Annihilation Prediction for Lanchester-Type Models of Modern Warfare

JAMES G. TAYLOR and GERALD G. BROWN

Naval Postgraduate School, Monterey, California

This paper introduces important new functions for analytic solution of Lanchester-type equations of modern warfare for combat between two homogeneous forces modeled by power attrition-rate coefficients with "no offset." Tabulations of these *Lanchester-Clifford-Schläfli* (or LCS) *functions* allow one to study this particular variable-coefficient model almost as easily and thoroughly as Lanchester's classic constant-coefficient one. LCS functions allow one to obtain important information (in particular, force-annihilation prediction) *without* having to spend the time and effort of computing force-level trajectories. The choice of these particular functions is based on theoretical considerations that apply in general to Lanchester-type equations of modern warfare and provide guidance for developing other canonical functions. Moreover, our new LCS functions also provide valuable information about related variable-coefficient models. Also, we introduce an important transformation of the battle's time scale that not only simplifies the force-level equations, but also shows that *relative fire effectiveness* and *intensity of combat* are the only two weapon-system parameters determining the course of such variable-coefficient Lanchester-type combat.

IN AN EARLIER paper (Taylor and Brown [1976]), we showed how to solve variable-coefficient Lanchester-type equations of modern warfare for combat between two homogeneous forces. In that paper, we introduced canonical hyperbolic-like Lanchester functions for constructing the solution. Unfortunately, with only these previous results one is limited to computing force-level trajectories and cannot gain a real understanding of qualitative model behavior (e.g. force annihilation) without extensive numerical computations (and only then for specific values of model parameters). Since the appearance of our earlier work, several mathematical discoveries (Taylor and Comstock [1977], Taylor [1979b]) have provided new qualitative insight about the behavior of this combat model. We wish to show here how these new results allow parametric analysis of combat modeled by power attrition-rate coefficients with somewhat the same facility as allowed by F. W. Lanchester's classic constant-coefficient model. In order to obtain this analysis capa-

bility, however, one must redefine the Lanchester-Clifford-Schläfli (or LCS) functions, which we introduced in Taylor and Brown [1976].

It is important for the military OR analyst to have a clear understanding of how the initial force ratio and weapon-system-capability parameters interact to determine a battle's outcome. One is consequently interested in developing insights into the dynamics of combat by explicitly portraying the relation between the various factors of the combat-attrition process and battle outcome. Modeling battle termination is a somewhat controversial topic (Taylor [1979a]), and no mathematical theory exists for other than determining zero points of solutions (i.e. force annihilation) to such differential-equation models (Taylor [1979b]). However, it is of considerable utility just to be able to easily predict the occurrence of force annihilation in simulated Lanchester-type combat. One is always interested in determining what conditions lead to the annihilation of an enemy force, since such an occurrence (of course) guarantees victory. Although actual battles rarely go completely to annihilation, a commander may decide to terminate an engagement once he anticipates that annihilation is possible, and hence force-annihilation conditions may be useful in modeling engagement termination. Additionally, a commander would seek to avoid engagements in which his own force could be annihilated, and such conditions may provide, for example, valuable information for the modeling of engagement avoidance.

In our earlier paper (Taylor and Brown [1976]), we gave various examples of hyperbolic-like Lanchester functions (in particular, the LCS functions, which arise from power attrition-rate coefficients with "no offset"). Subsequent research by Taylor and Comstock has revealed, however, that these canonical LCS functions must be redefined to permit force-annihilation prediction from initial conditions without having to spend the time and effort to compute force-level trajectories. It then became obvious that the entire topic of representing the solution to such Lanchester-type equations in terms of general Lanchester functions (GLF) should be critically reexamined. Consequently, we developed new general considerations for the selection of canonical Lanchester functions (Taylor and Brown [1977a]). Based on these considerations, we also developed new LCS functions for the special case of power attrition-rate coefficients with "no offset" (modeling, for example, weapon systems with the same maximum effective range) which are presented here. These power Lanchester (i.e. LCS) functions are significant, not only because they correspond to attrition-rate coefficients modeling a large class of combat situations of interest, but also because they yield valuable information about other related canonical Lanchester functions, e.g. the offset power Lanchester functions (see Taylor and Brown [1978] and Section 10 below). With the availability of tabulations of these new LCS func-

tions, one can study this model almost as easily and thoroughly as Lanchester's classic constant-coefficient one. Such models are important for developing insights into the dynamics of combat (Bonder and Honig [1971], Taylor [1980a]).

The results of this paper are also important for understanding complex operational differential-equation models that are widely used in both the United States and also NATO countries as defense-planning tools (see Huber et al. [1975, 1979], Taylor [1979a]). The modern high-speed, large-scale digital computer has made it possible to develop and use such complex Lanchester-type combat models (e.g. see Bonder and Farrell [1970]; Bonder and Honig; Command and Control Technical Center (CCTC) [1979]). Nevertheless, a simple combat model such as we consider here may yield a clearer understanding of important relations that are difficult to perceive in a more complex model, and such insights can provide valuable guidance for subsequent higher-resolution computerized investigations. As Geoffrion [1976] has emphasized, one can use a *simplified auxiliary model* for understanding the basic dynamics and behavior of a large-scale *complex operational model*. Furthermore, one can fit an analytical model to data generated from a detailed combat simulation, and thus a simple analytical model like the one considered here may provide an economical framework for summarizing simulation output data (see Ignall et al. [1978] for a lucid discussion of this modeling strategy in a nonmilitary context).

1. VARIABLE-COEFFICIENT LANCHESTER-TYPE EQUATIONS OF MODERN WARFARE

We consider the following *variable-coefficient Lanchester-type equations of modern warfare* for combat between two homogeneous forces for x and $y > 0$ (see p. 45 of Taylor and Brown [1976] for further discussion)

$$\begin{cases} dx/dt = -a(t)y & \text{with} \quad x(0) = x_0, \\ dy/dt = -b(t)x & \text{with} \quad y(0) = y_0, \end{cases} \tag{1.1}$$

where $t = 0$ denotes the time at which the battle begins, $x(t)$ and $y(t)$ denote the numbers of X and Y at time t, and $a(t)$ and $b(t)$ denote time-dependent Lanchester attrition-rate coefficients, which do not explicitly depend on x and y. In particular, both a and b depend explicitly upon time (perhaps via an intermediate variable such as range $r(t)$), but a does not directly depend on the number of targets x. Although combat between two military forces is a complex random process, such a deterministic model of the combat attrition process is frequently employed to provide insights into the dynamics of combat (e.g. see Weiss [1957]; Bonder and Farrell; Bonder and Honig; Taylor and Parry [1975], Taylor [1980a]). Moreover, current large-scale operational models (e.g. see Bonder and

Farrell, Bonder and Honig, CCTC) more or less take (1.1) as the point of departure for their development through the process of model enrichment (see Morris [1967] for a lucid discussion of this enrichment process). For example, in the detailed VECTOR-2 operational model (e.g. see CCTC), the attrition-rate coefficients are nonautonomous and depend (in quite a complicated fashion) on, not only the engagement conditions (e.g. range between firer and target, target and/or firer motion, posture, etc.), but also the number of firers and targets.

Equations 1.1 are usually taken to model combat in which both sides use aimed fire and target acquisition times are independent of the number of firers and targets (see Taylor [1974, 1980a], Taylor and Brown [1976] for further details). Other forms of Lanchester-type equations appear in the literature, but we will not consider them here (see Dolansky [1964], Taylor [1974, 1979a, 1980a]). The Lanchester attrition-rate coefficients $a(t)$ and $b(t)$ depend on such variables as force separation, tactical posture of targets, rate of target acquisition, firing doctrine, firing rate, and so forth (e.g. see Bonder [1965, 1967, 1970]; Bonder and Farrell). Bonder [1965] (see also Bonder and Farrell) has stressed the importance for evaluating weapon systems of such variable coefficient differential combat models to represent temporal variations in firepower on the battlefield.

We assume that $a(t)$ and $b(t)$ are defined, positive, and continuous for $t_0 < t < +\infty$ with $t_0 \leq 0$ (see Taylor and Brown [1976], Taylor [1979b] for further discussion). We further assume that $a(t)$ and $b(t)$ are such that their right-hand limits exist at t_0, with $+\infty$ allowed as a possibility: we define $a(t_0)$ as $\lim_{t \to t_0+} a(t)$ and similarly define $b(t_0)$. Note that the values 0 and $+\infty$ are possible for $a(t)$ and $b(t)$ only at $t = t_0$. For convenience, we introduce the notation $a(t) \in L(t_0, T)$ to mean that $\int_{t_0}^{T} a(t)dt$ exists. We also assume that $a(t)$ and $b(t) \in L(t_0, T)$ for any finite $T \geq t_0$. It follows that, for example, $a(t) \notin L(t_0, +\infty)$ implies that $\lim_{T \to +\infty} \int_{t_0}^{T} a(t)dt = +\infty$. We will further take $a(t)$ and $b(t)$ to be given in the form $a(t) = k_a g(t)$ and $b(t) = k_b h(t)$, where k_a and k_b are positive constants chosen so that $a(t)/b(t) \equiv k_a/k_b$ if and only if $g(t) \equiv h(t)$ (see Taylor [1979b, 1982]). In other words, k_a and k_b are basically "scale factors," which are useful for parametric study of battle outcomes as related to various system parameters. This factorization of $a(t)$ and $b(t)$ is not used directly in (1.1), but is implicit in constructing the general Lanchester functions used to represent the analytical solution to (1.1) (see pp. 441 and 448 of Taylor [1979b]). It is also convenient to introduce the *combat-intensity parameter* λ_I and the *relative-fire-effectiveness parameter* λ_R defined by

$$\lambda_I = \sqrt{k_a k_b}, \quad \text{and} \quad \lambda_R = k_a/k_b. \tag{1.2}$$

(See Taylor and Brown [1978], Taylor [1979a, 1980b] for further details.)

The X force level as a function of time, $x(t)$, may be represented as (Taylor and Brown [1976])

$$x(t) = x_0\{C_Y(0)C_X(t) - S_Y(0)S_X(t)\} \qquad (1.3)$$
$$- y_0\sqrt{\lambda_R}\{C_X(0)S_X(t) - S_X(0)C_X(t)\},$$

where the *hyperbolic-like general Lanchester functions* (GLF) $C_X(t)$ and $S_X(t)$ are linearly independent solutions to the *X force-level equation*

$$d^2x/dt^2 - \{[1/a(t)]da/dt\}dx/dt - a(t)b(t)x = 0, \qquad (1.4)$$

with initial conditions $C_X(t_0) = 1$, $\{1/a(t_0)\}dC_X/dt(t_0) = 0$, $S_X(t_0) = 0$, and $\{1/a(t_0)\}dS_X/dt(t_0) = 1/\sqrt{\lambda_R}$. When (for example) $a(t_0) = 0$ or $+\infty$, an initial value such as $\{1/a(t_0)\}dC_X/dt(t_0)$ should be interpreted as $\lim_{t \to t_0^+}[\{1/a(t)\}dC_X/dt(t)]$. Taylor and Comstock, and Taylor [1979b] introduced and studied exponential-like GLF. In Taylor and Brown [1976], we have discussed the representation of force levels in terms of GLF and have shown that these two types (i.e. hyperbolic-like and exponential-like GLF) are essentially the only kinds of GLF, but that the hyperbolic-like ones are to be preferred.

2. GENERAL FORCE-ANNIHILATION-PREDICTION CONDITIONS

The following theorem generalizes Lanchester's famous square law to variable-coefficient combat (see Taylor [1979b] for proof of a more general result).

THEOREM 1 (Taylor and Comstock). *Assume that either $a(t) \notin L(0, +\infty)$ or $b(t) \notin L(0, +\infty)$. Then the X force will be annihilated in finite time if and only if*

$$x_0/y_0 < \sqrt{\lambda_R}\{[C_X(0) - Q^*S_X(0)]/[Q^*C_Y(0) - S_Y(0)]\}, \qquad (2.1)$$

where the parity-condition parameter Q^* is unique and given by

$$\lim_{t \to +\infty}[S_X(t)/C_X(t)] = 1/Q^* = 1/\{\lim_{t \to +\infty}[S_Y(t)/C_Y(t)]\}. \qquad (2.2)$$

An answer to the seemingly simple question "Who will be annihilated in battle?" requires a significant extension of the theory of the real zeros of nonoscillatory (in the strict sense) solutions to the general second-order linear differential equation (Taylor [1979b]). Furthermore, consideration of Theorem 1 shows that the power Lanchester (or LCS) functions introduced in Taylor and Brown [1976] were inappropriately defined (see Taylor and Brown [1977a] for further details). It is, therefore, the purpose of this paper to appropriately redefine the power Lanchester functions in light of Theorem 1.

3. COMBAT MODELED WITH POWER ATTRITION-RATE COEFFICIENTS

A large class of tactical situations of interest can be modeled with the following *general power attrition-rate coefficients*

$$a(t) = k_a(t + K_S)^\mu, \quad \text{and} \quad b(t) = k_b(t + K_S + K_O)^\nu, \quad (3.1)$$

where $\mu, \nu, K_S, K_O \geq 0$. Taylor and Brown [1976] discuss the modeling roles of K_S and K_O. We will call K_S the *starting parameter*, since it allows us to model battles that begin within the maximum effective ranges of both opponents. We will call K_O the *offset parameter*, since it allows us to model battles between opposing weapon systems with different maximum effective ranges. We also observe that $t_0 = -K_S$.

The above nomenclature is motivated by Bonder's [1965] model of a constant-speed attack against a static defensive position

$$dx/dt = -\alpha(r)y, \quad \text{and} \quad dy/dt = -\beta(r)x, \quad (3.2)$$

where r denotes the range between opposing forces, and $\alpha(r)$ and $\beta(r)$ denote range-dependent attrition-rate coefficients. Range is related to time by

$$r(t) = r_0 - vt, \quad (3.3)$$

where $0 \leq t \leq r_0/v$, r_0 denotes the opening range of battle, and $v > 0$ denotes the constant attack speed. For example, consider the constant-speed attack of a homogeneous Y force against the static defensive position of a homogeneous X force (see Figure 1). The basic idea is that force separation (i.e. range between the opposing forces) changes over time and that the fire effectiveness of (for example) a single Y firer, denoted as $\alpha(r)$, depends on the force separation.

In many cases of tactical interest, we may model the fire effectiveness of the Y weapon system as a function of range with (see pp. 196–200 of Bonder and Farrell)

$$\alpha(r) = \begin{cases} \alpha_0(1 - r/r_a)^\mu & \text{for} \quad 0 \leq r \leq r_a, \\ 0 & \text{for} \quad r_a \leq r, \end{cases} \quad (3.4)$$

where r_a denotes the maximum effective range of the Y weapon system and $\mu \geq 0$ models the range dependency of Y's attrition-rate coefficient (see Figure 2). We model $\beta(r)$ similarly, with corresponding quantities r_β and ν.

Substituting (3.3) and (3.4) into (3.2), we find that $K_O = (r_\beta - r_a)/v$ and $K_S = (r_a - r_0)/v$, and that $k_a = a_0(v/r)^\mu$ and $k_b = \beta_0(v/r_\beta)^\nu$. Thus, K_O, $K_S \geq 0$ if and only if $r_\beta \geq r_a \geq r_0$. Moreover, for this particular application (and this situation is typical), the attrition-rate coefficients are techni-

cally not defined in the tactical scenario for $t > t_{max}$. Nevertheless, one can conceptually embed Bonder's tactical model with these time-dependent attrition-rate coefficients in the mathematical model in which (3.1) are assumed to hold for all $t \geq 0$ as long as one is careful not to use any quantities computed from the mathematical model outside range of definition of the tactical scenario. In particular, one must verify that force annihilation occurs within the range of definition of the tactical scenario (e.g. before t_{max} for Bonder's tactical model), and this requirement generates the need of computing the time at which force annihilation occurs in the mathematical model in which the coefficients are assumed to hold for all $t \geq 0$. We raise this point again in Sections 6 and 8 and illustrate it in the examples given in Section 9.
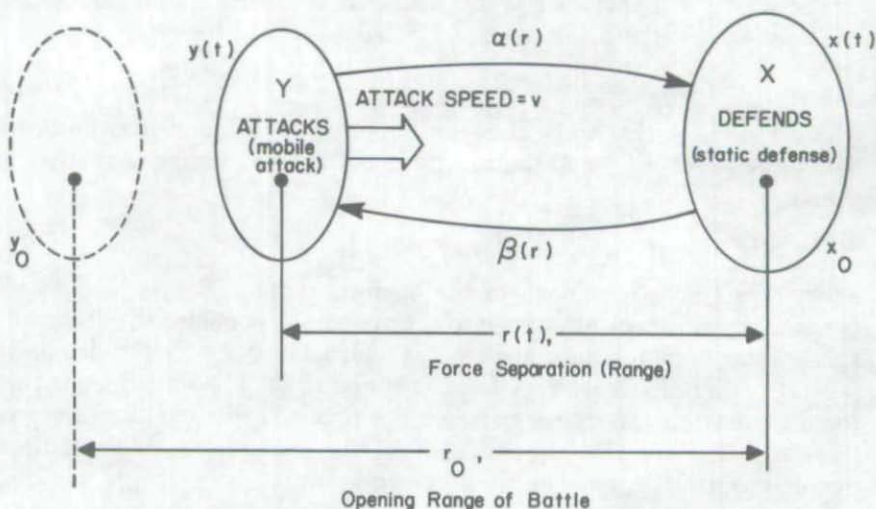

Figure 1. Diagram of Bonder's constant-speed attack model. Force separation, $r(t)$, is given by $r(t) = r_0 - vt$.

When the offset parameter is equal to zero (i.e. $K_O = 0$), the coefficients (3.1) reduce to the following *power attrition-rate coefficients with "no offset"*

$$a(t) = k_a(t + K_S)^\mu, \quad \text{and} \quad b(t) = k_b(t + K_S)^\nu. \quad (3.5)$$

As we have just seen above in Bonder's model, these coefficients model (for example) combat between weapon systems with the same maximum effective range, so that there is no "offset" in the capabilities of the opposing systems to "reach out" on the battlefield. It is the purpose of this paper to introduce new power Lanchester functions that facilitate force-annihilation prediction (and also determination of how long the

battle will last) for "aimed-fire" combat modeled by the power attrition-rate coefficients with "no offset" (3.5). The results of this section show how the physical characteristics of the weapon systems and environment are related to these coefficients.

4. A TRANSFORMATION TO NORMALIZE THE BATTLE'S TIME SCALE BY THE INTENSITY OF COMBAT

In this section we show how transformation of the battle's time scale
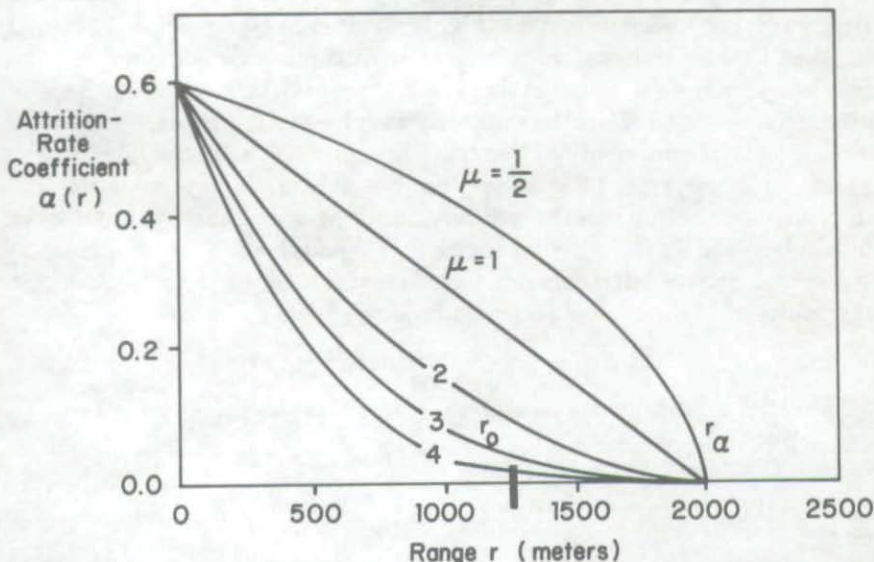

Figure 2. Dependence of Y's attrition-rate coefficient $\alpha(r)$ on the exponent μ with the maximum effective range of the weapon system and kill rate at zero range held constant. [*Notes*: (1) The maximum effective range of the system is denoted as $r_a = 2000$ meters. (2) $\alpha(0) = \alpha_0 = 0.6X$ casualties/(unit time × number of Y firers) denotes the weapon-system kill rate for Y at zero force separation (range). (3) The opening range of battle is denoted as $r_0 = 1250$ meters and (as shown) $r_0 < r_a$.]

provides important insight into the parametric dependence of the course of combat. Accordingly, we introduce the new independent variable τ defined by (see Taylor and Brown [1976, 1977a], Taylor [1979b] for further details)

$$\tau(t) = \int_{t_0}^{t} \sqrt{a(s)b(s)}\,ds, \qquad (4.1)$$

and let τ_0 denote $\tau(0)$. As is readily seen, this transformation is well

defined and invertible. We observe that $t_0 \leq 0$ implies that $\tau_0 \geq 0$. If we denote the "average intensity of combat" as $\sqrt{a(t)b(t)}$, then

$$\sqrt{a(t)b(t)}t = \left\{ (1/t) \int_0^t \sqrt{a(s)b(s)}ds \right\}t = \tau - \tau_0. \quad (4.2)$$

The substitution (4.1) transforms (1.4) into

$$d^2x/d\tau^2 - (\tfrac{1}{2})\{d \ln R(t)/d\tau\}dx/d\tau - x = 0, \quad (4.3)$$

with initial conditions $x(\tau_0) = x_0$ and $\{1/\sqrt{R(0)}\}dx/d\tau(\tau_0) = -y_0$, where $R(t) = a(t)/b(t)$. Equation 4.3 is highly significant (see Section 10 below) because it clearly shows that the course of combat depends on just the two weapon-system parameters: (1) $R(t) = a(t)/b(t)$, the relative fire effectiveness (Y to X) of the opposing weapon-system types, and (2) $I(t) = \sqrt{a(t)b(t)}$, the intensity of combat (through (4.1), which relates $I(t)$ to τ). In particular, from (4.3) we see that the nature of temporal variations in relative fire effectiveness will have a significant effect on the course of combat (see Taylor [1980b] for further details).

For the power attrition-rate coefficients with no offset (3.5), the transformed X force-level equation becomes

$$d^2x/d\tau^2 + \{(2q - 1)/\tau\}dx/d\tau - x = 0, \quad (4.4)$$

with initial conditions $x(\tau_0) = x_0$ and $\{(\tau/2)^{2q-1}dx/d\tau\}_{\tau=\tau_0} = -y_0\sqrt{\lambda_R}\{\lambda_I/(\mu + \nu + 2)\}$. Here

$$q = (\nu + 1)/(\mu + \nu + 2), \quad (4.5)$$

and

$$\tau(t) = \{2\lambda_I/(\mu + \nu + 2)\}(t + K_S)^{(\mu+\nu+2)/2}. \quad (4.6)$$

Let us observe that $0 < q < 1$ when μ and $\nu > -1$. Furthermore, $q > \tfrac{1}{2}$ if and only if $dR/dt < 0$, i.e. $R(t)$ is a strictly decreasing function of time.

5. LANCHESTER-CLIFFORD-SCHLÄFLI (LCS) FUNCTIONS

Consider the function $F_\alpha(x)$ defined by the power series

$$F_\alpha(x) = \Gamma(a) \sum_{k=0}^{\infty} (x/2)^{2k}/\{k!\Gamma(k + \alpha)\}. \quad (5.1)$$

For $\alpha \neq 0, -1, -2, \cdots$, the radius of convergence for $F_\alpha(x)$ is infinite by the ratio test for convergence of power series (Knopp [1956]). Hence, $F_\alpha(z)$ is an entire function of the complex variable $z = x + iy$ with an essential singularity at the point of infinity. Now consider the function $H_\alpha(x)$ defined by the infinite series

$$H_\alpha(x) = \Gamma(\alpha) \sum_{k=0}^{\infty} (x/2)^{2(k+\alpha)}/\{k!\Gamma(k + \alpha + 1)\}. \quad (5.2)$$

Observing that $H_\alpha(x) = (1/\alpha)(x/2)^{2\alpha}F_{\alpha+1}(x)$, we see that for $\alpha > 0$ the

infinite series (5.2) is uniformly convergent on compact subsets of the complex plane. One can also readily deduce the recursive relation $F_\alpha(x) = F_{\alpha+1}(x) + \{(x/2)^2/[\alpha(\alpha + 1)]\}F_{\alpha+2}(x)$. We will call the functions $F_\alpha(x)$ and $H_\alpha(x)$ *Lanchester-Clifford-Schläfli* (LCS) functions. Other properties are readily deduced and are given in Table I.

Although the solution of the X force-level Equation 1.4 with the power attrition-rate coefficients (3.5) may be expressed in terms of known higher transcendental functions (see Taylor [1974], Taylor and Brown [1976], Taylor and Comstock), we have chosen to introduce the LCS functions, since tabulations of these other functions are only available for a very restrictive range of parameter values of interest in Lanchester combat theory. For example, we can construct such solutions with modified Bessel functions of the first kind of fractional order, but tabulations of these (e.g. see Abramowitz and Stegun [1964]) exist only for a restrictive set of values of the order p (i.e. $p = \pm\frac{1}{4}, +\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm$

TABLE I

PROPERTIES OF THE LCS FUNCTIONS $F_\alpha(x)$ AND $H_\alpha(x)$

1. $dF_\alpha/dx = (x/2)^{1-2\alpha}H_\alpha(x)$.
2. $dH_\alpha/dx = (x/2)^{2\alpha-1}F_\alpha(x)$.
3. $F_\alpha(0) = 1$.
4. $H_\alpha(0) = 0$ for $\alpha > 0$.
5. $F_\alpha(x)F_{1-\alpha}(x) - H_\alpha(x)H_{1-\alpha}(x) = 1$ for all x where α is neither an integer nor zero.
6. $dF_\alpha/dx(0) = 0$.
7. $\{(x/2)^{1-2\alpha}dH_\alpha/dx(x)\}_{x=0} = 1$.
8. $F_{1/2}(x) = \cosh x$.
9. $H_{1/2}(x) = \sinh x$.

¾), where $p = (\mu + 1)/(\mu + \nu + 2)$. Furthermore, there are no tabulations of functions corresponding to the quotient of two GLF. Consequently, we have introduced our new LCS functions, which provide much of the information desired about such battles. The naming of our LCS functions follows from the facts that a function similar to $F_\alpha(x)$ was introduced by Ludwig Schläfli [1867/68] and that a related one appears in a posthumous fragment of the great English geometer William Kingdon Clifford [1882].

The function $F_\alpha(x)$ satisfies the linear second-order ordinary differential equation

$$d^2F_\alpha/dx^2 + \{(2\alpha - 1)/x\}dF_\alpha/dx - F_\alpha = 0, \qquad (5.3)$$

with initial conditions $F_\alpha(0) = 1$ and $dF_\alpha/dx(0) = 0$, while $H_\alpha(x)$ satisfies

$$d^2H_\alpha/dx^2 - \{(2\alpha - 1)/x\}dH_\alpha/dx - H_\alpha = 0, \qquad (5.4)$$

with initial conditions (for $\alpha > 0$) $H_\alpha(0) = 0$ and $\{(x/2)^{1-2\alpha}dH_\alpha/dx(x)\}_{x=0}$

= 1. Thus, $\{F_\alpha, H_{1-\alpha}\}$ is a fundamental system of solutions to

$$d^2F/dx^2 + \{(2\alpha - 1)/x\}dF/dx - F = 0, \tag{5.5}$$

with Wronskian $W(F_\alpha, H_{1-\alpha}) = (x/2)^{1-2\alpha}$. It follows that the GLF for the X and Y force-level equations for combat modeled by (1.1) with attrition-rate coefficients (3.5) are given by

$$C_X(t) = F_q(\tau), \qquad S_X(t) = \{\lambda_I/(\mu + \nu + 2)\}^{2q-1}H_p(\tau), \tag{5.6}$$

$$C_Y(t) = F_p(\tau), \qquad S_Y(t) = \{\lambda_I/(\mu + \nu + 2)\}^{1-2q}H_q(\tau), \tag{5.7}$$

where q is given by (4.5), $p = 1 - q$, and $\tau(t)$ is given by (4.6).

If we define

$$T_\alpha(x) = H_{1-\alpha}(x)/F_\alpha(x) \tag{5.8}$$

and let $T_X(t) = S_X(t)/C_X(t)$ denote a hyperbolic-like GLF corresponding to the hyperbolic tangent, then

$$T_X(t) = \{\lambda_I/(\mu + \nu + 2)\}^{2q-1}T_q(\tau). \tag{5.9}$$

Taylor [1979b] shows that for μ and $\nu > -1$, $T_X(t)$ is a strictly increasing function with range $[0, 1/Q^*)$ for $t \in [0, +\infty)$ and that

$$Q^* = \{\Gamma(q)/\Gamma(p)\}\{\lambda_I/(\mu + \nu + 2)\}^{1-2q}. \tag{5.10}$$

Consequently (2.2) and (5.9) yield that $T_\alpha(x)$ is strictly increasing with $T_\alpha(0) = 0$ and

$$\lim_{x\to+\infty}T_\alpha(x) = \Gamma(1 - \alpha)/\Gamma(\alpha). \tag{5.11}$$

6. USE OF LCS FUNCTIONS FOR ANALYZING COMBAT

The Lanchester-Clifford-Schläfli (LCS) functions $F_\alpha(x)$ and $H_\alpha(x)$ are very useful for analyzing "aimed-fire" combat modeled by the power attrition-rate coefficients with "no offset" (3.5). Here, we assume that the attrition-rate coefficients (3.5) hold for all $t \geq 0$ in the mathematical model (1.1). Recall (see Section 3 above) that one must be careful not to use any results computed from the mathematical model out of the range of definition of the tactical scenario describing the tactical situation considered in any particular application. For such combat, these LCS functions may be used to (1) compute force levels as a function of time, (2) predict force annihilation, and (3) compute the time of force annihilation. We will now show how to obtain this information.

According to (1.3), (5.6), and (5.7), we may write the X force level as

$$x(t) = x_0\{F_p(\tau_0)F_q(\tau) - H_q(\tau_0)H_p(\tau)\}$$
$$- y_0\sqrt{\lambda_R}\{\lambda_I/(\mu + \nu + 2)\}^{2q-1}\{F_q(\tau_0)H_p(\tau) - H_p(\tau_0)F_q(\tau)\}. \tag{6.1}$$

From (5.11) and (6.1) (see Taylor [1979b] for details), we may conclude the following force-annihilation-prediction result. (Alternatively, we may substitute (5.6), (5.7), and (5.10) into Theorem 1 to obtain Theorem 2.)

THEOREM 2: *Consider combat between two homogeneous forces modeled for all $t \geq 0$ by (1.1) with power attrition-rate coefficients (3.5). Assume that μ and $\nu > -1$. Then the X force will be annihilated in finite time if and only if*

$$
\begin{aligned}
x_0/y_0 < \sqrt{\lambda_R}(\lambda_I/(\mu + \nu + 2))^{q-p}(\Gamma(p)/\Gamma(q))\{[F_q(\tau_0) \\
- [\Gamma(q)/\Gamma(p)]H_p(\tau_0)]/[F_p(\tau_0) - [\Gamma(p)/\Gamma(q)]H_q(\tau_0)]\}.
\end{aligned}
\tag{6.2}
$$

When $\tau_0 = 0$ (i.e. $K_S = 0$), the X force will be annihilated to finite time if and only if

$$
x_0/y_0 < \sqrt{\lambda_R}(\lambda_I/(\mu + \nu + 2))^{q-p}(\Gamma(p)/\Gamma(q)).
\tag{6.2a}
$$

However, one must verify that force annihilation does not occur out of the range of definition of the tactical scenario for any particular application (e.g. after t_{max} for Bonder's constant-speed attack model considered in Section 3 above). Turning now to the determination of the time at which annihilation occurs for the mathematical model in which the attrition-rate coefficients (3.5) have been assumed to hold for all $t \geq 0$, we see that when (6.2) is satisfied, the time to annihilate the X force, t_a^X, is determined by $x(t_a^X) = 0$. It follows that

$$
\begin{aligned}
T_q[\tau(t_a^X)] = [x_0 F_p(\tau_0) + y_0\sqrt{\lambda_R}[\lambda_I/(\mu + \nu + 2)]^{q-p} \\
\cdot H_p(\tau_0)]/[x_0 H_q(\tau_0) + y_0\sqrt{\lambda_R}[\lambda_I/(\mu + \nu + 2)]^{q-p}F_q(\tau_0)],
\end{aligned}
\tag{6.3}
$$

or, more explicitly,

$$
\begin{aligned}
t_a^X = \tau^{-1}\{T_q^{-1}[[x_0 F_p(\tau_0) + y_0\sqrt{\lambda_R}[\lambda_I/(\mu + \nu + 2)]^{q-p} \\
\cdot H_p(\tau_0)]/[x_0 H_q(\tau_0) + y_0\sqrt{\lambda_R}[\lambda_I/(\mu + \nu + 2)]^{q-p}F_q(\tau_0)]]\},
\end{aligned}
\tag{6.4}
$$

where τ^{-1} and T_q^{-1} denote inverse functions.

7. AVAILABLE TABULATIONS OF LCS FUNCTIONS

Tabulations of the Lanchester-Clifford-Schläfli functions, which are given in Taylor and Brown [1977b, 1977c], are available from the National Technical Information Service. These reports contain five-decimal-place tables of the hyperbolic-like LCS functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for values of the argument $x = 0.00(0.01)2.00(0.1)10.0$ and various values of the order α. The short table (Taylor and Brown [1977c]) contains tabulations for eleven values of α in the range $(0, 1)$ corresponding to μ, $\nu = 0, 1, 2, 3$; while the longer table (Taylor and Brown

[1977b]) contains tabulations for 26 values of α in the same range corresponding to μ, $\nu = 0$, $\frac{1}{4}$, $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, 3. As we have seen above in Section 3 (see (3.2), (3.4), and Figure 2), such values of μ and ν allow one to analyze, for example, a wide variety of range capabilities for weapon systems in Bonder's [1965] constant-speed attack model (3.2). These tables have been calculated by the recursive methods given in Section 8 of Taylor and Brown [1976].

8. OUTLINE OF COMPUTATIONAL PROCEDURE

The above-mentioned tabulations of these new LCS functions make the analysis of an important class of Lanchester-type battles a comparatively easy matter. Before we consider numerical examples to show that insights may be easily obtained into the dynamics of combat, let us outline the general computational procedure (based on the results given in Section 6) that one follows in the analysis of such combat. Accordingly, the basic steps involved are as follows:

(1) Determine from (6.2) whether the X force can be annihilated,
(2) If annihilation is possible, determine the time of the X force's annihilation as follows:
  (a) Compute $T_q(\tau_a{}^X)$ by (6.3) [here $\tau_a{}^X = \tau(t_a{}^X)$],
  (b) Using interpolation, determine $\tau_a{}^X$ from the appropriate tabulation of T_q, and
  (c) Using (4.5), compute $t_a{}^X = \tau^{-1}(\tau_a{}^X)$.

Note from the above that these two determinations involve only the initial force ratio $u_0 = x_0/y_0$ (and not the individual initial force levels themselves). Additionally, one must verify that such numerical results hold within the range of definition of the tactical scenario in any particular application. For example, in the examples of the following section, we applied the above computational procedure to Bonder's constant-speed attack model for which the tactical scenario is only defined for $0 \leq t \leq t_{max} = r_0/v$. In these examples, when the X force is not annihilated within this given time t_{max}, we calculated the final X force level by (6.1) with the help of our tabulations.

9. NUMERICAL EXAMPLES

In the section we examine a couple of numerical examples to show how our results lead to insights about the dynamics of combat between two homogeneous forces. As in Taylor [1974] and Taylor and Brown [1976], we consider Bonder's [1965] model (3.2) for the constant-speed attack against a static defensive position. We will focus on the new results of this paper [in particular, the prediction of battle outcome from initial conditions without explicitly computing the force-level trajectories] and

will follow the computational procedure outlined in the previous section. Here the tactical scenario is defined only for $0 \leq t \leq t_{max}$, since the constant-speed attack ends at $t_{max} = r_0/v$. Hence, for $t > t_{max}$ one must not use any results from the mathematical model in which the attrition-rate coefficients (3.5) have been assumed to hold for all time. From the input data given in Table II, we compute the parameter values shown in Table III. We observe from Table 8B of Taylor and Brown [1977b] and Table III above, the predicted agreement between $\Gamma(1 - \alpha)/\Gamma(\alpha)$ and the limiting value of $T_a(x)$ as $x \rightarrow +\infty$ (see (5.11)) for $\alpha = q = \%$ (recall (4.5)). We now consider two cases: (I) $r_0 = 2000$ meters, and (II) $r_0 = 1250$ meters. The interested reader can find these examples worked out in even more detail in Taylor and Brown [1977b, 1977c].

When $r_0 = 2000$ meters (see Figure 3 of Taylor [1974]), we have $K_S = 0$ and $\tau_0 = 0$. The maximum time that the battle can last is $t_{max} = 14.91$ minutes, since at this time the attackers reach their final objective (i.e the defensive position). We now consider the qualitative behavior of the

TABLE II
INPUT DATA FOR NUMERICAL EXAMPLES

$\mu = 1, \nu = 2$.
$\alpha_0 = 0.06X$ casualties/minute/Y unit.
$\beta_0 = 0.6Y$ casualties/minute/X unit.
$r_a = r_\beta = 2000$ meters.
$v = 5$ miles/hour.

$\mu = 1$, $\nu = 2$ force-level trajectory shown in Figure 3 of Taylor [1974]. Theorem 2 tells us that X can be annihilated if and only if $x_0/y_0 < 0.420$. By (6.3), the annihilation time of the X force is given by $T_q[\tau(t_a^X)] = 3.544 \, x_0/y_0$. For $x_0 = 10$, $y_0 = 30$, we have $T_q(\tau_a^X) = 1.8122$ so that from Table 8A of Taylor and Brown [1977b] (using linear interpolation) we obtain $\tau_a^X = 1.009$. Hence, (4.6) yields $t_a^X = 14.24$ minutes and $r_a^X = 89.8$ meters. Further results are given in Table IV.

When $r_0 = 1250$ meters (see Figure 3 of Taylor and Brown [1976]), we have $K_S = 5.5923$ meters, $\tau_0 = 0.0975$, and $t_{max} = 9.32$ minutes. In this case (again, for $\mu = 1$, $\nu = 2$), X can be annihilated if and only if $x_0/y_0 < 0.382$ with (from (6.3)) the annihilation time of the X force given by $T_q(\tau_a^X) = (3.565u_0 + 0.223)/(0.156u_0 + 1.004)$, where $u_0 = x_0/y_0$. Some further numerical results are given in Table V. Again, these parametric results should be contrasted with the single $\mu = 1$, $\nu = 2$ force-level trajectory shown in Figure 3 of Taylor and Brown [1976].

10. FINAL REMARKS

In Sections 6 and 9, we have seen how our new definition of power Lanchester functions (guided by the general requirements for GLF given

in Taylor and Brown [1977a]) allows one to conveniently obtain much valuable information about the model (1.1) with attrition-rate coefficients (3.5) *without explicitly computing the entire force-level trajectories.* In his well-known survey paper on the Lanchester theory of combat, Dolansky suggested the development of such outcome-predicting relations without solving in detail and/or computing force-level trajectories as one of several problems for further research. Our Theorem 2 is a step toward resolving this problem (see also Taylor and Parry; Taylor and Comstock; Taylor [1979a]). Previously, one was limited to being able to compute only force-level trajectories, but now we can tell who is going to be annihilated (and when) without explicitly computing the trajectories.

We have answered questions about qualitative model behavior (e.g. force annihilation), not only for specific values of, for example, initial force levels, but also for the entire possible range of values for the initial force ratio (i.e. parametric analysis of model behavior). The results of this paper may be used for other parametric analyses (see Bonder [1971] for a lucid discussion of the importance of such analyses), e.g. parametric

TABLE III

PARAMETER VALUES FOR NUMERICAL EXAMPLES

$k_a = 4.0233 \times 10^{-3}$ X casualties/(minute)^p/Y unit.
$k_b = 2.6979 \times 10^{-3}$ Y casualties/(minute)^q/X unit.
$p = 2/5, q = 3/5$.
$\Gamma(p)/\Gamma(q) = 1.48951$.
$K_{O=0}$.

dependence of battle outcome on attrition-rate coefficients. Thus, our new results now allow one to develop important insights into the dynamics of combat between two homogeneous forces with temporal variations in fire effectiveness. With the availability (Taylor and Brown [1977b, 1977c]) of tabulations of the LCS functions, one can now analyze combat modeled by the power attrition-rate coefficients (3.5) with somewhat the same facility as he can for the constant-coefficient case and thus aid in parameter analyses.

In his classic paper, Lanchester [1914] considered constant fire effectiveness for individual firers and deduced his famous square law

$$\beta\{x_0^2 - x^2(t)\} = \alpha\{y_0^2 - y^2(t)\}, \tag{10.1}$$

where α and β denote the constant attrition-rate coefficients. It follows from (10.1) that (provided there is no "time limit" for combat)

X *will be annihilated if and only if* $x_0/y_0 < \sqrt{\alpha/\beta}$. (10.2)

Thus, we see that equality of Lanchester-type fighting strengths depends

TABLE IV

ANNIHILATION OF THE X FORCE AS A FUNCTION OF THE INITIAL FORCE RATIO
FOR $r_0 = 2000$ METERS

(x_0/y_0)	$t_a{}^X$(Minutes)	$r_a{}^X$(Meters)
0.333	14.24	89.8
0.250	11.61	443.2
0.200	10.19	633.2

on two parameters: (I) initial force ratio, and (II) relative effectiveness. When the timing of military actions is also considered, we add a third parameter, the intensity of combat $= \sqrt{\alpha\beta}$, to this list of significant combat parameters. No such simple relation like the square law (10.1), which yields (10.2), holds in general for variable attrition-rate coefficients. However, by transforming the independent variable t to normalize the battle's time scale by the intensity of combat, we found that the course of such variable-coefficient combat depends on only the two weapon-system parameters: (I) relative fire effectiveness, $R(t) = a(t)/b(t)$, and (II) intensity of combat, $I(t) = \sqrt{a(t)b(t)}$. This way of viewing the attrition-rate coefficients $a(t)$ and $b(t)$ is both intuitively appealing and also important because under some circumstances relative fire effectiveness (i.e. only one parameter) plays the major role in determining battle outcome (e.g. when $a(t)/b(t) \equiv$ constant, the intensity of combat does not influence the outcome of battle (provided that there is no time limitation)). (See also Taylor [1980b].) Moreover, we did extend (10.2) to combat modeled with the power attrition-rate coefficients with "no offset" (3.5) (see Theorem 2). This is the *first* time that such a generalization of the square law has been obtained for the variable-coefficient Lanchester-type model (1.1) with $a(t)/b(t) \neq$ constant. We observe that for $K_S > 0$, this "exact" outcome-prediction relation (i.e. necessary and sufficient condition for force annihilation) involves higher transcendental functions (here, the LCS functions) and is complementary to the sufficient condition (involving only elementary functions) given by Taylor and Parry for $K_S > 0$.

Work by Bonder [1965, 1967, 1970], Clark [1969], Barfoot [1969], and

TABLE V

ANNIHILATION OF THE X FORCE AS A FUNCTION OF THE INITIAL FORCE RATIO
FOR $r_0 = 1250$ METERS

(x_0/y_0)	$t_a{}^X$(Minutes)	$r_a{}^X$(Meters)
0.333	10.63	—[a]
0.250	7.56	235.9
0.200	6.17	422.8

[a] $t_{max} = 9.32$ minutes and $x_f = x(r = 0) = 1.35$.

Bonder and Farrell on the prediction of Lanchester attrition-rate coefficients (see Taylor and Brown [1976] for further discussion and references) has generated interest in variable-coefficient Lanchester-type models. Interest in the power attrition-rate coefficients with "no offset" (3.5) is provided by S. Bonder's [1965] model (3.2) and his examination of predicted attrition rates for various weapon systems (see pp. 196–200 of Bonder and Farrell). However useful our results may be in their own right, they have far greater import: (I) they are a model for the treatment of other Lanchester functions and their tabulations, and (II) they may be used in the numerical determination of the parity-condition parameter Q^* for related attrition-rate coefficients (e.g. (3.1) with $K_0 > 0$). In Taylor and Brown [1978], we show how our tabulations of the LCS functions play a key role in the numerical determination of the parity condition parameter Q^* for the general power attrition-rate coefficients (3.1) with positive "offset" (i.e. $K_0 > 0$).

We have extended our mathematical theory (Taylor and Brown [1976]) of variable-coefficient Lanchester-type equations of "modern warfare" for combat between two homogeneous forces in order to be able to more thoroughly analyze such models (see also Taylor and Brown [1977a]). The classic ordinary-differential-equation theories (e.g. see Hille [1969]) were inadequate to supply all the answers sought about such combat models (Taylor [1979b]). The mathematical theory of the model (1.1) with coefficients (3.5) is now nearly as complete as that of the constant-coefficient model. Such results as we have given here are very useful for understanding the dynamics of combat, i.e. how the trading of casualties will be projected over time. H. K. Weiss [1959] has emphasized that such a simplified combat model is particularly valuable when it leads to a clearer understanding of such significant relationships that would tend to be obscured in a more complex model.

ACKNOWLEDGMENTS

REFERENCES

ABRAMOWITZ, M., AND I. A. STEGUN. 1964. *Handbook of Mathematical Functions*. National Bureau of Standards Applied Mathematics Series, No. 55, Washington, D.C.

BARFOOT, C. B. 1969. The Lanchester Attrition-Rate Coefficient: Some Comments on Seth Bonder's Paper and a Suggested Alternate Method. *Opns. Res.* **17**, 888–894.

BONDER, S. 1965. A Generalized Lanchester Model to Predict Weapon Performance in Dynamic Combat, Ph.D. Thesis, The Ohio State University, Columbus, Ohio (also available from University Microfilms International, P.O. Box 1764, Ann Arbor, Mich. 48106 as Publication No. 65-9341).

BONDER, S. 1967. The Lanchester Attrition-Rate Coefficient. *Opns. Res.* **15**, 221–232.

BONDER, S. 1970. The Mean Lanchester Attrition Rate. *Opns. Res.* **18**, 179–181.

BONDER, S. 1971. Systems Analysis: A Purely Intellectual Activity. *Milit. Rev.* **51**, No. 2, 14–23.

BONDER, S., AND R. L. FARRELL (eds.). 1970. Development of Models for Defense Systems Planning, Report No. SRL 2147 TR 70-2, Systems Research Laboratory, The University of Michigan, Ann Arbor, Mich., September (also available from Nat. Tech. Info. Service, U. S. Department of Commerce, 5285 Port Royal Road, Springfield, Va. as AD 714 677).

BONDER, S., AND J. G. HONIG. 1971. An Analytical Model of Ground Combat: Design and Application. In *Proceedings of the Tenth Annual U. S. Army Operations Research Symposium*, pp. 319–394, Durham, N.C.

CLARK, G. M. 1969. The Combat Analysis Model, Ph.D. thesis, The Ohio State University, Columbus, Ohio (also available from University Microfilms International as Publication No. 69-15, 905).

CLIFFORD, W. K. 1882. *Mathematical Papers*. Macmillan and Co., London (reprinted by Chelsea Publishing Co., New York, 1968).

COMMAND AND CONTROL TECHNICAL CENTER (CCTC). 1979. VECTOR-2 System for Simulation of Theater-Level Combat, TM 201-79, Washington, D.C., January.

DOLANSKY, L. 1964. Present State of the Lanchester Theory of Combat. *Opns. Res.* **12**, 344–358.

GEOFFRION, A. M. 1976. The Purpose of Mathematical Programming Is Insight, Not Numbers. *Interfaces* **7**, No. 1, 81–92.

HILLE, E. 1969. *Lectures on Ordinary Differential Equations*. Addison-Wesley, Reading, Mass.

HUBER, R. K., L. F. JONES AND E. REINE (eds.). 1975. *Military Strategy and Tactics*. Plenum Press, New York.

HUBER, R. K., K. NIEMEYER AND H. W. HOFMANN (eds.). 1979. *Operationsanalytische Spiele für die Verteidigung/Operational Research Games for Defense*. R. Oldenbourg Verlag, München.

IGNALL, E. J., P. KOLESAR AND W. E. WALKER. 1978. Using Simulation to Develop and Validate Analytic Models: Some Case Studies. *Opns. Res.* **26**, 237–253.

KNOPP, K. 1956. *Infinite Sequences and Series.* Dover, New York.

LANCHESTER, F. W. 1914. Aircraft in Warfare: The Dawn of the Fourth Arm—No. V, The Principle of Concentration. *Engineering* **98**, 422–423 (reprinted in *The World of Mathematics*, pp. 2138–2148, J. Newman (ed.). Simon & Schuster, New York, 1956).

MORRIS, W. T. 1967. On the Art of Modelling. *Mgmt. Sci.* **13**, B-707–B-717.

SCHLÄFLI, L. 1867/68. Sulle relazioni tra diversi integrali definiti che giovano ad esprimere la soluzione generale della equazione di Riccati. *Ann. Mat. pura Appl.* [2] **1**, 232–242 (also in *Gesammelte Mathematische Abhandlungen, Band III*, pp. 85–94. Birkhäuser, Basel, 1956).

TAYLOR, J. G. 1974. Solving Lanchester-Type Equations for "Modern Warfare" with Variable Coefficients. *Opns. Res.* **22**, 756–770.

TAYLOR, J. G. 1979a. Recent Developments in the Lanchester Theory of Combat. In *Operational Research '78, Proceedings of the Eighth IFORS International Conference on Operational Research*, pp. 773–806, K. B. Haley (ed.). North-Holland, Amsterdam.

TAYLOR, J. G. 1979b. Prediction of Zero Points of Solutions to Lanchester-Type Differential Combat Equations for Modern Warfare. *SIAM J. Appl. Math.* **36**, 438–456.

TAYLOR, J. G. 1980a. *Force-on-Force Attrition Modelling.* Military Applications Section of ORSA, Alexandria, Va.

TAYLOR, J. G. 1980b. Dependence of the Parity-Condition Parameter on the Combat-Intensity Parameter for Lanchester-Type Equations of Modern Warfare. *OR Spektrum* **1**, 199–205.

TAYLOR, J. G. 1982. Analysis of the Functional Dependence of the Modified Parity-Condition Parameter on Its Argument for an Important Class of Lanchester-Type Equations for Modern Warfare. *Int. J. Syst. Sci.* **13**, 115–136.

TAYLOR, J. G., AND G. G. BROWN. 1976. Canonical Methods in the Solution of Variable-Coefficient Lanchester-Type Equations of Modern Warfare. *Opns. Res.* **24**, 44–69.

TAYLOR, J. G., AND G. G. BROWN. 1977a. Further Canonical Methods in the Solution of Variable-Coefficient Lanchester-Type Equations of Modern Warfare, Tech. Report NPS55-77-27, Naval Postgraduate School, Monterey, Calif., June (also available from Nat. Tech. Info. Service as AD A044 302).

TAYLOR, J. G., AND G. G. BROWN. 1977b. A Table of Lanchester-Clifford-Schläfli Functions, Tech. Report NPS55-77-39, Naval Postgraduate School, Monterey, Calif., October (also available from Nat. Tech. Info. Service as AD A050 248).

TAYLOR, J. G., AND G. G. BROWN. 1977c. A Short Table of Lanchester-Clifford-Schläfli Functions, Tech. Report NPS55-77-42, Naval Postgraduate School, Monterey, Calif., October (also available from Nat. Tech. Info. Service as AD A049 863).

TAYLOR, J. G., AND G. G. BROWN. 1978. Numerical Determination of the Parity Condition Parameter for Lanchester-Type Equations of Modern Warfare. *Comput. Opns. Res.* **5**, 227–242.

TAYLOR, J. G., AND C. COMSTOCK. 1977. Force-Annihilation Conditions for Variable-Coefficient Lanchester-Type Equations of Modern Warfare. *Naval Res. Log. Quart.* **24**, 349–371.

TAYLOR, J. G., AND S. H. PARRY. 1975. Force-Ratio Considerations for Some Lanchester-Type Models of Warfare. *Opns. Res.* **23,** 522–533.

WEISS, H. K. 1957. Lanchester-Type Models of Warfare. In *Proceedings of the First International Conference on Operational Research,* pp. 82–98, M. Davies, R. T. Eddison, and T. Page (eds.). Operations Research Society of America, Baltimore, Md.

WEISS, H. K. 1959. Some Differential Games of Tactical Interest and the Value of a Supporting Weapon System. *Opns. Res.* **7,** 180–196.