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An n Monopole Solution with $4n-1$ Degrees of Freedom

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Abstract. An exact static monopole solution, possessing n units of magnetic charge and $(4n-1)$ degrees of freedom, is constructed, generalising the recent work of Ward on two monopole solutions. The equations solved are those of an $SU(2)$ gauge theory with adjoint representation Higgs field in the (BPS) limit of vanishing Higgs potential. The number of degrees of freedom is maximal for self-dual solutions. The construction is described in a deductive way, within the framework of the Atiyah-Ward formalism for self-dual gauge fields.

1. Introduction

Gauge field theories in which the symmetry group G is spontaneously broken, by the agency of a Higgs field in the adjoint representation, possess classical solutions with the natural interpretation of magnetic monopoles [1, 2]. (For a review see e.g. [3]). The magnetic charge of these solutions is quantised in that, for topological reasons, it has to be an integral multiple of 4π, in suitable units. We shall call a solution with magnetic charge $4\pi n$ an n monopole solution. In the limit in which the potential describing the self-interaction of the Higgs field vanishes, the Bogomol'nyi-Prasad-Sommerfield (BPS.) limit [4, 5], it is possible to produce some exact static finite energy solutions of the equations of motion, in terms of elementary functions. The first example, a charge one $SU(2)$ monopole, was spherically symmetric [5]. This has been generalised to obtain spherically symmetric solutions for larger gauge groups [6]. Recently, following a paper in which Ward constructed an axis symmetric two monopole solution [7], axis symmetric solutions of arbitrary charge have been proposed [8]. Further Ward has now produced a reasonably general solution of charge two [9]. In this paper we extend Ward's result to higher charge, analysing the construction in a way which we hope makes it appear rather natural.

In the BPS. limit the equations of motion are implied by the Bogomol'nyi equations

$$\mathbf{B}_i = \pm D_i \Phi, \tag{1.1}$$

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where the $SU(2)$ generalised magnetic field $\mathbf{B}_i = -\frac{1}{2}\varepsilon_{ijk}\mathbf{F}_{jk}$, and

$$\mathbf{F}_{ij} = \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + [\mathbf{A}_i, \mathbf{A}_j], \tag{1.2}$$

$$D_i \boldsymbol{\Phi} = \partial_i \boldsymbol{\Phi} + [\mathbf{A}_i, \boldsymbol{\Phi}], \tag{1.3}$$

provided that the fields are static,

$$\partial_0 \mathbf{A}_i = \partial_0 \boldsymbol{\Phi} = 0. \tag{1.4}$$

(The gauge potentials, $\mathbf{A}_i = \frac{1}{2} i A_i^a \sigma^a$, and the Higgs field, $\boldsymbol{\Phi} = \frac{1}{2} i \Phi^a \sigma^a$, are written as antihermitian matrices: σ^a, $a = 1, 2, 3$ denote the Pauli matrices.) It is the first order Eq. (1.1) to which static solutions have been found, subject to the boundary conditions that

$$\Phi^2 = -2\,\mathrm{Tr}(\boldsymbol{\Phi}^2) \to 1 \tag{1.5}$$

as the Euclidean spatial distance from the origin, $r \to \infty$, and that the energy be finite. Condition (1.5) reflects the vestigial influence of the Higgs potential in the BPS. limit. In this limit the energy equals the modulus of the magnetic charge. (This charge is positive if Eq. (1.1) holds with a plus sign.) We shall produce solutions for whcih

$$\Phi^2 = 1 - 2l/r + O(r^{-2}) \quad \text{as } r \to \infty. \tag{1.6}$$

where l is a positive integer. The magnetic charge is given by the flux out of a large sphere of $B_i = \Phi^a B_i^a = \frac{1}{2}\partial_i(\Phi^2)$, using Eq. (1.1). Hence, if Eq. (1.6) holds, the magnetic charge is $4\pi l$.

Ward's approach to solving Eq. (1.1) has been to exploit its relation to the self-duality equations for a pure $SU(2)$ gauge theory in four dimensional Euclidean space,

$$\mathbf{F}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\mathbf{F}_{\gamma\delta}. \tag{1.7}$$

If the gauge potentials are independent of Euclidean time, i.e. $\partial_0 \mathbf{A}_\alpha = 0$, Eq. (1.7) becomes

$$\partial_i \mathbf{A}_0 + [\mathbf{A}_i, \mathbf{A}_0] = -\frac{1}{2}\varepsilon_{ijk}\mathbf{F}_{jk}. \tag{1.8}$$

Thus Eqs. (1.1) and (1.8) are equivalent if $\mathbf{A}_0$ is identified with $\boldsymbol{\Phi}$. Much is known about the self-duality equations. Four years ago Ward [10] established a correspondence between self-dual Euclidean gauge fields and certain analytic vector bundles. This was discussed further by Atiyah and Ward [11] who showed that this correspondence led to a series of ansätze A_l, $l = 1, 2, \ldots$. These ansätze were explicitly constructed in [2]. However they have not proved a very fruitful approach for constructing the finite action self-dual gauge fields on four dimensional Euclidean space $\mathbb{R}^4$, which are called instantons. Atiyah, Drinfeld, Hitchin and Manin produced an algebraic construction of these [13, 14], again based on Ward's correspondence. The ansatz A_1 had been previously discovered and used by 't Hooft and others [15] to produce instanton solutions. Manton [16] discussed its application to the BPS. monopole equations, but showed it yielded only the spherically symmetric one monopole solution [5]. The recent multi-monopole solutions have used the ansatz A_l and have charge $4\pi l$, and this remains true for our solutions. Weinberg [17] has shown that any such solution belongs to a $(4l-1)$-parameter family of solutions; our solutions have this maximal number of degrees of freedom though we have certainly not proved that all l monopole solutions can be obtained from A_l in the way we outline.

The paper is organised as follows. In the next section we summarise what we need to know about the Atiyah-Ward ansätze and their construction. In Sect. 3 we discuss the boundary conditions and their realisation in terms of the transition function g, which plays the central rôle in the Atiyah-Ward construction. Having established a suitably general form for g which, under certain conditions, builds in the boundary conditions, in Sect. 4 we discuss the implications for g of the reality conditions (i.e. the conditions which ensure the existence of a gauge in which the gauge potentials A_i^a and the Higgs field Φ^a are real). A $(4l-1)$-parameter family of possibilities for g is constructed which have the property that certain functions, $\Delta_r(x)$, used in constructing the ansatz, are nonsingular. However, to ensure that the gauge fields are everywhere nonsingular is much more difficult and we have only been able to do it by appealing to an argument of Ward, which exploits the existence of a known solution to guarantee nonsingularity for nearby solutions. In Sect. 5 we explain how rotations affect the description of our solutions and we discuss the way the known one and two monopole solutions fit into our discussion. Finally, in Sect. 6, we summarise the construction.

2. The Atiyah-Ward Ansätze

Ward [10] established a correspondence between self-dual gauge fields and certain analytic vector bundles. Such a bundle can be described by a matrix-valued transition function $g(\omega, \pi)$, depending on two complex two-spinors ω, π, which satisfies the conditions

$$g(\lambda\omega, \lambda\pi) = g(\omega, \pi), \tag{2.1}$$

$$\det g = 1, \tag{2.2}$$

and other properties which we shall elaborate (see also [12]). It is convenient to represent the points of four dimensional Euclidean space by quaternions $x = x^0 - i\mathbf{x}\cdot\boldsymbol{\sigma}$. Then, if we relate ω and π by

$$\omega = x\pi, \tag{2.3}$$

for a fixed x, $g(x\pi, \pi)$ is a function of the single variable $\zeta = \pi_1/\pi_2$ by virtue of Eq. (2.1). This function must be regular in some annular neighbourhood $\{\zeta : 1 - \varepsilon_1 < |\zeta| < 1 + \varepsilon_2\}$ of $|\zeta| = 1$. Further, for each fixed x, it must be possible to "split" g:

$$g(x\pi, \pi) = h(x, \zeta)\, k(x, \zeta)^{-1}, \tag{2.4}$$

where $h(x,\zeta)$ is regular in $|\zeta|>1-\varepsilon$, whilst $k(x,\zeta)$ is regular in $|\zeta|<1+\varepsilon$, for some $\varepsilon>0$. These conditions are sufficient for g to correspond to a self-dual gauge field.

The transition function g' will describe gauge equivalent self-dual fields if and only if

$$g' = v_\infty g v_0, \tag{2.5}$$

where v_∞, v_0 possess the properties of g expressed by Eqs. (2.1) and (2.2), and, in addition, are regular functions of ω and π in $|\zeta|>1-\varepsilon$ and $|\zeta|<1+\varepsilon$, respectively, for some $\varepsilon>0$. In this case we shall write $g'\approx g$.

Atiyah and Ward [11] argued that, at least for instanton solutions, any suitable transition matrix

$$g \approx \begin{pmatrix} \zeta^l & \rho(x,\zeta) \\ 0 & \zeta^{-l} \end{pmatrix}, \tag{2.6}$$

where ρ is only really a function of ζ and

$$2\mu = i\omega_2/\pi_2 = (x_1+ix_2)\zeta + ix_0 - x_3, \tag{2.7}$$

$$2\nu = i\omega_1/\pi_1 = (x_1-ix_2)/\zeta + ix_0 + x_3. \tag{2.8}$$

We shall assume Eq. (2.6) also holds for monopole solutions. As a consequence of this limited dependence, ρ and its Laurent coefficients

$$\Delta_s(x) = \frac{1}{2\pi i}\oint \frac{d\zeta}{\zeta}\zeta^s \rho(x,\zeta) \tag{2.9}$$

satisfy the four-dimensional Laplace equation

$$\partial^2\rho = \partial^2 \Delta_r = 0. \tag{2.10}$$

[The contour integral in Eq. (2.9) and subsequent equations is taken round the unit circle, $|\zeta|=1$, unless otherwise stated.] Because of the analyticity conditions stated above $\rho(x,\zeta)$ must be analytic in an annular neighbourhood of $|\zeta|=1$. Additionally the splitting condition of Eq. (2.4) is equivalent to

$$\det D^{(l)} \neq 0, \tag{2.11}$$

where $D^{(l)}$ is the banded $l\times l$ matrix

$$D^{(l)}_{rs} = \Delta_{r+s-l-1} \qquad 1 \leqq r,\ s \leqq l. \tag{2.12}$$

In this case, the gauge potentials are given by

$$A_\alpha = \frac{-i}{2\Lambda_2}\begin{pmatrix} \eta^3_{\alpha\beta}\partial_\beta \Lambda_2 & \eta^{1-i2}_{\alpha\beta}\partial_\beta \Lambda_1 \\ \eta^{1+i2}_{\alpha\beta}\partial_\beta \Lambda_3 & -\eta^3_{\alpha\beta}\partial_\beta \Lambda_2 \end{pmatrix}, \tag{2.13}$$

where

$$\eta^a_{\alpha\beta} = \varepsilon_{0a\alpha\beta} + \delta_{a\alpha}\delta_{0\beta} - \delta_{a\beta}\delta_{0\alpha}, \tag{2.14}$$

and

$$\Lambda_1 = (D^{(l)-1})_{11}, \quad \Lambda_2 = (D^{(l)-1})_{1l}, \quad \Lambda_3 = (D^{(l)-1})_{ll}. \tag{2.15}$$

For Eq. (2.13) to be nonsingular we need $\det D^{(l-1)} \neq 0$ in addition to (2.11). If this does not hold we must use another gauge.

Finally we need conditions for reality, x_0-independence and the correct asymptotic behaviour of the Higgs field, if we are to obtain a monopole solution. To obtain solutions which are independent of x_0, in some gauge, we need g to be equivalent, in the sense of Eq. (2.5), to a function, $g_0(\gamma, \zeta)$, of ζ and

$$\gamma = \mu - \nu \tag{2.16}$$

only [7]. The reality condition takes the form [7]

$$g_1(\gamma, \zeta)^\dagger = g_1(\bar{\gamma}, -1/\bar{\zeta}), \tag{2.17}$$

for some $g_1 \approx g_0$. This implies the weaker condition

$$g_0(\gamma, \zeta)^\dagger \approx g_0(\bar{\gamma}, -1/\bar{\zeta}). \tag{2.18}$$

Eq. (2.18) would also hold if $g_0 \approx g_2$, where

$$g_2(\gamma, \zeta)^\dagger = -g_2(\bar{\gamma}, -1/\bar{\zeta}), \tag{2.19}$$

but this can be shown to yield a real $SU(1,1)$ solution. The asymptotic condition is discussed in the next section.

3. The Asymptotic Condition on the Higgs Field

In the examples of the construction of monopole solutions, using the Atiyah-Ward ansatz, so far given [16, 7–9], the dependence of ρ in Eq. (2.9) is of the form e^{ix_0}, so that

$$\Delta_s(x) = e^{ix_0} \tilde{\Delta}_s(\mathbf{x}). \tag{3.1}$$

In this case we can use a remarkable formula of Prasad [8, 18] for the length of the Higgs field

$$\Phi^2 = 1 - \nabla^2 \ln D^{(l)}, \tag{3.2}$$

where ∇^2 is the three-dimensional Laplacian operator. Equation (2.10) then implies

$$\nabla^2 \tilde{\Delta}_s = \tilde{\Delta}_s. \tag{3.3}$$

Provided that its angular behaviour is not too wild, $\tilde{\Delta}_s$ will satisfy

$$\tilde{\Delta}_s(\mathbf{x}) \sim \frac{e^r}{r} \delta_s(\theta, \phi), \qquad \text{as } r \to \infty, \tag{3.4}$$

using spherical polar coordinates r, θ, ϕ. If this condition holds for $|s| \leqq l-1$, the Higgs field has the required asymptotic behaviour (1.6) and so we have an l-monopole solution. These heuristic considerations lead us to seek a g as in Eq. (2.9) with ρ having an e^{ix_0} dependence on x_0. We shall then prove (3.4) does indeed hold for our eventual choice.

Previous work suggests we consider

$$g_0(\gamma,\zeta)=\begin{pmatrix} e^{\gamma}\zeta^{l} & f(\gamma,\zeta) \\ 0 & e^{-\gamma}\zeta^{-l}\end{pmatrix}$$
$$=\begin{pmatrix} e^{-\nu} & 0 \\ 0 & e^{\nu}\end{pmatrix}\begin{pmatrix} \zeta^{l} & \rho \\ 0 & \zeta^{-l}\end{pmatrix}\begin{pmatrix} e^{\mu} & 0 \\ 0 & e^{-\mu}\end{pmatrix}. \tag{3.5}$$

Hence such a g_0 is equivalent to a g of the form of Eq. (2.9) with

$$\rho(x,\zeta)=e^{\mu+\nu}f(\gamma,\zeta), \tag{3.6}$$

which possesses the desired x_0 dependence.

4. Reality and Regularity

Assuming a transition matrix $g \approx g_0$ as in Eq. (3.5) and (3.6), we seek to determine conditions on f which will ensure that the resulting gauge fields are real and smooth. First we consider which f's yield the same (or gauge equivalent) solutions and then we impose reality and regularity conditions.

(a) Equivalence. Replacing $f(\gamma,\zeta)$ by $f'(\gamma,\zeta)$ will make no essential difference if and only if

$$\begin{pmatrix} a_\infty & b_\infty \\ c_\infty & d_\infty\end{pmatrix}\begin{pmatrix} e^{\gamma}\zeta^{l} & f' \\ 0 & e^{-\gamma}\zeta^{-l}\end{pmatrix}=\begin{pmatrix} e^{\gamma}\zeta^{l} & f \\ 0 & e^{-\gamma}\zeta^{l}\end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0\end{pmatrix}, \tag{4.1}$$

where a_0, b_0, c_0, d_0 are functions of ζ and γ regular in $|\zeta|<1+\varepsilon$ at fixed x, and a_∞, b_∞, c_∞, d_∞ are also functions of ζ and γ but are regular in $|\zeta|>1-\varepsilon$ at fixed x, for some $\varepsilon>0$, and

$$a_0 d_0 - b_0 c_0 = a_\infty d_\infty - b_\infty c_\infty = 1. \tag{4.2}$$

Since $l>0$, it is easy to deduce, from Liouville's theorem, that $c_0=c_\infty=0$ and, hence, that $a_0=a_\infty$ and $d_0=d_\infty$ and that both are constant. Thus the condition for f and f' to be equivalent is

$$f'=kf+\beta_0 e^{\gamma}\zeta^{l}+\beta_\infty e^{-\gamma}\zeta^{-l}, \tag{4.3}$$

with k a nonzero constant and $\beta_0(\gamma,\zeta)$, $\beta_\infty(\gamma,\zeta)$ regular in $|\zeta|<1+\varepsilon$ and $|\zeta|>1-\varepsilon$, respectively.

(b) Reality. The reality condition (2.18) takes the form

$$\begin{pmatrix} a_\infty & b_\infty \\ c_\infty & d_\infty\end{pmatrix}\begin{pmatrix} \zeta^{l}e^{\gamma} & f \\ 0 & \zeta^{-l}e^{-\gamma}\end{pmatrix}=\begin{pmatrix} \zeta^{l}e^{-\gamma} & \bar{f} \\ 0 & \zeta^{-l}e^{\gamma}\end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0\end{pmatrix}, \tag{4.4}$$

where a_0, b_0, c_0, d_0 and a_∞, b_∞, c_∞, d_∞ enjoy the regularity properties stated after Eq. (4.1), and satisfy Eq. (4.2), and where

$$\bar{f} \equiv \overline{f(\bar{\gamma}, -1/\bar{\zeta})}. \tag{4.5}$$

To obtain Eq. (4.4) we have used

$$g(\bar{\gamma}, -\bar{\zeta}^{-1})^{\dagger} = \begin{pmatrix} 0 & (-1)^l \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \zeta^l e^{-\gamma} & \bar{f} \\ 0 & \zeta^{-l} e^{\gamma} \end{pmatrix} \begin{pmatrix} 0 & (-1)^l \\ 1 & 0 \end{pmatrix}$$
$$\approx \begin{pmatrix} \zeta^l e^{-\gamma} & \bar{f} \\ 0 & \zeta^{-l} e^{\gamma} \end{pmatrix}. \tag{4.6}$$

Thus

$$\psi = c_{\infty} \zeta^l = c_0 \zeta^{-l} \tag{4.7}$$

must be of the form

$$\psi(\gamma, \zeta) = \sum_{r=-l}^{l} \Psi_{-r}(x) \zeta^r. \tag{4.8}$$

Also

$$f = \frac{d_0 e^{\gamma} - d_{\infty} e^{-\gamma}}{\psi}, \qquad \bar{f} = \frac{a_{\infty} e^{\gamma} - a_0 e^{-\gamma}}{\psi}. \tag{4.9}$$

Equations (4.8) and (4.9), together with Eqs. (4.2) and the regularity requirements, contain all the reality condition (2.18). It follows that

$$\psi = \pm \bar{\psi}, \tag{4.10}$$

$$a_0 = \pm \bar{d}_{\infty}, \qquad d_0 = \pm \bar{a}_{\infty}, \tag{4.11}$$

where $\bar{\psi}$, etc., are defined as in Eq. (4.5), are signs must be chosen consistently. So $a_0(\zeta, \gamma\zeta)$, $d_0(\zeta, \gamma\zeta)$ are analytic functions of ζ and $\gamma\zeta$ for $|\zeta| \leqq 1 + \varepsilon$ and all finite values of $\gamma\zeta$. Let us suppose a_0, d_0 are non-zero in this region. (This property, possessed by previous examples [7, 9], is quite possibly necessary for a smooth solution.) Then, using Eq. (4.2),

$$f = (d_0 e^{\gamma} - e^{-\gamma}/a_{\infty})/\psi - \zeta^{-l} b_{\infty}/a_{\infty}$$
$$\approx (d_0 e^{\gamma} - e^{-\gamma}/a_{\infty})/\psi, \tag{4.12}$$

in this sense of Eq. (4.3). Our assumption about the absence of zeros of d_0 means we can write

$$d_0 = e^{\delta}, \tag{4.13}$$

where $\delta(\zeta, \gamma\zeta)$ is regular for $|\zeta| < 1 + \varepsilon$ and all finite values of $\zeta\gamma$. Clearly we can write

$$\delta = \chi + \zeta^l \psi \delta', \tag{4.13}$$

where δ' is regular in $|\zeta| < 1 + \varepsilon$ and χ is a polynomial of degree $l-1$ in $\gamma\zeta$, as $\zeta^l \psi$ is a polynomial of degree l in $\gamma\zeta$. Then, in the sense of Eq. (4.3),

$$f \approx (e^{\gamma + \chi} \mp e^{-\gamma - \bar{\chi}})/\psi. \tag{4.15}$$

Here

$$\psi = \sum_{s=0}^{l} \psi_{l-s}(\zeta) \gamma^s, \tag{4.16}$$

with $\psi_s(\zeta)\,\zeta^s$ being a polynomial of degree $2s$ in ζ satisfying

$$\psi_s(-1/\bar{\zeta}) = \overline{\pm\psi_s(\zeta)}. \tag{4.17}$$

So ψ_s has $(2s+1)$ real degrees of freedom, leaving ψ with $(l+1)^2$. We can use the constant k in Eq. (4.3) to fix $\psi_0=1$, or i depending on the choice of sign in Eq. (4.17), leaving l^2+2l. This means there must be $(l-1)^2$ constraints on ψ for it to produce a smooth solution, because we know such a solution to have $(4l-1)$ degrees of freedom [17]. As yet χ can be any polynomial

$$\chi = \sum_{s=0}^{l-1} \chi_s(\zeta)\,(\zeta\gamma)^s, \tag{4.18}$$

where the coefficients $\chi_s(\zeta)$ are regular for $|\zeta|<1+\varepsilon$.

The full reality condition (2.17) requires the upper sign in Eq. (4.10), etc., if l is odd and the lower if l is even; see Sect. 6.

(c) Regularity. As given by Eq. (4.15), f may have x-dependent singularities viewed as a function of ζ at fixed x. These tend to be incompatible with f being analytic in an annular neighbourhood of $|\zeta|=1$ for each x. To avoid them we seek to choose χ so that

$$e^{\gamma+\chi} = \pm e^{-\gamma-\bar{\chi}} \quad \text{when } \psi=0. \tag{4.19}$$

Write

$$\psi = \prod_{s=1}^{l} (\gamma-\gamma_s(\zeta)), \tag{4.20}$$

so that $\gamma_s(\zeta)$, $1\leqq s\leqq l$ are the roots of $\psi=0$ regarded as an equation of degree l in γ. If $\gamma_s(\zeta)$ is a root so is $\overline{\gamma_s(-1/\bar{\zeta})}$. We need to choose χ so that

$$e^{\Theta(\gamma,\zeta)} = \pm 1, \quad \text{for } \gamma=\gamma_s(\zeta),\ 1\leqq s\leqq l, \tag{4.21}$$

where

$$\Theta = 2\gamma+\chi+\bar{\chi}. \tag{4.22}$$

Now Θ must be a polynomial of degree $l-1$ in γ, satisfying

$$\Theta = \bar{\Theta}, \tag{4.23}$$

with $\bar{\Theta}$ defined as in Eq. (4.5). Further, if

$$\Theta(\gamma,\zeta) = 2\pi i \sum_{r=0}^{l-1} \Theta_r(\zeta)\,\gamma^r, \tag{4.24}$$

the fact that χ is a polynomial in $\zeta\gamma$ rather than γ and that $\bar{\chi}$ is a polynomial in γ/ζ, means that certain of the Laurent coefficients of $\theta_r(\zeta)$ must be absent or restricted:

$$\oint \Theta_r(\zeta)\,\zeta^s \frac{d\zeta}{\zeta} = 0, \quad |s|<r,\ 2\leqq r\leqq l-1, \tag{4.25}$$

$$\oint \Theta_1(\zeta)\frac{d\zeta}{\zeta} = 2. \tag{4.26}$$

Equation (4.21) says that $\Theta = 2\pi i n_s$ when $\gamma = \gamma_s$, $1 \leqq s \leqq l$, where the n_s are integers if we take the upper sign in Eq. (4.21) and half odd integers if we take the lower one.

Given n_s, $1 \leqq s \leqq l$, Θ is uniquely determined,

$$\Theta(\gamma, \zeta) = 2\pi i \sum_{r=1}^{l} n_r \prod_{s \neq r} \frac{(\gamma - \gamma_s)}{(\gamma_r - \gamma_s)}, \tag{4.27}$$

as it is a polynomial of degree $l-1$. To satisfy Eq. (4.23) we need

$$n_r = -n_s \quad \text{if } \gamma_r(\zeta) = \overline{\gamma_s(-1/\bar{\zeta})}. \tag{4.28}$$

Given the choice of the discrete parameters n_s, Eqs. (4.25) and (4.26) constitute the desired $(l-1)^2$ constraints on the coefficients of ψ. This leaves it with $4l-1$ degrees of freedom.

For polynomials ψ satisfying the constraints of Eqs. (4.25–7) we stand some chance of obtaining a nonsingular solution. Provided that $\gamma_r(\zeta)$ is regular in some annular neighbourhood of $|\zeta| = 1$, we can indeed split Θ as in Eq. (4.22), explicitly,

$$\chi(\gamma, \zeta) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \frac{(\xi + \zeta)}{2(\xi - \zeta)} \Theta(\gamma, \xi) - \gamma, \tag{4.29}$$

where $|\zeta| < |\xi|$. (An arbitrary pure imaginary number can be added to χ but this just corresponds to multiplying f by a number of unit modulus and so makes an irrelevant change.) Then we can verify that the heuristic arguments of Sect. 3 on asymptotic behaviour are not misleading. To study the asymptotic behaviour, write

$$e^{\chi} = \chi^0 + \zeta^l \psi R^0, \tag{4.30}$$

where, in a similar fashion to Eq. (4.13) χ^0 is a polynomial in $\zeta\gamma$ of order $l-1$,

$$\chi^0 = \sum_{r=0}^{l-1} \chi_r^0(\zeta)(\zeta\gamma)^r, \tag{4.31}$$

and χ_r^0, R^0 are regular in $|\zeta| < 1 + \varepsilon$. Similarly we can write

$$e^{-\bar{\chi}} = \chi^\infty + \zeta^{-l} \psi R^\infty, \tag{4.32}$$

where χ^∞ is a polynomial of order $l-1$ in γ/ζ. Then

$$f \approx (\chi^0 e^{\gamma} \mp \chi^\infty e^{-\gamma})/\psi \tag{4.33}$$

and, uisng Eq. (3.6), it can be shown that

$$\Delta_s(x) = e^{i x_0} \frac{e^r}{r} [1 + O(r^{-1})] \, \delta_s(\theta, \phi) \tag{4.34}$$

as $r \to \infty$.

However more is needed to establish that the gauge fields are smooth. But we can exploit an argument of Ward [9] to use a known smooth solution corresponding to a specific choice $\psi^{(0)}$ of ψ. Then, for ψ sufficiently close to $\psi^{(0)}$ (in the sense that the coefficients of the polynomials are close to one another), the resulting solutions are smooth. The determinant (2.11), which should be nonzero, is a continuous function of ψ at each point $\mathbf{x}$ of space. This means that condition (2.11) will be maintained in any compact region of space by a sufficiently small variation of ψ from $\psi^{(0)}$. At large distance the functions $\delta_s(\theta, \phi)$ of Eq. (3.4) are continuously dependent on ψ so that we can prove asymptotic regularity for ψ near to $\psi^{(0)}$. Thus we have smooth solutions for ψ in some neighbourhood of $\psi = \psi^{(0)}$.

Prasad and Rossi [8] claim to have established the existence of certain smooth axis symmetric solutions for each positive l. These correspond to the following:

$$\gamma_s = \frac{i\pi}{2}(l+1-2s), \qquad n_s = \tfrac{1}{2}(l+1-2s) \;\; 1 \leqq s \leqq l, \tag{4.35}$$

and we take the upper or lower sign in Eq. (4.10), etc., as l is odd or even, respectively. In this case $\chi = 0$. The above techniques provide families of solutions, in the neighbourhood of each these, with $4l-1$ degrees of freedom.

It is convenient to use the phase ambiguity in f, resulting from the equivalence relation (4.3), to multiply ψ by i if l is even so that it satisfies $\psi = \bar{\psi}$ for all l.

5. Rotations and the Two Monopole Solution

It is illuminating to consider the effects of rotations in this formalism. We shall see how the rotational properties of one and two monopole solutions emerge from the above discussion.

The effect of the rotation corresponding to the $SU(2)$ matrix u is

$$x \to x' = u x u^{-1}, \tag{5.1}$$

$$\pi \to \pi' = u\pi, \qquad \omega \to \omega' = u\omega. \tag{5.2}$$

Hence, if we write

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5.3}$$

where $a = \bar{d}$, $b = -\bar{c}$, $|a|^2 = 1 - |b|^2$, we have

$$\zeta \to \zeta' = (a\zeta + b)/(c\zeta + d) \tag{5.4}$$

and

$$\gamma \to \gamma' = \gamma/(a + b/\zeta)\, c\zeta + d). \tag{5.5}$$

The transition matrix, g', for the rotated solution is equivalent to

$$\begin{pmatrix} (\zeta')^l & \rho(x',\zeta') \\ 0 & (\zeta')^{-l} \end{pmatrix}$$
$$= \begin{pmatrix} (a+b/\zeta)^l & 0 \\ 0 & (a+b/\zeta)^{-l} \end{pmatrix} \begin{pmatrix} \zeta^l & \rho'(x,\zeta) \\ 0 & \zeta^{-l} \end{pmatrix} \begin{pmatrix} (c\zeta+d)^{-l} & 0 \\ 0 & (c\zeta+d)^l \end{pmatrix}, \tag{5.6}$$

where

$$\rho'(x,\zeta) = \frac{\rho(x',\zeta')}{(a+b/\zeta)^l (c\zeta+d)^l}. \tag{5.7}$$

Thus with ψ given as in Eq. (4.17), the corresponding function after rotation is

$$\psi'(\gamma,\zeta) = (a+b/\zeta)^l (c\zeta+d)^l \psi(\gamma',\zeta') = \sum_{s=0}^{l} \psi'_{l-s}(\zeta)\,\gamma^s, \tag{5.8}$$

where

$$\psi'_s(\zeta) = (a+b/\zeta)^s (c\zeta+d)^s \psi_s(\zeta'). \tag{5.9}$$

For the single monopole, $l=1$,

$$\psi(\gamma,\zeta) = \gamma + \eta(\zeta), \tag{5.10}$$

where

$$\eta(\zeta) = \eta_1\zeta + \eta_0 - \bar{\eta}_1/\zeta, \tag{5.11}$$

η_0 being real. ψ_1 can be removed by a translation, which has the effect

$$x \to x' = x - y, \qquad \zeta \to \zeta' = \zeta \tag{5.12}$$

$$\gamma \to \gamma' - \eta, \tag{5.13}$$

where η is obtained from γ by repalcing x by y:

$$\eta = \tfrac{1}{2}(y_1 + iy_2)\zeta - y_3 - \tfrac{1}{2}(y_1 - iy_2)/\zeta. \tag{5.14}$$

After such a translation, the rotation has no effect; this result corresponds to the spherical symmetry of the one monopole solution.

For the two monopole solution, $l=2$,

$$\psi(\gamma,\zeta) = \gamma^2 + 2\eta\gamma + \lambda, \tag{5.15}$$

where η is as in Eq. (5.11) and

$$\lambda = \lambda_2\zeta^2 + \lambda_1\zeta + \lambda_0 - \bar{\lambda}_1/\zeta + \bar{\lambda}_2/\zeta^2 \tag{5.16}$$

with λ_0 real. λ and η must together satisfy one constraint corresponding to Eq. (4.25), together with further unknown inequalities to ensure smoothness. This leaves seven parameters. By a translation we can arrange

$$\psi = \gamma^2 + \lambda', \tag{5.17}$$

where $\lambda' = \lambda - \eta^2$. By a rotation we can reduce λ' so that the coefficients of ζ^2 and ζ^{-2} are absent, a further so that the coefficients of ζ and ζ^{-1} are real. This yields the case discussed by Ward [9].

6. Summary

In Sect. 4 we showed (subject to certain assumptions about the positions of zeros of the functions a_0, etc.) that the transition matrix for an l monopole solution could be obtained from a polynomial $\psi(\gamma, \zeta)$ of degree l in γ, such that $\zeta^l \psi(\gamma, \zeta)$ is a polynomial of degree $2l$ in ζ at fixed x. There are certain restrictions on ψ. There is a reality condition

$$\overline{\psi(\gamma, \zeta)} = \psi(\bar{\gamma}, -1/\bar{\zeta}). \tag{6.1}$$

We can normalise ψ so that the coefficient of γ^l is 1, without loss of generality, leaving $l^2 + 2l$ degrees of freedom. To formulate the remaining conditions, we factorise ψ in Eq. (4.20) and define a polynomial Θ, of degree $l - 1$ in γ by Eq. (4.27). This involves the choice of l constants n_s which are such that $n_s + \frac{1}{2}l + \frac{1}{2}$ is integral, $1 \leqq s \leqq l$. Then Eqs. (4.25) and (4.26) provide $(l-1)^2$ further constraints on Θ and so, implicity, on ψ. It is not yet clear which choices of the constants n_s and what range of the parameters in ψ yield non-singular solutions, but in the known cases the n_s are chosen so that they are as small as possible consistent with their being distinct.

Then the transition matrix for the solution is of the form

$$g \approx \begin{pmatrix} \zeta^l e^{\Theta} & (e^{\Theta} \mp e^{-\Theta})/\psi \\ 0 & \zeta^{-l} e^{-\Theta} \end{pmatrix}, \tag{6.2}$$

which is equivalent to a g of Eq. (2.6), where

$$\rho(x, \zeta) = e^{\mu + \nu}(e^{\gamma + \chi} \mp e^{-\gamma - \bar{\chi}})/\psi. \tag{6.3}$$

In Eq. (6.3), μ, ν are defined in Eqs. (2.7), (2.8), $\gamma = \mu - \nu$ and χ is defined by Eq. (4.29). The prescription of Eqs. (2.9–15) then yields $\mathbf{A}_i$ and $\boldsymbol{\Phi}$. (The minus or plus signs are taken depending on whether the n_s are integral or not.)

To verify the reality conditions note that

$$g \approx g_1 = \begin{pmatrix} (e^{\Theta} \mp e^{-\Theta})/\psi & \mp \zeta^l e^{-\Theta} \\ \zeta^{-l} e^{-\Theta} & \psi e^{-\Theta} \end{pmatrix}, \tag{6.4}$$

using relations similar to those in [7] and [9], showing that (2.17) holds if we take the upper or lower sign as l is odd or even. With the other sign, $g_2 = i\sigma_3 g_1$ satisfies (2.19) showing this choice yields an $SU(1,1)$ solution.

General explicit formulae will be much more difficult to obtain for $l > 2$ than for the $l = 2$ case considered by Ward [9]. He pointed out that that case required the solution of a quartic and the use of elliptic integrals. For $l \geqq 3$ we pass beyond this comparatively elementary domain and we are involved with the solution of sixth and higher order equations, which cannot be solved explicitly in the same sense as the quartic, and with generalisations of elliptic integrals. It seems likely that the central remaining problem on which progress can be made is that of determining for which range of ψ and choice of n_s, the solutions are nonsingular.

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