological constant and in open models, employing the cold dark matter model and the *COBE*-normalization for the power spectrum of the density perturbation. The implications of our calculations for observation are discussed. These anisotropies in $d_{\rm L}$ may cause uncertainties in determining cosmological parameters, e.g., the deceleration parameter q_0, via the magnitude-redshift relation.

We found that the effects on the $d_{\rm L}$-z relation of the LSS are divided into three types: the peculiar velocity effect, gravitational lensing and the Sachs-Wolfe effect. We show that, for lower redshifts, the peculiar velocity effect is dominant, while around $z \gtrsim 0.5$, the gravitational lensing is dominant, though the amplitude is rather small, affecting the estimate of q_0 by at most about 5%.

§1. Introduction

The large-scale structure of the universe has been a recent topic of great interest both in observational and theoretical cosmology. For two decades, we have accumulated a large amount of observational data on the distribution of galaxies in the nearby universe by redshift surveys.[1)-3)] We also have recently succeeded in obtaining data on the cosmic microwave background (CMB) anisotropies using *COBE*.[4)] By analyzing these data, great progress has been made in our understanding of the structure of the universe, and it has been revealed that our universe can be described well by a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) model with small inhomogeneities. With regard to the parameters necessary to specify the model, however, no definitive conclusion has been submitted so far in spite of a great amount of research employing various approaches. One reason for this is the lack of our observational information concerning the universe at rather high redshifts, but there are currently two plans for further redshift surveys to overcome this difficulty:[5),6)] the Sloan Digital Sky Survey (SDSS), which will obtain redshifts for 10^6 galaxies and 10^5 quasars, and the Two-Degree Field Survey (2dF), which will measure 250,000 galaxies at even deeper redshifts over a limited region of the sky. With these studies, it is certain that we will obtain in the near future

*) E-mail: sugiura@tap.scphys.kyoto-u.ac.jp
**) E-mail: naoshi@tap.scphys.kyoto-u.ac.jp
***) E-mail: sasaki@vega.ess.sci.osaka-u.ac.jp

much observational data which will allow us to derive more quantitative information about the large-scale features of the universe.

Important information we can obtain from such observations is the effect of the large-scale structure on the appearance of luminous sources. The effect of the density inhomogeneities on the luminosity distance ($d_{\rm L}$) was studied by Sasaki [7] using the gauge invariant perturbation method. He derived a general formula to relate the inhomogeneities of the gravitational potential (Ψ) with fluctuations of $d_{\rm L}$, and applied it to the Einstein-de Sitter universe, giving an analytic expression, though no numerical evaluation was done because of the lack of reliable data regarding the large-scale behavior of Ψ. Now that we have obtained quantitative information concerning Ψ from the large-scale structure observations and the *COBE* data, it is worthwhile evaluating the magnitude of the effect in some cosmological models, as we do in this paper.

Another motivation for investigating the relation between the luminosity distance and the large-scale structure comes from the recent reports on the method to determine the cosmological parameters from the new observational data of supernovae (SNe), which have been systematically searched for at high redshifts $z = 0.3$–0.6.[8],[9] By studying the magnitude-redshift (m-z) relation of type Ia SNe, the authors give attempts to constrain the density parameter and the cosmological constant. In determining the cosmological parameters using the m-z relation, one needs to know the absolute luminosity of the sources. Past attempts to use galaxies as "standard candles" have failed due to large uncertainties in the luminosity evolution of galaxies. It is possible, however, to determine the peak luminosity of SNe accurately by studying the observed correlation between their lightcurve shape and peak luminosity. The effective dispersion in the luminosity distance can be now reduced to about 10%. Thus it is often emphasized that supernovae act as good standard candles. As the dispersion becomes small, however, the inhomogeneities of the density of the universe may begin to work as an extra noise in determining the luminosity distance.

In this paper, we give a detailed investigation of the anisotropies in the luminosity distance caused by the large-scale perturbations in FRW models. Following the general formula derived by Sasaki,[7] we rewrite the relation between $d_{\rm L}$ and Ψ in a form which can be used also for an open universe. Then we make numerical evaluations of the anisotropies in $d_{\rm L}$ on spatially flat models with cosmological constants and open models. We employ the cold dark matter (CDM) model, utilizing the information of Ψ obtained by the 4yr *COBE* data for the normalization of the power spectrum of the large-scale density perturbations.[10)-12)] Though the effects of the LSS may be expected to be small, it is necessary to firmly constrain the amplitude of the effect. At the same time, the observational data carry information on the power spectrum of density fluctuations. It may be possible, at least in principle, that one can use the data of the distance-redshift relation for constraining both the background cosmology and the power spectrum of the density perturbation.

The anisotropies in the $d_{\rm L}$-z relation have two consequences. First, they lead to errors in determining the luminosity distance. This effect may become important when an accurate measurement of the distance is needed, such as the case in determining the deceleration parameter. We will show that the contribution of the LSS to the error amounts to 5% for sources at $z \gtrsim 0.5$. This value is small compared to the instrinsic dispersions of the luminosities of the currently known sources, e.g., type Ia SNe at high redshifts, $z \sim 0.5$.

Secondly, we point out that such anisotropy patterns give us some information about the cosmological models because the magnitude and characters of these patterns are model-dependent. Thus, if such anisotropy patterns are identified by deep surveys of the universe in the future, we will have another test for the cosmological model. It is already known that information regarding the dark matter distribution can be obtained by observing the ellipticities of galaxy images caused by the shear effect of gravitational lensing. [13] In this paper, we focus on the magnification and demagnification of the luminosity by the Ricci focusing effect. This effect plays a complementary role to the shear effect in studying the inhomogeneities of the universe.

Since the luminosity distance is related to the angular distance, the discussion here is also valid with regard to an angular size or angular separation, such as the influence of the large-scale structure on the separation angle of double images of a distant luminous source. However, one may need a separate discussion for local non-linear structures, such as large voids or clusters.

This paper is organized as follows. In §2, the perturbed d_L-z relation is derived following Sasaki. [7] Applications to spatially flat models and open models are discussed in §3. The remaining sections are devoted for discussion and concluding remarks.

§2. Formalism

2.1. *Basic equations*

In this subsection, we give the basic equations which will be used in the following sections.

The line element of the spacetime is assumed to be written in the form

$$d\hat{s}^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = a(\eta)^2 ds^2 = a(\eta)^2 g_{\mu\nu} dx^\mu dx^\nu, \qquad x^\mu = (\eta, x^i), \tag{2·1}$$

where $\hat{g}_{\mu\nu}$ is the physical spacetime metric, $a(\eta)$ is the scale factor, and η is the conformal time. In the following, we use the rescaled metric $g_{\mu\nu}$ and other rescaled tensors, denoting the original tensors defined in the physical spacetime with hats (e.g., $\hat{g}_{\mu\nu}$). For example, we introduce a new velocity variable u^μ on the conformally transformed spacetime by $u^\mu = a\hat{u}^\mu$.

We consider the propagation of light rays emitted by a distant source such as a star or a galaxy. This is described by geometric optics, since the scale of change in the curvature we are considering is much larger than the wavelength. First, the energy momentum tensor of a congruence of photons with 4-momentum $\hat{k}^\mu$ is given by

$$\hat{T}^{\mu\nu} = \frac{1}{8\pi}\mathcal{A}^2 \hat{k}^\mu \hat{k}^\nu, \tag{2·2}$$

where $\mathcal{A}$ is the scalar amplitude of the wave. We introduce a null vector k^μ conformal to $\hat{k}^\mu$ and an affine parameter λ, defined by

$$k^\mu = a^2 \hat{k}^\mu \equiv \frac{dx^\mu}{d\lambda}. \tag{2·3}$$

Using the equation of motion $\hat{\nabla}_\nu \hat{T}^{\mu\nu} = 0$ and the geodesic equation $k^\nu \nabla_\nu k^\mu = 0$ (which follows from $\hat{k}^\nu \hat{\nabla}_\nu \hat{k}^\mu = 0$ by the conformal invariance of the null geodesic equation), we obtain the propagation equation for the scalar amplitude as

$$\nabla_\mu(\mathcal{A}^2 a^2 k^\mu) = 2\mathcal{A}a\left[\frac{d}{d\lambda}(\mathcal{A}a) + \frac{1}{2}\mathcal{A}a\theta\right] = 0, \tag{2·4}$$

where we define the expansion of the null congruence as $\theta \equiv \nabla^\nu k_\nu$. This equation is interpreted to represent the conservation of photon number. By differentiating this with respect to λ, we obtain the propagation equation for θ, i.e., the Raychaudhuri equation,

$$\frac{d}{d\lambda}\theta = -R_{\mu\nu}k^\mu k^\nu - \frac{1}{2}\theta^2 - 2\sigma^2; \quad \sigma^2 \equiv \frac{1}{2}\left[k^{(\alpha;\beta)}k_{(\alpha;\beta)} - \frac{1}{2}\theta^2\right], \tag{2·5}$$

$$k_{(\alpha;\beta)} \equiv \frac{1}{2}(k_{\alpha;\beta} - k_{\beta;\alpha}), \tag{2·6}$$

where σ is the shear of the congruence. In the following, we set $\sigma = 0$. In the FRW spacetime, the shear term vanishes for all times if it does initially. When we consider the linear perturbation on the FRW universe, σ acts as the higher order correction in Eq. (2·5) (see, e.g., Nakamura [18]). Also note that the vorticity vanishes in the geometric optic description.

Next, we define the luminosity distance. The absolute luminosity L_s of a spherically symmetric source with radius R_s is given by

$$L_\mathrm{s} = 4\pi R_\mathrm{s}^2 f(\lambda_\mathrm{s}), \tag{2·7}$$

where λ_s is the conformal affine parameter evaluated at the source and f is the amplitude of the energy flux of the source measured by an observer comoving with the source whose 4-velocity is $\hat{u}^\mu$:

$$f = \frac{1}{8\pi}\mathcal{A}^2(-\hat{k}_\nu \hat{u}^\nu)^2 \equiv \frac{1}{8\pi}\mathcal{A}^2 w^2. \tag{2·8}$$

Here we defined the energy of the photon w .

Then the luminosity distance to the source measured by an observer at $\lambda = 0$ is defined by

$$d_\mathrm{L}^2 = \frac{L_\mathrm{s}}{4\pi f(0)} = \frac{f(\lambda_\mathrm{s})}{f(0)}R_\mathrm{s}^2 = \frac{\mathcal{A}^2 w^2|_\mathrm{s}}{\mathcal{A}^2 w^2|_0}R_\mathrm{s}^2. \tag{2·9}$$

Recalling the definition of the redshift $w_\mathrm{s}/w_0 = 1 + z$, we can write d_L in terms of z as

$$d_\mathrm{L} = \frac{\mathcal{A}(\lambda_\mathrm{s})}{\mathcal{A}(0)}(1 + z(\lambda_\mathrm{s}))R_\mathrm{s}. \tag{2·10}$$

In the next section, we derive the anisotropies in d_L using these equations. We will rewrite the right-hand side of Eq. (2·10) in terms of z employing a linear perturbation method.

2.2. *Anisotropies in luminosity distance*

If the universe were completely homogeneous and isotropic, the luminosity distance would be the same at the same redshift regardless of the direction one is looking at; i.e., the $d_{\rm L}$-z relation has no anisotropies. However, small fluctuations in the density of the universe, which have led to complex structures, cause the anisotropies in the $d_{\rm L}$-z relation. We derive an expression of the anisotropies in the $d_{\rm L}$-z relation using a linear perturbation method.

For notational convenience, we denote perturbed quantities with a tilde ($\tilde{}$) and unperturbed ones without one in the following. For example, we express the metric as

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}. \tag{2·11}$$

For convenience, we introduce a new null vector, $\tilde{K}^\mu$, by

$$\tilde{K}^\mu = -\frac{1}{\tilde{w}(\lambda_{\rm s})a[\tilde{\eta}(\lambda_{\rm s})]}\tilde{k}^\mu; \tag{2·12}$$

namely, it is the past-directed null vector along a ray of light which is normalized as

$$(\tilde{g}_{\mu\nu}\tilde{K}^\mu\tilde{u}^\nu)_{\lambda_{\rm s}} = \frac{1}{\tilde{w}(\lambda_{\rm s})a[\tilde{\eta}(\lambda_{\rm s})]}(-\tilde{g}_{\mu\nu}\tilde{k}^\mu\tilde{u}^\nu)_{\lambda_{\rm s}} = 1. \tag{2·13}$$

In the following, λ denotes a new affine parameter associated with $\tilde{K}^\mu$.

First, let us consider light propagation in the unperturbed background. The spatial part of the background metric is homogeneous and isotropic:

$$g_{ij}dx^i dx^j = \gamma_{ij}dx^i dx^j = \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2. \tag{2·14}$$

Then the Ricci tensor takes the form

$$R_{\mu\nu} = 2K\gamma_{ij}\delta^i_\mu\delta^j_\nu. \tag{2·15}$$

The wave vector which is normalized correspondingly with (2·13) is

$$K^\mu = \frac{dx^\mu}{d\lambda} = (-1, \gamma^i), \tag{2·16}$$

where γ^i satisfies $\gamma_{ij}\gamma^i\gamma^j = 1$, and $\gamma^i_{|j}\gamma^j = 0$, since $k^\mu k_\mu = 0$ and $k^\nu\nabla_\nu k^\mu = 0$. Then, inserting these quantities into Eq. (2·5), we obtain the following equation for θ:

$$\frac{d}{d\lambda}\theta = -2K - \frac{1}{2}\theta^2. \tag{2·17}$$

Integration of this equation yields

$$\theta_{\rm L} = 2\sqrt{-K}\coth\sqrt{-K}(\lambda-\lambda_{\rm s}-\Delta\lambda_{\rm s}) = \begin{cases} 2\sqrt{-K}\coth\sqrt{-K}(\lambda-\lambda_{\rm s}-\Delta\lambda_{\rm s}), & K<0, \\ 2(\lambda-\lambda_{\rm s}-\Delta\lambda_{\rm s})^{-1}, & K=0, \\ 2\sqrt{K}\cot\sqrt{K}(\lambda-\lambda_{\rm s}-\Delta\lambda_{\rm s}), & K>0, \end{cases} \tag{2·18}$$

where the suffix L indicates the boundary condition $\theta_{\rm L}=0$ at $\lambda=\lambda_s+\Delta\lambda$, and $\Delta\lambda_{\rm s}$ is the infinitesimal affine parameter corresponding to the radius of the source $R_{\rm s}=a[\tilde{\eta}(\lambda_{\rm s})]\Delta\lambda_{\rm s}$.

Next, we consider the perturbed quantities. The linear perturbation of Eq. (2·17) gives

$$\frac{d}{d\lambda}\delta\theta_{\rm L}=-\theta_{\rm L}\delta\theta_{\rm L}-\delta(R_{\mu\nu}K^\mu K^\nu)_\lambda. \tag{2·19}$$

Integrating this with the boundary condition $\delta\theta_{\rm L}(\lambda_{\rm s})=0$, we obtain

$$\delta\theta_{\rm L}(\lambda)=\frac{1}{{\rm sh}^2\sqrt{-K}(\lambda-\lambda_{\rm s}-\Delta\lambda_{\rm s})}\int_\lambda^{\lambda_{\rm s}}d\bar{\lambda}{\rm sh}^2\sqrt{-K}(\bar{\lambda}-\lambda_{\rm s}-\Delta\lambda_{\rm s})\delta(R_{\mu\nu}K^\mu K^\nu)_{\bar{\lambda}}. \tag{2·20}$$

Equation (2·4) is integrated with the aid of Eq. (2·18), yielding

$$\frac{\mathcal{A}(\lambda_{\rm s})a[\tilde{\eta}(\lambda_{\rm s})]}{\mathcal{A}(0)a[\tilde{\eta}(0)]}=\frac{{\rm sh}\sqrt{-K}(\lambda_{\rm s}+\Delta\lambda_{\rm s})}{{\rm sh}\sqrt{-K}\Delta\lambda_{\rm s}}\exp\left[-\frac{1}{2}\int_0^{\lambda_{\rm s}}d\lambda\delta\theta_{\rm L}(\lambda)\right]. \tag{2·21}$$

Inserting Eq. (2·21) and the relation $R_{\rm s}=a[\tilde{\eta}(\lambda_{\rm s})]\Delta\lambda_{\rm s}$ into Eq. (2·10) and taking the point source limit, $\Delta\lambda_{\rm s}\to 0$, the perturbed luminosity distance can be expressed as a function of $\lambda_{\rm s}$:

$$\tilde{d}_{\rm L}(\lambda_{\rm s})=a[\tilde{\eta}(0)]\frac{{\rm sh}\sqrt{-K}\lambda_{\rm s}}{\sqrt{-K}}[1+\tilde{z}(\lambda_{\rm s})]\exp\left[-\frac{1}{2}\int_0^{\lambda_{\rm s}}d\lambda\delta\theta_{\rm L}(\lambda)\right]. \tag{2·22}$$

For comparison with observational data, we should express this perturbed luminosity distance as a function of the redshift z. We first note that

$$\begin{aligned}1+\tilde{z}&=\frac{\tilde{w}(\lambda_{\rm s})}{\tilde{w}(0)}=\frac{a[\tilde{\eta}(0)]}{a[\tilde{\eta}(\lambda_{\rm s})]}\frac{(\tilde{K}_\mu\tilde{u}^\mu)_{\lambda_{\rm s}}}{(\tilde{K}_\mu\tilde{u}^\mu)_0}\\&=\frac{a[\eta(0)]}{a[\eta(\lambda_{\rm s})]}\left\{1-\left[\frac{a'}{a}\delta\eta+\frac{d}{d\lambda}\delta\eta-A-(\beta_i+v_i)\gamma^i\right]_0^{\lambda_{\rm s}}\right\},\end{aligned} \tag{2·23}$$

where $[\cdot\cdot]_0^{\lambda_{\rm s}}$ denotes the difference in the quantity inside the brackets evaluated at $\lambda=\lambda_{\rm s}$ and $\lambda=0$, and the prime ($'$) denotes differentiation with respect to η. For simplicity, we denote in the following the quantities evaluated at $\lambda=\lambda_{\rm s}$ by using the suffix "s" and those evaluated at the observer by the suffix 0. The metric perturbation variables are defined as

$$\delta g_{0\mu}=(-2A,\beta_i). \tag{2·24}$$

The variable v_i is the spatial component of the perturbed 4-velocity:

$$u^\mu=(1,\mathbf{0}),\ \delta u^\mu=(-A,v^i). \tag{2·25}$$

The time component of δu is derived from the normalization $\tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu=-1$.

Next we replace $\lambda_{\rm s}$ in Eqs. (2·22) and (2·23) by $\lambda_{\rm s}(z)+\delta\lambda_{\rm s}$, where $\lambda_{\rm s}(z)$ is the affine parameter at z in the unperturbed universe, which is defined implicitly by the relation

$$1+z=\frac{a(\eta_0)}{a(\eta_0-\lambda_{\rm s})}. \tag{2·26}$$

Keeping these relations in mind, we obtain the $d_{\rm L}$-z relation from Eq. (2·22) as

$$\tilde{d}_{\rm L}(z,\gamma^i) = d_{\rm L}(z)\left[1+\left(\frac{a'}{a}\delta\eta\right)_0 + \sqrt{-K}\coth\sqrt{-K}\lambda_{\rm s}\delta\lambda_{\rm s} - \frac{1}{2}\int_0^{\lambda_{\rm s}}\delta\theta_{\rm L}(\lambda)d\lambda\right], \tag{2·27}$$

where $d_{\rm L}$ is the luminosity distance in the unperturbed background,

$$d_{\rm L}(\eta) = a_0\frac{{\rm sh}\sqrt{-K}\lambda_{\rm s}}{\sqrt{-K}}(1+z), \tag{2·28}$$

and $\delta\lambda_{\rm s}$ is given by

$$\delta\lambda_{\rm s} = \delta\eta(\lambda_{\rm s}) - \frac{1}{(a'/a)_{\rm s}}\left[\frac{a'}{a}\delta\eta + \frac{d}{d\lambda}\delta\eta - A - (\beta_i+v_i)\gamma^i\right]_0. \tag{2·29}$$

Finally, we define the fluctuation of the luminosity distance to be the difference between the perturbed luminosity distance and the angular average of that compared at the same redshift. That is,

$$\Delta d_{\rm L} \equiv \tilde{d}_{\rm L}(z) - \langle\tilde{d}_{\rm L}(z)\rangle, \tag{2·30}$$

where

$$\langle\tilde{d}_{\rm L}(z)\rangle = \frac{1}{4\pi}\int d\Omega\,\tilde{d}_{\rm L}(z). \tag{2·31}$$

This quantity is the whole sky average of the samples of luminosity distance at the same redshift z. If there are no intrinsic errors in $d_{\rm L}$, we would obtain $\Delta d_{\rm L}$, which directly carries the information of the inhomogeneities of the universe. However, real observational data always contain intrinsic errors in $d_{\rm L}$. We will discuss this point later.

We decompose $\Delta d_{\rm L}$ into multipole components associated with the conventional spherical harmonic function $Y_{lm}(\Omega)$, as is usually done for analyzing anisotropies of a physical quantity (e.g., the anisotropies of CMB):

$$\frac{\Delta d_{\rm L}}{d_{\rm L}} = \sum_{l,m}C_{lm}(z)Y_{lm}(\Omega). \tag{2·32}$$

Then the coefficient C_{lm} represents the anisotropies in the luminosity distance. Note that we can omit the terms involving $\delta\eta_0$ since it is irrelevant for the discussion of anisotropies in the $d_{\rm L}$-z relation (it contributes only to the monopole component, which will be subtracted by $\langle\tilde{d}_{\rm L}(z)\rangle$).

In the next section we derive an explicit expression of the perturbed luminosity distance in the spatially flat model and open model.

§3. Application to cosmological models

3.1. *Flat universe*

We assume the background universe is spatially flat with or without a cosmological constant and is matter-dominated. When $K=0$, we can write

$$\frac{\Delta d_{\rm L}}{d_{\rm L}} = \frac{\delta\lambda_{\rm s}}{\lambda_{\rm s}} - \frac{1}{2}I_{\rm L}, \tag{3·1}$$

where $\delta\lambda_s$ is given by Eq. (2·29) and

$$-\frac{1}{2}I_L = \frac{1}{2\lambda_s}\int_0^{\lambda_s} d\lambda(\lambda-\lambda_s)\lambda\delta(R_{\mu\nu}K^\mu K^\nu)_\lambda. \tag{3·2}$$

When the spacetime inhomogeneities are due to density perturbations, perturbations of all the quantities can be described by functions which are scalars with respect to the spatial indices. We define such scalars $B, \mathcal{R}, H_T$ and v by

$$\beta_i \equiv B_{|i}\,, \tag{3·3}$$

$$\delta g_{ij} \equiv H_{ij} \equiv 2\mathcal{R}\gamma_{ij} + 2H_{T|ij}\,, \tag{3·4}$$

$$v_i \equiv -v_{|i}\,. \tag{3·5}$$

Here we have assumed that the velocities of the source and the observer are of cosmological origin.

According to the method of the gauge-invariant cosmological perturbation theory, the metric perturbations are most conveniently described by the variables

$$\Psi \equiv A - \frac{1}{a}\left[a({H_T}' - B)\right]'\,, \tag{3·6}$$

$$\Phi \equiv \mathcal{R} - \frac{a'}{a}({H_T}' - B)\,. \tag{3·7}$$

We also use the gauge-invariant combination

$$V \equiv v - H_T'. \tag{3·8}$$

These equations suggest that the longitudinal gauge (${H_T}' = B = 0$) is convenient for further calculations. Thus, taking the longitudinal gauge, we rewrite the expression of the fluctuation of the luminosity distance in terms of Ψ, which is the Newtonian potential in the present case (and Φ denotes the curvature fluctuation). First, the perturbation of the curvature term in (3·2) is

$$\delta(R_{\mu\nu}K^\mu K^\nu)_\lambda = 2\Psi_{|i}^{|i} + 2\Psi'' - 4\Psi'_{|i}\gamma^i. \tag{3·9}$$

Here we have used the relation $\Psi + \Phi = 0$, which follows from the fact that the pressure is negligible.

To evaluate $\delta\lambda_s$ in (2·29), we need $\delta\eta_s$ and $\frac{d}{d\lambda}\delta\eta|_0$. These can be obtained as follows. The time component of the geodesic equation gives

$$\frac{d^2}{d\lambda^2}\delta\eta = 2\frac{d}{d\lambda}\Psi + 2\Psi'. \tag{3·10}$$

Integrating this, we obtain

$$\frac{d}{d\lambda}\delta\eta\Big|_s - \frac{d}{d\lambda}\delta\eta\Big|_0 = 2(\Psi_s - \Psi_0) + 2\int_0^{\lambda_s} d\lambda\Psi'(\eta)d\lambda, \tag{3·11}$$

$$\delta\eta_s - \delta\eta_0 = \lambda_s\frac{d}{d\lambda}\delta\eta\Big|_0 + 2\int_0^{\lambda_s} d\lambda\left[\Psi(\lambda) - \Psi_0\right] + 2\int_0^{\lambda_s} d\lambda\int_0^{\lambda}\Psi'(\bar\eta)d\bar\lambda. \tag{3·12}$$

From the normalization of the null vector $\tilde{K}^\mu$, we obtain

$$\frac{d}{d\lambda}\delta\eta\Big|_{\rm s} = \Psi_{\rm s} - V_{{\rm s}|i}\gamma^i. \tag{3·13}$$

Therefore,

$$\frac{d}{d\lambda}\delta\eta\Big|_0 = -\Psi_{\rm s} + 2\Psi_0 - 2\int_0^{\lambda_{\rm s}} \Psi'(\eta)d\lambda - V_{{\rm s}|i}\gamma^i. \tag{3·14}$$

Now substituting $\delta\eta_{\rm s}$ in Eq. (3·12) and $\frac{d}{d\lambda}\delta\eta\Big|_0$ in Eq. (3·14) into Eq. (2·29), and $\delta(R_{\mu\nu}K^\mu K)$ in Eq. (3·9) into Eq. (3·2), we obtain an expression for the fluctuation of the luminosity distance in terms of the gravitational potential as

$$\begin{aligned}\frac{\Delta d_{\rm L}}{d_{\rm L}} =& \frac{1}{\lambda_{\rm s}}\int_0^{\lambda_{\rm s}} d\lambda\left[(\lambda-\lambda_{\rm s})\lambda\left(\Psi_{|i}^{|i} + \Psi'' - 2\Psi'_{|i}\gamma^i\right)\right] + \left(\frac{2}{\lambda_{\rm s}(a'/a)_{\rm s}} - 2\right)\int_0^{\lambda_{\rm s}}\Psi'(\eta)d\lambda \\ &+\frac{2}{\lambda_{\rm s}}\left[\int_0^{\lambda_{\rm s}}\Psi d\lambda + \int_0^{\lambda_{\rm s}} d\lambda\int_0^{\lambda}\Psi'(\bar\eta)d\bar\lambda\right] \\ &-\frac{1}{\lambda_{\rm s}(a'/a)_{\rm s}}(\Psi_0 - \Psi_{\rm s}) - \Psi_{\rm s} - \frac{1}{\lambda_{\rm s}(a'/a)_{\rm s}}(V_{0|i}\gamma^i - V_{{\rm s}|i}\gamma^i) - V_{{\rm s}|i}\gamma^i,\end{aligned} \tag{3·15}$$

where $\lambda_{\rm s}$ is defined by Eq. (2·26) and the velocity V is given in terms of Ψ through the relation

$$V = \frac{2}{3}\left(\frac{a\Psi}{H_0^2\Omega_0 a_0}\right)', \tag{3·16}$$

where Ω_0 is the density parameter and H_0 is the Hubble constant $H_0 = 100$ $h$km/s/Mpc.

For computation of C_{lm}, it is convenient to expand Ψ using $Y_{lm}(\Omega)$ as

$$\Psi(\boldsymbol{x}) = \frac{2}{\pi}\sum_{lm} Y_{lm}(\Omega)\int k^2 dk \Psi_{lm}(k) j_l(kr). \tag{3·17}$$

Then, with the replacements

$$\Psi(x)\Big|_{|i}^{|i} \to j_l(kr)(-k^2)\Psi_{lm}(k)\,, \tag{3·18}$$

$$\gamma^i\Psi_{|i}(x) \to \frac{d}{dr}j_l(kr)\Psi_{lm}(k)\,, \tag{3·19}$$

we immediately obtain the expression for C_{lm}:

$$C_{lm} = \int d\Omega\frac{\Delta d_{\rm L}}{d_{\rm L}}Y^*_{lm}(\Omega) \equiv \frac{2}{\pi}\int k^2 dk Q_{lm}, \tag{3·20}$$

$$Q_{lm} \equiv G_{lm} + P_{lm} + V_{lm}, \tag{3·21}$$

$$\begin{aligned}G_{lm} =& \frac{1}{\lambda_{\rm s}}\int_0^{\lambda_{\rm s}} d\lambda(\lambda-\lambda_{\rm s})\lambda \\ &\times\left[(-k^2\Psi_{lm}(k,\eta) + \Psi''_{lm}(k,\eta))j_l(kr) - 2j_l'(kr)\Psi'_{lm}(k,\eta)\right],\end{aligned} \tag{3·22}$$

$$
\begin{aligned}
P_{lm} =& \left(\frac{2}{\lambda_{\rm s}(a'/a)_{\rm s}} - 2\right) \int_0^{\lambda_{\rm s}} d\lambda j_l(kr) \Psi'_{lm}(k,\eta) \\
&+ \frac{2}{\lambda_{\rm s}} \int_0^{\lambda_{\rm s}} d\lambda j_l(kr) \Psi_{lm}(k,\eta) + \frac{2}{\lambda_{\rm s}} \int_0^{\lambda_{\rm s}} d\lambda \int_0^{\lambda} d\bar{\lambda} j_l(kr) \Psi'_{lm}(k,\bar{\eta}) \\
&+ \left(\frac{1}{\lambda_{\rm s}(a'/a)_{\rm s}} - 1\right) j_l(kr_{\rm s}) \Psi_{lm}(k,\eta_{\rm s}), \qquad (3{\cdot}23)
\end{aligned}
$$

$$
\begin{aligned}
V_{lm} =& \frac{2}{3\Omega_0} \frac{1}{\lambda_{\rm s}(a'/a)_{\rm s}} \Big(j'_l(0) \left[\Psi_{lm}(k,\eta_0) + \Psi'_{lm}(k,\eta_0)\right] \\
&- j'_l(kr_{\rm s}) \left[b_{\rm s} \Psi_{lm}(k,\eta_{\rm s}) + (\frac{a_{\rm s}}{a_0}) \Psi'_{lm}(k,\eta_{\rm s}) \right] \Big) \\
&- \frac{2}{3\Omega_0} j'_l(kr_{\rm s}) \left[b_{\rm s} \Psi_{lm}(k,\eta_{\rm s}) + (\frac{a_{\rm s}}{a_0}) \Psi'_{lm}(k,\eta_{\rm s}) \right], \qquad (3{\cdot}24)
\end{aligned}
$$

where $r = \eta_0 - \eta$, $j'_l(0)$ is equal to $k/3$ for $l = 1$ and vanishes for $l \geq 2$, and $b_{\rm s} \equiv \sqrt{(1-\Omega_0)(a_{\rm s}/a_0)^4 + (a_{\rm s}/a_0)\Omega_0}$. Here we have set $H_0 = 1$ for simplicity.

3.2. *Open universe*

We consider the case in which the background is an open universe without a cosmological constant.

Equation (2·27) leads to

$$
\frac{\Delta d_{\rm L}}{d_{\rm L}} = \sqrt{-K} \delta\lambda_{\rm s} \coth \sqrt{-K} \lambda_{\rm s} - \frac{1}{2} I_{\rm L}, \qquad (3{\cdot}25)
$$

where $\delta\lambda_{\rm s}$ is given by Eq. (2·29) and

$$
\begin{aligned}
-\frac{1}{2} I_{\rm L} =& \frac{1}{2\sqrt{-K}} \int_0^{\lambda_{\rm s}} d\lambda [\coth \sqrt{-K} \lambda_{\rm s} + \coth \sqrt{-K} (\lambda - \lambda_{\rm s})] \\
&\times {\rm sh}^2 \sqrt{-K} (\lambda - \lambda_{\rm s}) \left[(\delta R_{\mu\nu} K^\mu K^\nu)_\lambda + 4K \frac{d}{d\lambda} \delta\chi \right]. \qquad (3{\cdot}26)
\end{aligned}
$$

The last term proportional to K is specific to an open model. The coordinate χ is defined in Appendix A, and the term including $\delta\chi$ comes from the fluctuation in the 4-momentum vector of the photon. To evaluate this term, we need the radial component of the geodesic equation, which reduces to

$$
\frac{d^2}{d\lambda^2} \delta\chi = -\Psi' \frac{d\chi}{d\lambda}, \qquad (3{\cdot}27)
$$

in the longitudinal gauge. Following the discussion in §3.1, we obtain an expression for the fluctuation of the luminosity distance in an open universe corresponding to Eq. (3·15)

$$
\begin{aligned}
\frac{\Delta d_{\rm L}}{d_{\rm L}} =& \frac{\sqrt{-K}}{{\rm sh}(\sqrt{-K}\lambda_{\rm s})} \int_0^{\lambda_{\rm s}} d\lambda \frac{{\rm sh}\sqrt{-K}(\lambda - \lambda_{\rm s})}{\sqrt{-K}} \frac{{\rm sh}\sqrt{-K}(\lambda - \lambda_{\rm s}) + {\rm sh}\sqrt{-K}\lambda_{\rm s}}{\sqrt{-K}} \\
&\times \left(\Psi_{|i}^{|i} + \Psi'' - 2\Psi'_{|i}\gamma^i - 2K \int_0^{\lambda} \Psi'(\bar{\eta}) d\bar{\lambda} \right)
\end{aligned}
$$

$$+\frac{2\sqrt{-K}}{\mathrm{sh}\sqrt{-K}\lambda_{\mathrm{s}}}\left[\left(\frac{1}{(a'/a)_{\mathrm{s}}}-\lambda_{\mathrm{s}}\right)\int_0^{\lambda_{\mathrm{s}}}\Psi'(\eta)d\lambda+\int_0^{\lambda_{\mathrm{s}}}\Psi(\eta)d\lambda\right.$$
$$\left.+\int_0^{\lambda_{\mathrm{s}}}d\lambda\int_0^{\lambda_{\mathrm{s}}}\Psi'(\bar{\eta})d\bar{\lambda}\right]$$
$$-(\Psi_0-\Psi_{\mathrm{s}})\left(\frac{\sqrt{-K}}{(a'/a)_{\mathrm{s}}\mathrm{sh}(\sqrt{-K}\lambda_{\mathrm{s}})}\right)-\frac{\Psi_{\mathrm{s}}\sqrt{-K}\lambda_{\mathrm{s}}}{\mathrm{sh}(\sqrt{-K}\lambda_{\mathrm{s}})}$$
$$-(V_{0|i}\gamma^i-V_{\mathrm{s}|i}\gamma^i)\frac{\sqrt{-K}}{(a'/a)_{\mathrm{s}}\mathrm{sh}(\sqrt{-K}\lambda_{\mathrm{s}})}-\frac{V_{\mathrm{s}|i}\gamma^i\sqrt{-K}\lambda_{\mathrm{s}}}{\mathrm{sh}(\sqrt{-K}\lambda_{\mathrm{s}})}\ . \tag{3·28}$$

Utilizing the expansion defined in Appendix A, we can obtain the multipole components as in the spatially flat case. (We do not display these here to avoid wasting of the space and perhaps trying the reader's patience).

3.3. *Spectrum of fluctuations*

The fluctuations in our universe are often presented in terms of a power spectrum of the density perturbation Δ, which is related to the gravitational potential Ψ through the Poisson equation

$$\Psi^{|i}{}_{|i}=\frac{3}{2}\frac{a_0}{a}\Omega_0H_0^2\Delta. \tag{3·29}$$

The present matter power spectrum is given by the transfer function $T(k)$ for the CDM model[10),14)] as

$$|\Delta_k(\eta_0)|^2=A_{\mathrm{N}}kT^2(k)\,, \tag{3·30}$$

$$T(k)=\frac{\ln(1+2.34q)}{2.34q}[1+3.89q+(16.1q)^2+(5.46q)^3+(6.71q)^4]^{-1/4}, \tag{3·31}$$

with

$$q=\frac{k(T_0/2.7\mathrm{K})^2}{\Omega_0h^2\exp(-\Omega_{\mathrm{B}}-\sqrt{h/0.5}\Omega_{\mathrm{B}}/\Omega_0)\mathrm{Mpc}^{-1}}\,, \tag{3·32}$$

where Ω_{B} is the baryon density parameter (we adopt $\Omega_{\mathrm{B}}=0.015h^{-2}$, which is predicted by the Big Bang nucleosynthesis theory), and T_0 is the present temperature of the CMB. We assumed the Harrison-Zeldovich spectrum.

With these we can calculate the expectation value of $|C_{lm}|^2$ as

$$C_l\equiv\frac{1}{2l+1}\sum_{m=-l}^{l}\langle|C_{lm}|^2\rangle \tag{3·33}$$
$$=16\pi^2A_{\mathrm{N}}\int\frac{dk}{k}\left(\frac{3\Omega_0H_0^2}{2a_0}\right)^2\frac{1}{2l+1}\sum_{m=-l}^{l}|D_{lm}(k,z)|^2T^2(k), \tag{3·34}$$

where we have defined $D_{lm}(k,z)$ by $Q_{lm}=D_{lm}(k,z)\Psi_{lm}(k,\eta_0)$. Then the anisotropies in the luminosity distance can be simply written as

$$\left\langle\left|\frac{\Delta d_{\mathrm{L}}}{d_{\mathrm{L}}}\right|^2\right\rangle=\sum_l\frac{2l+1}{4\pi}C_l(z). \tag{3·35}$$

§4. Results and discussion

We numerically calculated the multipole components of the anisotropies in the luminosity distance C_l according to the expression derived in the preceding sections. The normalization factor A_N in Eqs. (3·30) and (3·34) is determined from the observational data of the CMB anisotropies.[10)-12)] The parameters we employed are

$$\text{(a) } \Omega_0 = 1.0,\ \Omega_\Lambda = 0.0,\ h = 0.5, 0.8,\ A_{\rm N}h^4 = 2.6 \times 10^3\ {\rm Mpc}^4,$$
$$\text{(b) } \Omega_0 = 0.3,\ \Omega_\Lambda = 0.7,\ h = 0.5, 0.8,\ A_{\rm N}h^4 = 1.5 \times 10^4\ {\rm Mpc}^4,$$
$$\text{(c) } \Omega_0 = 0.3,\ \Omega_\Lambda = 0.0,\ h = 0.8,\quad A_{\rm N}h^4 = 3.5 \times 10^3\ {\rm Mpc}^4,$$

where $\Omega_\Lambda = \Lambda/3H_0^2$.

Let us first explain the physical processes which produce the fluctuations in the luminosity distance. The expressions in Eqs. (3·1) and (3·25) have two terms. The first term represents the fluctuation of the redshift z. It comes from two physical processes: one represents the peculiar velocity [the terms including V in Eqs. (3·15) and (3·28)], and the other represents the fluctuations in the gravitational potential, which correspond to the Sachs-Wolfe effect in the CMB anisotropies. We denote their contributions to C_l as V_l and P_l, respectively [namely, P_l and V_l are defined from P_{lm} in Eq. (3·23) and V_{lm} in Eq. (3·24) in the same way as in Eq. (3·33)]. The second term in Eqs. (3·1) and (3·25) can be written in the form of the integration of the fluctuations along the path of the ray. It describes the situation that the photons are reflected by the fluctuations of the gravitational potential, that is, magnification or demagnification of their luminosity by the gravitational lensing. We denote this effect as G_l [which comes from G_{lm} in Eq. (3·22)].

Each effect (G_l, V_l, P_l) is plotted separately in the figures in order to understand how each behaves when we change the cosmological parameters and the observing redshift. We have not plotted the total C_l, which we can read off the dominant component.

4.1. *Comparison with recent work*

We plot G_l,V_l and P_l in the Einstein-de Sitter model for sources of various redshifts in Fig. 1. We give results for only from $l = 1$ to 50 in this paper, since our main interest is in the large-scale correlation of the fluctuations, though in principle we can calculate any large l.

Our result for G_l can be compared with the results of other works in which the amplification of the luminosity by the gravitational lensing effect was calculated using different methods.[15)-18)] We can calculate essentially the same quantity by summing up the contributions from small scales where the density fluctuation becomes non-linear, i.e., by summing G_l up to sufficiently large l. In order to confirm consistency with previous works, we have calculated the rms flux amplification $\sigma_A(= 2|\Delta d_{\rm L}/d_{\rm L}|)$ presented by Frieman[16)] by summing G_l up to $l = 3000$ in the flat model with $\Omega_0 = 1.0$ and $h = 0.5$. We have obtained almost the same value using the power spectrum he adopted. Moreover, we find from Fig. 1 that G_l becomes smaller for lower redshifts, which is also claimed in the previous work. Therefore, our results for the gravitational lensing effect exhibit good agreement with the previous work.

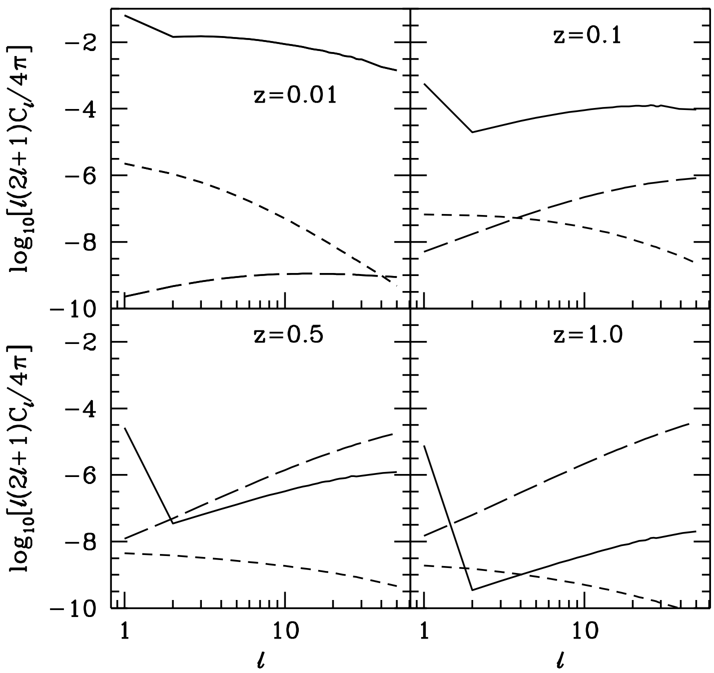

Fig. 1. Multipole moments of anisotropies in $d_{\rm L}$. We have plotted the velocity term V_l (solid line), the gravitational lensing term G_l(long dashed), and the gravitational redshift term P_l(short dashed) for the Einstein de-Sitter model with $(\Omega_0, h) = (1.0, 0.5)$ for sources at different redshifts. One can see that the velocity effect V_l dominates for lower redshifts, while the gravitational lensing G_l dominates at higher redshifts $z \geq 0.5$.

When we discuss the $d_{\rm L}$-z relation, however, we also have to take into account the effect of fluctuations of the redshift caused by the peculiar velocity (V_l) and the Sachs-Wolfe effect(P_l) in addition to the amplification effect; one of the advantages of our formulation is that all these effects are naturally included. We find that the Sachs-Wolfe effect is practically negligible for most cases, but the peculiar velocity gives significant contributions to the fluctuations in $d_{\rm L}$ for low redshift sources (see Fig. 1). At higher redshifts around $z \gtrsim 0.5$, the gravitational lensing effect dominates. One may notice that the dipole moment of the peculiar velocity effect, V_1, is particularly large. This corredponds to the fact that the peculiar velocity of the observer contributes to the dipole moment.

We will discuss the implications of these anisotropies in the $d_{\rm L}$-z relation in the later sections after studying the parameter-dependence of C_l in detail in the next section.

4.2. *Cosmological parameter dependence*

First, we study the z-dependence of each effect in Fig. 1. The gravitational lensing effect G_l increases slightly as z increases since the photons are affected by

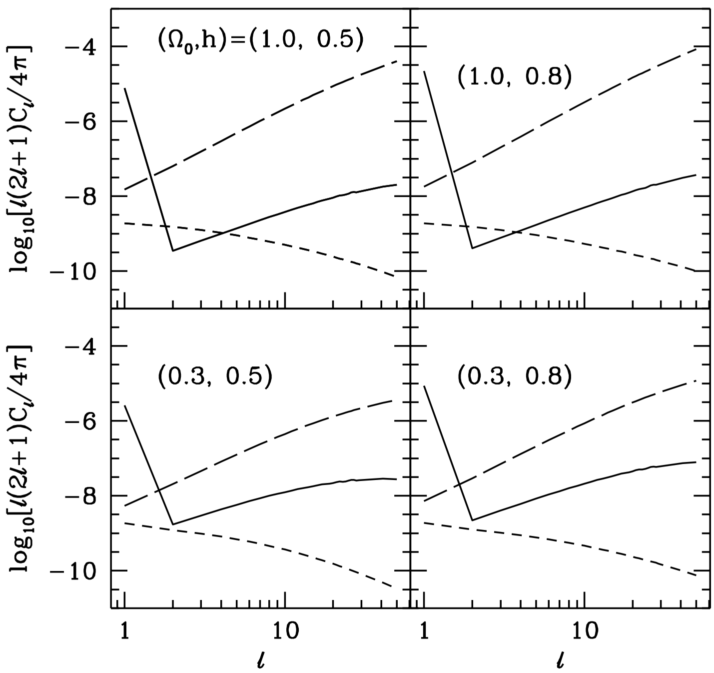

Fig. 2. Comparison of the flat model with different (Ω_0, h) for the redshift of the source $z = 1.0$. The lines have the same identifications as in Fig. 1.

the curvature fluctuations for a longer time. On the other hand, the peculiar velocity effect V_l decreases with increasing z. This corresponds to the fact that the peculiar velocity becomes fairly small compared to the Hubble expansion for high z. In total, the anisotropies in $d_{\rm L}$ become smaller for higher z.

Next, let us investigate the cosmological parameter dependence (Fig. 2). If one fixes Ω_0 and varies H_0, larger H_0 yields slightly larger G_l. This comes from the change in the shape of the transfer function; the transfer function we employ here becomes larger for larger H_0 on small scales when one fixes the large-scale amplitude. This pushes up G_l in the larger H_0 model. Now we examine the dependence on Ω_0 with H_0 fixed. We can see that G_l becomes larger for a high density universe. There are two competing factors. The spectrum of $\Psi(\eta_0)$ in the large Ω_0 model has larger values on all scales: owing to the factor ${\Omega_0}^2$ in Eq. (3·34), the large-scale amplitude is larger for model (a) by a factor of 2. In addition, the transfer function begins to decrease at a larger scale in model (b). Contrastingly, the range of the integration in G_l is longer for the smaller Ω_0 model, which would push up G_l. It turns out, however, this cannot complement the low amplitude of the fluctuations in Ψ. Thus, G_l becomes larger in the larger Ω_0 model.

With regard to V_l, the situation is somewhat complicated. For $l = 1$, the term including $j_l'(0)$ in Eq. (3·24) gives the dominant contribution to V_1. This term

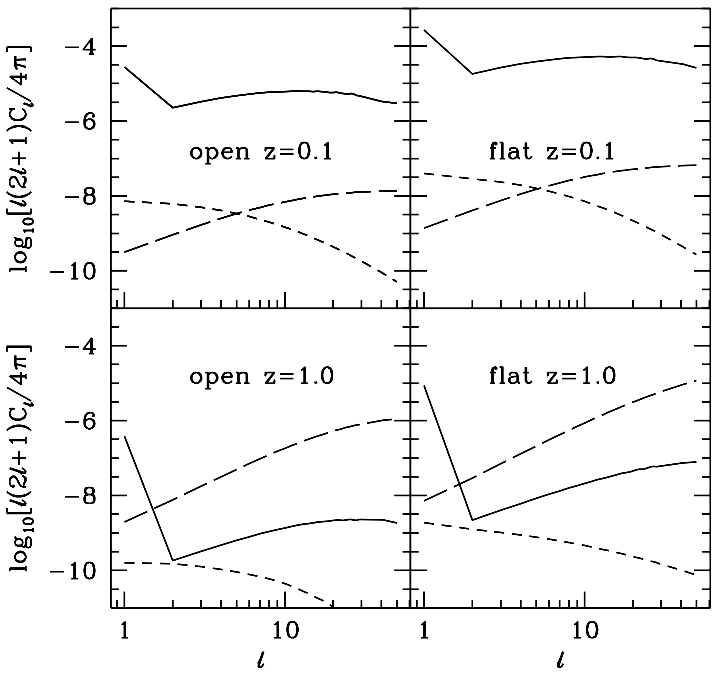

Fig. 3. Comparison between the open and flat models with $(\Omega_0, h) = (0.3, 0.8)$ for the redshift of the source $z = 0.1$ and $z = 1.0$. The lines have thes same identifications as in Fig. 1. One can see that the spectrum is almost the same, with a lower amplitude for the open model.

represents the peculiar motion of the observer. Thus, the amplitude of this term is determined by density fluctuations on rather small scales. Since the power spectrum we employ here gives a larger value on small scales for the larger Ω_0 model, V_1 for the $\Omega_0 = 1.0$ model is slightly larger than the $\Omega_0 = 0.3$ model in Fig. 2.

For $l \geq 2$, on the other hand, the second term in Eq. (3·24) is the dominant term since $j_l'(0) = 0$. The factor $(\Psi(\eta_s){\Omega_0}^{-1}/\lambda_s)^2$ is about ten times larger for model (b) at $z = 1.0$ on large scales, and it turns out that the contributions on small scales in the model (a) cannot turn over the low amplitude on larger scales. Thus V_l becomes larger when $l \geq 2$ for the smaller Ω_0 model in Fig. 2.

The open universe gives smaller values than in the flat model, reflecting the small amplitude of Ψ (Fig. 3). One can also see that the shape of the spectrum of C_l is almost the same as in the flat model, which implies that the effect of the spatial curvature is not significant for $z < 1.0$.

4.3. *Smoothing and bulk velocity*

We have seen that the dipole term is dominated by the velocity effect. Then, we can relate this dipole moment of the fluctuation of the luminosity distance with the bulk motion of our local group of galaxies relative to the CMB rest frame, by smoothing out the smaller scale fluctuations. For small z, the redshift is expressed

Table I. Bulk velocity v obtained from the dipole moment of $\Delta d_{\rm L}$ for $z = 0.02, 0.01$.

Ω_0	1.0	1.0	0.3	0.3	0.3
Ω_Λ	0.0	0.0	0.7	0.7	0.0
h	0.5	0.8	0.5	0.8	0.8
v (km s^{-1})	220,230	270,310	120,90	190,170	67,60

by the velocity as $z = v/c$. Then the peculiar velocity is given by

$$V_{\rm pec} \equiv \Delta v = c\Delta z, \qquad (4\cdot1)$$

where c is the speed of light. Since $z = H_0 d_{\rm L}/c$ for small z, the expectation value of the square of the peculiar velocity can be expressed in terms of the fluctuation of the luminosity distance as

$$\langle V_{\rm pec}^2 \rangle = 3(H_0 d_{\rm L})^2 \left\langle \left| \frac{\Delta d_{\rm L}}{d_{\rm L}} \right|^2 \right\rangle. \qquad (4\cdot2)$$

The numerical factor 3 comes from the fact that the fluctuation of the luminosity distance corresponds only to the radial component of the peculiar velocity. If we smooth out fluctuations whose scale is smaller than $d_{\rm L}$, the fluctuations in $d_{\rm L}$ reflect the velocity fluctuations on scales larger than $d_{\rm L}$. Thus we obtain the bulk velocity of the sphere of radius $d_{\rm L}$ relative to the CMB rest frame. We take a window function of the form [19] $W(kR) = [3j_1(kR)/kR]\exp(-k^2 r_{\rm s}^2/2)$ with $r_{\rm s} = 12\,h^{-1}$Mpc. The numerical results are given in Table I. These results are consistent with other calculations (see, e.g., Sugiyama [10]).

4.4. *Implications for observation*

The anisotropies in the $d_{\rm L}$-z relation provide information regarding the inhomogeneities of the universe. If there are no errors in estimating the luminosity distance, the analysis presented in this paper is available: gather samples of $d_{\rm L}$ in different directions at the same redshift, transform the data into C_l, and compare with theoretical prediction. However, the observations of the luminosities of distant sources such as Type Ia SNe contain intrinsic errors of various origins. Then, the effect of the LSS may be regarded as an extra contribution to the errors in the luminosity distance. Therefore, here we discuss how these anisotropies in $d_{\rm L}$ affect the determination of the cosmological parameters via the redshifts-magnitude relation. Recently, attempts to measure the deceleration parameter q_0 have been made using distant Type Ia supernovae. [8],[9] The authors of those works argue that Ia SNe act as good standard candles with an effective dispersion of absolute magnitude $\sigma_M = 0.1 \sim 0.2$ mag, which makes a highly accurate measurement of q_0 possible. There is a claim, however, that the large-scale structure of the universe prevents the accurate determination of q_0 from these standard candles. [16],[17] They estimate the expected error caused by the gravitational lensing amplification, showing that the effect is negligible for most cases, especially at low redshifts. Here we study the relation between the uncertainty of q_0 and the fluctuations of $d_{\rm L}$ in our formalism including all the effects. The $d_{\rm L}$-z relation for small z reduces to

$$d_{\rm L} = \frac{c}{H_0}\left(z + \frac{1}{2}(1-q_0)z^2 + \cdots\right). \tag{4·3}$$

Then, we obtain

$$|\delta q_0| \simeq \frac{2}{z}\left|\frac{\Delta d_{\rm L}}{d_{\rm L}}\right|. \tag{4·4}$$

This means that for low z the fluctuations of $d_{\rm L}$ are amplified by a factor of $2z^{-1}$, leading to large uncertainties of q_0. Our calculations give $\Delta d_{\rm L}/d_{\rm L} \sim 10^{-1}$ for $z = 0.01$, and 10^{-2} for $z = 0.1$, leading to $\delta q_0 \sim 1$ and 0.1, respectively. This indicates that it is impossible in principle to constrain the parameter with practical precision by observing these low z samples. This fact is, however, quite trivial when one is reminded that the peculiar velocity obscures the redshift for sources at smaller redshifts.

On the other hand, for $z = 0.5$ the uncertainty in the value of q_0 is at most ~ 0.05, even considering the non-linear effect. Considering the fact that σ_M yields $|\delta q_0| \simeq \frac{0.9}{z}|\sigma_M|$, the effect of LSS on sources at high redshifts around 0.5 is small compared with the intrinsic dispersion in the absolute magnitude $\sigma_M = 0.1 \sim 0.2$ of standard candles such as type Ia SNe.

§5. Summary

We investigated the effects of large-scale structures on the luminosity distance by employing a linear perturbation method. We found that there are three effects: the peculiar velocity, gravitational lensing, and the Sachs-Wolfe effect. The dependence of these effects on the cosmological parameters has been clarified. We showed that the dominant contribution is the velocity effect for low z and the gravitational lensing effect for high z. We can relate the dipole moment of the fluctuation for low z with the bulk velocity of the local group by smoothing the small-scale fluctuations.

These anisotropies in $d_{\rm L}$ also cause uncertainties in determining the cosmological parameters via the magnitude-redshift relation. We showed that the effect is small for $z \gtrsim 0.5$, compared with the intrinsic dispersion in the absolute magnitude of currently known standard candles such as type Ia SNe. On the other hand, the uncertainty amounts to $\delta q = 0.1$ for lower redshifts around 0.1 due to the effect of the peculiar velocity.

Appendix A
—— *Spherical Harmonics Expansion* ——

In this appendix, we give an eigenfunction of the Laplacian operator in the case $K < 0$ following Wilson.[20] The flat case follows by taking the limit $K \to 0$. The spatial components of the metric are taken to be

$$\gamma_{ij}dx^i dx^j = -K^{-1}[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)], \tag{A·1}$$

where θ and ϕ are the usual angular coordinates, and χ is a dimensionless radial coordinate. The distance from the origin is given by $r = \chi/(-K)^{1/2}$. The Laplacian of a function F is given by

$$\Delta F = -K \sinh^{-2}\chi \left[\frac{\partial}{\partial\chi}\left(\sinh^2\chi\frac{\partial F}{\partial\chi}\right) + \sin^{-1}\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \sin^{-2}\theta\frac{\partial^2 F}{\partial\phi^2}\right]. \tag{A·2}$$

We are interested in the solutions of the equation $\Delta F = -k^2 F$. It is well known that the spectrum of this operator is given for $k^2 \geq -K$. If we separate variables, the angular part of the solution is just a spherical harmonic. The radial part associated with $Y_{lm}(\Omega)$ is defined by

$$X_k^l(\chi) = \left(\frac{\pi}{2\sinh\chi}\right)^{1/2} (\nu^2+1)^{l/2} P_{i\nu-1/2}^{-(l+1/2)}(\cosh\chi) \tag{A·3}$$

$$= (-1)^{l+1} N_l^{-1} (\nu^2+1)^{l/2} \sinh^l\chi \frac{d^{l+1}(\cos\nu\chi)}{d(\cosh\chi)^{l+1}}, \tag{A·4}$$

with $N_l^{-1}(\nu) = \nu^2(\nu^2+1)\cdots(\nu^2+l^2)$, and ν is defined by $k^2 =: -K(\nu^2+1)$. In the limit $K \to 0$ with k and r fixed, this reduces to a spherical Bessel function $j_l(kr)$. With these, we can expand Ψ as

$$\Psi(\boldsymbol{x}) \equiv \frac{2}{\pi}\sum_{lm} Y_{lm}(\Omega) \int [k^2 dk] \Psi_{lm}(k) X_k^l(\chi), \tag{A·5}$$

$$\Psi_{lm}(k) = \int [d^3x] \Psi(\boldsymbol{x}) X_k^l(\chi) Y_{lm}^*(\Omega), \tag{A·6}$$

where

$$[k^2 dk] = d\nu N_l(\nu)(\nu^2+1)^{-l}, \tag{A·7}$$

$$[d^3x] = \sinh^2\chi d\chi \sin\theta d\theta d\phi\,. \tag{A·8}$$

Appendix B
—— Power Spectrum ——

When the fluctuation of the luminosity distance can be written in the form

$$\frac{\Delta d_{\rm L}}{d_{\rm L}} = \frac{2}{\pi}\sum_{lm} Y_{lm}(\Omega) \int [k^2 dk] D_{lm}(k,z) \Psi_{lm}(k,\eta_0), \tag{B·1}$$

we can immediately read the multipole component C_{lm} as

$$C_{lm} = \frac{2}{\pi}\int [k^2 dk] D_{lm}(k,z) \Psi_{lm}(k,\eta_0). \tag{B·2}$$

Then the averaged quantity is given by

$$\left\langle \left|\frac{\Delta d_{\rm L}}{d_{\rm L}}\right| \right\rangle^2 = \frac{1}{4\pi}\sum_{lm}\langle |C_{lm}|^2\rangle \tag{B·3}$$

$$= \frac{1}{\pi^3}\sum_{lm}\int [k^2 dk] |D_{lm}(k,z)|^2 |\Psi_{lm}(k,\eta_0)|^2, \tag{B·4}$$

where we have defined the power spectrum of Ψ by

$$\langle \Psi_{lm}(k,\eta_0)\Psi^*_{lm}(\bar{k},\eta_0)\rangle \equiv |\Psi_{lm}(k,\eta_0)|^2\delta_D(\bar{k}-k)/\bar{k}^2. \qquad \text{(B·5)}$$

In order to relate $|\Psi_{lm}(k,\eta_0)|^2$ with the density power spectrum, we proceed as follows. In the spatially flat case, we can give the Fourier transformation of Ψ:

$$\Psi(\boldsymbol{x}) = \int d^3\bar{k}\Psi(\bar{\boldsymbol{k}})\exp(i\bar{\boldsymbol{k}}\cdot\boldsymbol{x}) \qquad \text{(B·6)}$$

$$= \int d^3\bar{k}\Psi(\bar{\boldsymbol{k}})4\pi\sum_{\bar{l}\bar{m}} i^{\bar{l}} j_{\bar{l}}(\bar{k}r)Y_{\bar{l}\bar{m}}(\Omega)Y^*_{\bar{l}\bar{m}}(\bar{\Omega}_{\bar{k}}). \qquad \text{(B·7)}$$

Inserting this into Eq. (A·6), we obtain the relation between $\Psi(\boldsymbol{k})$ and $\Psi_{lm}(k)$,

$$\Psi_{lm}(k) = 2\pi^2 i^l \int d\Omega_k \Psi(\boldsymbol{k})Y^*_{lm}(\Omega_k). \qquad \text{(B·8)}$$

Note that inserting Eq. (B·8) into the expression of C_{lm} in §3, we recover the factor $4\pi i^l$ in the expression derived by Sasaki.[7)]

Then the relation between the power spectrums is

$$\langle \Psi_{lm}(k,\eta_0)\Psi^*_{lm}(\bar{k},\eta_0)\rangle = (2\pi^2)^2\int d\Omega_k \int d\bar{\Omega}_k \langle \Psi(\boldsymbol{k},\eta_0)\Psi^*(\bar{\boldsymbol{k}},\eta_0)\rangle Y_{lm}(\Omega_k)Y_{lm}(\bar{\Omega}_k)$$

$$= (2\pi^2)^2\delta_D(k-\bar{k})/\bar{k}^2|\Psi(k,\eta_0)|^2, \qquad \text{(B·9)}$$

where we have used

$$\langle \Psi(\boldsymbol{k})\Psi(\bar{\boldsymbol{k}})\rangle = |\Psi(k)|^2\delta(\boldsymbol{k}-\bar{\boldsymbol{k}}). \qquad \text{(B·10)}$$

Therefore, from the definition (B·5) and the Poisson equation, we obtain

$$|\Psi_{lm}(k,\eta_0)|^2 = (2\pi^2)^2|\Psi(k,\eta_0)|^2 = 4\pi^4\left(\frac{3\Omega_0 {H_0}^2}{2a_0k^2}\right)^2 A_{\rm N}kT^2. \qquad \text{(B·11)}$$

This expression leads to Eq. (3·35).

Acknowledgements

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