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Tome 46 N° 13 1er JUILLET 1985

LE JOURNAL DE PHYSIQUE - LETTRES

J. Physique Lett. **46** (1985) L-575 - L-583 1er JUILLET **1985,** PAGE L- 575

Classification
Physics Abstracts
05.40 — 47.25J

Anomalous diffusion on a selfsimilar hierarchical structure

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(*Reçu le 21 janvier 1985, accepté sous forme définitive le 9 mai 1985*)

Résumé. — Nous étudions la croissance temporelle des moments de la distribution de particules diffusant sur un fractal à portée de saut variable avec une coupure inférieure. Les paramètres essentiels sont : le taux de croissance, le facteur d'échelle de la longueur et celui du temps le long de la hiérarchie ; ce dernier critère est nouveau. Nous trouvons des lois de croissance algébriques et exponentielles et des corrections logarithmiques, ou un piégeage si la coupure est éliminée. Une augmentation anormale du taux de croissance de la variance $\sigma \propto t^{\theta}$, θ étant supérieur à 2, comme cela a déjà été observé pour la turbulence, est obtenue pour la première fois.

Abstract. — The temporal increase of the moments in diffusion on a fractal with variable hopping range and lower cut-off is given. The essential parameters are the growth ratio, the length scaling and, as a new feature, the time scaling along the hierarchy. We find algebraical or exponential increase, logarithmic corrections, or trapping if the cut-off is removed. For the first time anomalous enhancement of the variance increase $\sigma \propto t^{\theta}$, θ larger than 2, is obtained as observed in turbulence.

1. Introduction.

In normal diffusion the distribution of the particles is asymptotically Gaussian ; the variance increases linearly with time, other central moments either vanish (odd) or increase like powers of the variance (even)

$$\langle (\delta x)^m \rangle \propto t^{\theta_m}, \qquad \theta_m = m/2, \qquad m \text{ even integer}. \tag{1.1}$$

This normal diffusion is a very widespread phenomenon, since its physical basis — the statistical independence of successive steps during the random walk — is rather generic. But there are several mechanisms which result in « anomalous » diffusion, i.e. deviations from equation (1.1). Among those are long-lasting correlations implying $1 < \theta_2 \leqq 2$, fluctuating transition rates or hopping widths as in amorphous substances resulting in $0 < \theta_2 < 1$ [1], restrictions in the random walker's available sites due to percolation structure or fractal selfsimilarity also characterized by $\theta_2 < 1$ [2-4], and convective transport by the eddies in a fully developed turbulent fluid flow with $\theta_2 \cong 3$ [5].

Several of these mechanisms seem to determine turbulent relative diffusion (particle pair separation) simultaneously : long lasting correlations due to slow decay of the relevant eddies, a fractal, scaling structure of the eddies, fluctuating transition rates due to intermittency, and, in addition, two aspects which are usually not met : the hopping range depends on the level of the hierarchy, i.e. on the eddies' extension, and also the transition rate or hopping time-scale depends on that level.

These features of diffusive transport in fully developed turbulence have motivated us to introduce hierarchical models with variable range hopping and to investigate the anomalous diffusion on such models. In reference [6] we have studied a nested hierarchy of d-dimensional intervals whose levels scale with a spatial scale factor μ. The underlying idea is deterministic, chaos-induced diffusion as defined for example by discrete-time mapping by a broken linear map showing fractal selfsimilarity [7]. Under such mapping which enjoys a variable hopping range step function distributions remain step functions and their time development can be described in terms of transition probabilities. We found in [6] either exponential increase of the moments or trapping (where the lower order moments approach a finite limit). The physical reason for trapping is the possibility of transitions to arbitrary low levels below a given one.

We have modified that model in three respects to meet more closely turbulent transport in the inertial subrange. (i) We have introduced a lower cut-off of levels. (ii) There is mapping only to the *next* lower (and next higher, as before) level. (iii) The transition rates scale with the level number. These aspects reflect the Kolmogoroff dissipation length, the eddy decay into about half or twice as big ones, and the decreasing decay rates with increasing eddy size, respectively.

The coupling of the eddy size x, its energy $\sim v^2(x)$, and decay rate $\tau^{-1}(x)$ has a deep-lying physical origin unique to turbulence. According to our present understanding of fully developed turbulence (Richardson, Kolmogoroff, Oboukhoff, von Weizsäcker, Heisenberg, Onsager) the basic parameter in the inertial subrange is the energy dissipation rate $\varepsilon \sim v^2(x)/\tau(x)$. Its scale invariance implies $v \sim x^{1/3}$ and $\tau \sim x^{2/3}$. Denoting the spatial scaling in our hierarchy with μ and the rate constant's scaling with s these are related accordingly by

$$\mu^2 s^3 = 1 \tag{1.2}$$

if intermittency is neglected. Simple scaling would predict $\theta_m = m \ln \mu / \ln s^{-1}$ for our model, but we will find that this relation holds only for a restricted range of the model parameters. Deviations from this law as well as logarithmic corrections and exponential diffusion occur in other regions of the parameter space.

Like our original model in [6], henceforth denoted as I, also the modified model can be solved by analytical methods. In this Letter we present the main results together with a summarizing comparison with I. Only the basic steps of the derivation are indicated ; details are published either in [6] or in [8].

2. The model.

We consider a set of d-dimensional intervals arranged in levels $k = 0, 1, 2, \ldots$ (Fig. 1). The intervals of a given level are labelled by $i = 0, 1, 2, \ldots$; these are all of the same size. The spatial extension of adjacent levels scales with a factor $\mu > 1$; the size of the lowest level ($k = 0$) intervals is chosen arbitrarily, say 1. Phase points in a given interval (k, i) are assumed to cover it homogeneously ;

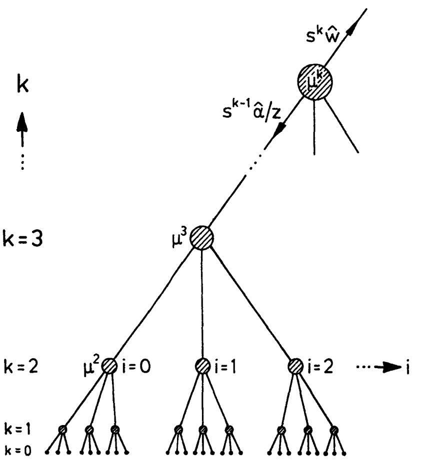


Fig. 1. — Vertex and transition structure of the hierarchical set of intervals with $z = 3$. For examples of realizations in the d-dimensional Euclidian space the reader is referred to [6].

$p_{k,i}(t)$ denotes the probability to be in this interval. After an infinitesimal change of the continuous time t the phase points in (k, i) either stay there, or are mapped upwards the hierarchy with rate $s^k \hat{w}$ to $(k + 1, [i/z])$, or downwards with transition rate $s^{k-1}\,\hat{\alpha}/z$ into each of the z subordinate intervals $(k - 1, iz + j)$, $j = 0, 1, ..., z - 1$. z denotes the branching ratio, $[i/z]$ is the largest integer smaller or equal to i/z. The fractal dimension D of the hierarchy is $D = \ln z/\ln \mu$; it is $D < d$, since $z\mu^{-d} < 1$.

The hierarchy is meant to represent the structure in the space of two-particle distances, one particle being attached to the origin. This space is *not* translational (but scale) invariant. According to these ideas the model is defined by the master equation for the probabilities

$$\begin{aligned} \dot{p}_{0,i} &= - \hat{w}p_{0,i} + (\hat{\alpha}/z)\, p_{1,[i/z]}\,, \quad k = 0\,, \\ \dot{p}_{k,i} &= - (\hat{w}s^k + \hat{\alpha}s^{k-1})\, p_{k,i} + (\hat{\alpha}/z)\, s^k\, p_{k+1,[i/z]} + \hat{w}s^{k-1} \sum_{j=0}^{z-1} p_{k-1,iz+j}\,, \quad k > 0\,. \end{aligned} \tag{2.1}$$

Henceforth we measure the time in units of $\hat{w}^{-1}$, the upward transition time. The remaining parameters are the branching ratio z, the rate scaling factor s, and

$$r = \hat{w}/\hat{\alpha} \tag{2.2}$$

which we denote as the « growth ratio ». If $r > 1$ the phase points are predominantly climbing upwards (with an expected exponential increase of the moments). If $r < 1$ they are mainly mapped downwards the level-hierarchy; this would yield trapping (cf. I), but in the present model that cannot happen due to the existence of a *finite* lowest level, $k = 0$. Still one expects suppression of the diffusive spreading, resulting in a power law increase of the moments. The borderline $r = 1$ represents a balance between up- and downward trend and will give rise to logarithmic corrections.

One starts at $t = 0$ with $p_{k,i}(0) = \delta_{k,0}\,\delta_{i,0}$, i.e. a homogeneous distribution of all phase points in the single interval (0, 0). The normalization $\sum_{k,i} p_{k,i} = 1$ is conserved for all times, as one easily verifies by summing (2.1) over all k, i.

A phase point which has reached but never exceeded a certain level $l \geqq 0$ up to time t can only be in one of the subsets (k, i) with $k \leqq l$ and $i \leqq i_{\max} = z^{l-k} - 1$ at time t. This observation suggest to introduce $q_k^{(l)}(t)$, defined as the probability to be in the kth level at time t without having ever exceeded level l up to this time. Of course, $q_k^{(l)} = 0$ if $k > l$. The sets $\{ q_k^{(l)}(t) \}$ and $\{ p_{k,i}(t) \}$ can be calculated from one another (cf. I).

The master equation for the $q_k^{(l)}(t)$ reads (using the dimensionless time already)

$$\begin{aligned} \dot{q}_0^{(l)} &= - q_0^{(l)} + r^{-1} q_1^{(l)}, \quad k = 0, \\ \dot{q}_k^{(l)} &= - (r^{-1} s^{k-1} + s^k) q_k^{(l)} + r^{-1} s^k q_{k+1}^{(l)} + s^{k-1} q_{k-1}^{(l)}, \quad k > 0. \end{aligned} \tag{2.3}$$

The initial condition is $q_k^{(l)}(0) = \delta_{k,0}$. The branching ratio z does not appear anymore. All quantities of physical interest which can be calculated from the q's depend on the two parameters s (transition rate scaling of adjacent levels) and r (growth ratio) only. Among these are the moments $\langle x^m \rangle (t)$, which are given by

$$\langle x^m \rangle (t) \cong \sum_{l=0}^{\infty} \mu^{ml} q^{(l)}(t) = \sum_{l=0}^{\infty} \mu^{ml}(\hat{q}^{(l)} - \hat{q}^{(l-1)}). \tag{2.4}$$

Here the notation is used

$$\hat{q}^{(l)}(t) = \sum_{k=0}^{l} q_k^{(l)}(t). \tag{2.5a}$$

the probability to be at any level $0 \leqq k \leqq l$ without having ever exceeded level l up to time t, and

$$q^{(l)}(t) = \hat{q}^{(l)}(t) - \hat{q}^{(l-1)}(t), \tag{2.5b}$$

the probability that a phase point has indeed reached level l in the hierarchy but without having exceeded it, up to time t. Since these points are spread over the interval $(l, 0)$ and its directly and indirectly subordinate intervals (k, i), $k \leqq l$, $i \leqq z^{l-k} - 1$, and thus over a region of linear extension of order μ^l, the moment is estimated by equation (2.4).

We conclude this section with the remark that the discrete time model in I can easily be transformed to the continuous time version presented here by putting

$$t = n/N, \qquad \alpha = \hat{\alpha}/N, \qquad w = \hat{w}/N, \qquad N \to \infty. \tag{2.6}$$

w and α/z denote the transition probabilities from a given set (k, i) to the next larger one and to one of the subordinate lower ones in a discrete time step $n \to n + 1$. The growth ratio $w/\alpha = \hat{w}/\hat{\alpha} = r$ is the same in the discrete or continuous version. The transition rates in I are assumed to be independent of the level k, i.e. $s = 1$; all results then depend on the growth ratio r only.

3. Eigenvalues.

The solution of the master equation (2.3) can be obtained in terms of eigenfunctions and eigenvalues. Note, that in I for the discrete time model *without* lower cut-off, i.e. $k = ..., -2, -1, 0, 1, 2, ...$ we gave the explicit solution not expanded into eigensolutions but, instead, as sums over binomial coefficient expressions.

Substituting $\dot{q}_k^{(l)} = - \lambda^{(l)} q_k^{(l)}$ in (2.3) defines $l + 1$ eigenvalues $\lambda_i^{(l)}$ and eigenvectors for fixed l which are numbered from $i = 0$ to $i = l$ with increasing size. All eigenvalues are positive. There are two important boundaries which separate the regions A, B, C, and D with different behaviour of the eigenvalues $\lambda_i^{(l)}$.

The heavy line separates the region $s < 1$ with ever decreasing transition rates along the hierarchy from that with $s > 1$, speading up transitions for higher levels. $\lambda_i^{(l)}$ behaves roughly like s^{l-i} for $s < 1$ and like s^i for $s > 1$. There are slight r-dependent deviations at $i \cong 0$ and $i \cong l$, which, apart from that for $\lambda_0^{(l)}$ stated in (3.2) are not essential for the diffusion behaviour. At the borderline $s = 1$ the eigenvalues are bounded from below and above by

$$(1 - r^{-1/2})^2 \leqq \lambda_i^{(l)} \leqq (1 + r^{-1/2})^2 \tag{3.1}$$

independent of i and l. Thus, with the exception of $r = 1$, equation (3.1) provides a lower non-zero bound for the λ's.

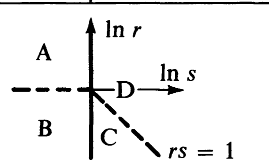
The other boundary (heavy broken line) singles out the region $r < 1$, $rs < 1$. In this region the description given above does not apply for the lowest eigenvalue. Instead it is given by

$$\lambda_0^{(l)} = c(rs)^l , \qquad c = (1 - r)(1 - rs) > 0 \tag{3.2}$$

for large l. Consequently, the system is governed by two time scales, a large one $\sim (rs)^{-l}$ and a shorter one ranging between $\sim s^{-l}$ and ~ 1.

The behaviour of the eigenvalues in the different parameter ranges and at their corresponding boundaries is summarized in table I.

Table I. — *Eigenvalues $\lambda_i^{(l)}$ and temporal behaviour of the moments $\langle x^m \rangle(t)$, $m \geqq 1$, in the various regions of the parameter space (r, s). The regions are displayed in the inset. The boundaries of two regions are labelled by the letters of the adjacent regions; for the centre $r = s = 1$ take DA. The constants are $c = (1 - r)(1 - rs)$, $a = (\mu^m - 1)(1 - \mu^{-m}/r)$. The exponents θ are defined in (4.7) through (4.9). l labels the levels, i the eigenvalues in increasing order ($i = 0, 1, ..., l$). The notion « bounded » refers to (3.3). Always $\mu > 1$, therefore $\mu^m rs \leqq 1$ does not exist in region D and its boundaries CD and DA; this part of the table is taken for the inset.*

Region	Eigenvalues $\lambda_i^{(l)}$ lowest eigenvalue ($i = 0$)	other eigenvalues ($i > 0$)	Diffusion behaviour of $\langle x^m \rangle(t)$ for $\mu^m rs < 1$	$\mu^m rs = 1$	$\mu^m rs > 1$
A			t^{θ_v}		
AB	$\sim s^l/l$	$\sim s^{l-i}$	$(t/\ln t)^{\theta_v}$	$t/\ln t$	$t^{\theta_v}/\ln t$
B	$c(rs)^l$		$t^{\theta_<}$	$t \ln t$	$t^{\theta_>}$
BC		bounded		t^2	e^{at}
C		$\sim s^i$			∞
CD	$\sim 1/l$				
D					
DA	bounded				e^{at}

4. Diffusion.

Diffusion is described by the moments (2.4). The $\hat{q}^{(l)}(t)$ can be evaluated without the knowledge of the eigenvectors. Performing the sum over k in the master equation (2.3) for fixed l yields

$$\dot{\hat{q}}^{(l)}(t) = - s^l q_l^{(l)}(t) . \tag{4.1}$$

Its meaning is that the decay of $\hat{q}^{(l)}(t)$ is only possible by the escape of points from the largest level l. $q_l^{(l)}(t)$ can be expressed by the eigenvalues,

$$q_l^{(l)}(t) = \sum_{i=0}^{l} c_i^{(l)} e^{-\lambda_i^{(l)} t} . \tag{4.2}$$

At $t = 0$ the probability $q_l^{(l)}(t)$ and all its derivatives up to the $(l - 1)$-th vanish. Furthermore, $\hat{q}^{(l)}(0) = 1$ and $\hat{q}^{(l)}(\infty) = 0$, hence from (4.1)

$$\int_0^\infty q_l^{(l)}(t)\, \mathrm{d}t = s^{-l} . \tag{4.3}$$

These properties determine the coefficients $c_i^{(l)}$ uniquely,

$$c_i^{(l)} = s^{-l} \lambda_i^{(l)} \prod_{\substack{j=0 \\ (j \neq i)}}^{l} \lambda_j^{(l)}/(\lambda_j^{(l)} - \lambda_i^{(l)}) . \tag{4.4}$$

With (4.4) in (4.2) we get for large l in the respective regions

$$q_l^{(l)}(t) = \begin{cases} g_s(s^l t) , & \mathrm{A} , \\ cr^l \, \mathrm{e}^{-c(rs)^l t} \, G_s(s^{l-1} t) , & \mathrm{B} , \\ cr^l \, \mathrm{e}^{-c(rs)^l t} \, G_{s-1}(t) , & \mathrm{C} , \\ s^{-l} g_{s-1}(t), & \mathrm{D} . \end{cases} \tag{4.5}$$

Here, $g_s(\tau) = \mathrm{d}G_s(\tau)/\mathrm{d}\tau$ and $G_s(\tau)$ is defined for $s < 1$ by

$$\begin{aligned} G_s(\tau) &= 1 - \pi_\infty(s) \sum_{n=0}^{\infty} (-1)^n \pi_n(s)\, s^{n(n+1)/2} \, \mathrm{e}^{-\tau/s^n} , \\ \pi_n(s) &= \prod_{i=1}^{n} (1 - s^i)^{-1} . \end{aligned} \tag{4.6}$$

$G_s(\tau)$ increases monotonously from 0 at $\tau = 0$ to 1 at $\tau = \infty$. All its derivatives vanish for $\tau = 0$. If s approaches 1, the transition gets sharper (Fig. 2).

From $q_l^{(l)}(t)$ (4.5) and $\hat{q}^{(l)}(t)$ *via* (4.1) the moments (2.4) can be estimated; a more precise and detailed elaboration is given elsewhere [8].

Because the hierarchy of levels has a lowest level $k = 0$, all phase points eventually will reach arbitrary high levels and will thus migrate to arbitrary large distances. Therefore the moments $\langle x^m \rangle(t)$ will increase with time for all r, s. The rate of increase is calculated from (2.4), (4.1),

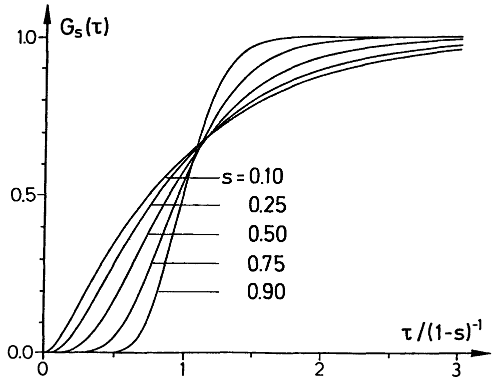


Fig. 2. — The function $G_s(\tau)$ for various values of the parameter $s \in (0, 1)$. Both the position and the width of the transition range increase with s as $\sim (1 - s)^{-1}$ and $\sim (1 - s^2)^{-1/2}$, the relative width decreases $\sim [(1 - s)/(1 + s)]^{1/2}$.

and (4.5). We distinguish the cases $\mu^m rs < 1, > 1$, and the marginal value $= 1$. In table I we have summarized the behaviour of $\langle x^m \rangle(t)$ *vs.* t in the various parameter regions for the three ranges of $\mu^m rs$. We obtained a power law increase, an exponentially strong increase of moments, or even infinite moments for all $t > 0$. There are logarithmic corrections at the border lines. The three relevant exponents are given by

$$\theta_\vee = \ln \mu^m / \ln s^{-1}, \tag{4.7}$$

$$\theta_< = \ln \mu^m / \ln (rs)^{-1} < 1, \tag{4.8}$$

$$\theta_> = \ln (\mu^m r) / \ln s^{-1} > 1. \tag{4.9}$$

In region B the exponent θ_m increases linearly with m with a slope κ

$$\kappa_< = \ln \mu / \ln (rs)^{-1}, \qquad \mu^m rs < 1. \tag{4.10}$$

At $\mu^{m_c} rs = 1$ there is a kink to another slope

$$\kappa_> = \ln \mu / \ln s^{-1}, \qquad \mu^m rs > 1. \tag{4.11}$$

At the marginal value $m_c = \ln (rs)^{-1} / \ln \mu$ there is a logarithmic correction. From the constraints $\mu^2 rs \lessgtr 1$ we conclude that $\theta_2 = \theta_<$ is smaller than one, while $\theta_2 = \theta_>$ describes a system with enhanced, though algebraic anomalous diffusion; $\theta_>$ may even exceed 2, the value known from highly correlated random walk.

In region A we also have $\theta_m \propto m$ with a slope

$$\kappa_\vee = \ln \mu / \ln s^{-1}, \qquad r > 1, \quad s < 1. \tag{4.12}$$

$\theta_\vee$ can be less or larger than unity.

While the *suppression* of moment growth for diffusion on fractals is known from recent work [2-4, 9], we have described here for the first time diffusion on selfsimilar structures with anomalously *enhanced* algebraic variance increase $\theta_2 > 2$. This happens if $s < 1$ and $\mu > s^{-1} \times \max(1, r^{-1/2})$.

5. Infinite fractal.

If there is no lower cut-off and the fractal extends to arbitrary small (and large) scales we found in [6] the phenomenon of trapping. We briefly give the results in the continuum limit (2.6) which are obtained from equations (3.17), (3.20), (3.37), and (3.38) of reference [6], model I. Since the transition rates in this model are independent of the level, one has $s = 1$.

If $r > 1$ we get exponential increase

$$\langle x^m \rangle(t) \propto \exp \Gamma_m t, \qquad \Gamma_m = (\mu^m - 1)(\mu^m r - 1)/(r\mu^m), \tag{5.1}$$

for all moments $m \geqq 1$.

If, instead, $r < 1$, one observes trapping. There is an asymptotic normalized distribution

$$q^{(l)}(t \to \infty) \equiv \sum_{k=-\infty}^{l} (q_k^{(l)} - q_k^{(l-1)}) = (1 - r)\, r^l, \qquad l = 0, 1, 2, \dots \tag{5.2}$$

for the probability to have reached but not exceeded level l. This implies finite moments

$$\langle x^m \rangle(t \to \infty) = (1 - r)/(1 - r\mu^m), \quad \text{if} \quad r\mu^m < 1. \tag{5.3}$$

Moments of sufficiently large order m, namely if $r\mu^m > 1$, are infinite. The approach to infinity is exponential, described also by (5.1).

The marginal case $r\mu^{m_c} = 1$ yields a moment which grows linearly in t.

$$\langle x^{m_c} \rangle(t) \propto t\,, \qquad m_c = \ln r^{-1}/\ln \mu\,. \tag{5.4}$$

If $m_c = 2$, this corresponds to normal diffusion. Note the difference to diffusion on the fractal *with lower cut-off* : If $s = 1$, $r < 1$, but $r\mu^{m_c} = 1$ we have (Table I, BC) enhanced diffusion $\langle x^{m_c} \rangle(t) \propto t^2$.

6. Applications.

What do we learn for turbulence ? Since larger eddies have larger turnover times we choose $s < 1$. The condition of scale invariant energy dissipation can be met by $\mu^2 = s^{-3}$, cf. (1.2). From table I, regions A, AB, and B we can formulate the following behaviour for the temporal increase of the variance ($m = 2$) for relative diffusion.

If the growth ratio r is larger than one, i.e. dominance of upward transitions,

$$\sigma_t \cong \langle x^2 \rangle(t) \propto t^3\,, \qquad r > 1 > s\,. \tag{6.1}$$

If downward transitions dominate, $r < 1$, there is less diffusion enhancement.

$$\sigma_t \propto \begin{cases} t^{\theta_>}\,, \quad \theta_> = 3 - \dfrac{\ln r^{-1}}{\ln s^{-1}} \in (1, 3)\,, & s^2 < r < 1\,, \\ t \ln t\,, & s^2 = r < 1\,, \\ t^{\theta_<}\,, \quad \theta_< = 3\Big/\left(1 + \dfrac{\ln r^{-1}}{\ln s^{-1}}\right) < 1\,, & r < s^2 < 1\,. \end{cases} \tag{6.2}$$

For balanced upward and downward trend, $r = 1$, we find

$$\sigma_t \propto t^3/\ln t\,, \quad s^2 < r = 1\,. \tag{6.3}$$

Remarkable are the logarithmic corrections and the general tendency of *reducing* the diffusion if the downward transitions dominate. Usually intermittency is suspected to *increase* the exponent of diffusion [5, 10, 11]. Thus one has to be aware of counteracting influences of dynamical fluctuations. Present data do not yet allow to make reliable conclusions on the deviations from $\theta = 3$.

In addition to the moments also correlation functions can be evaluated. For example, the probability to be in the subset (0, 0) at time t (provided one starts in (0, 0) at $t = 0$) is

$$p_{0,0}(t) = \sum_{l=0}^{\infty} z^{-l}(q_0^{(l)} - q_0^{(l-1)})\,. \tag{6.4}$$

In regions B and C we have

$$q_0^{(l)}(t) \cong (1 - r) \exp(-\, c(rs)^l\, t)\,. \tag{6.5}$$

The main contribution comes from $l \sim l_<(t)$ with $ct(rs)^{l_<} = 1$, leading to

$$p_{0,0}(t) \propto z^{-l_<(t)} \propto t^{-\nu}\,, \tag{6.6}$$

$$\nu = \ln z/\ln (rs)^{-1}\,. \tag{6.7}$$

Thus together with anomalous diffusion there is algebraically slow decay of correlations. The scaling relation between variance-exponent $\theta_<$ and autocorrelation exponent ν reads (if $\mu^2\, rs < 1$)

$$\nu = \theta_<\, D/2 \tag{6.8}$$

with D the fractal dimension, which characterizes the nested hierarchy over the range of scales above the cut-off. (This attributes the spectral (fracton) dimension [3, 9] $d_s = 2\nu = D\theta_<$ to the hierarchy.)

We have put emphasis on turbulent diffusion. But hierarchical models are also of interest for modelling glassy and metastable behaviour [12, 13]. In our model the lowest level $k = 0$ might be identified with states of local energy minima, whereas the states at higher levels correspond to saddle point states between adjacent valleys. To make contact with thermally activated transitions we put in equations (6.6) and (6.7).

$$rs = K \exp(-\varepsilon/T). \tag{6.9}$$

T is the temperature, ε the activation energy, and $K \geqq 1$ is the number of saddle points effectively contributing to the transitions between adjacent valleys. The exponent ν of correlation decay then displays a characteristic temperature dependence :

$$\nu = \frac{T \ln z}{\varepsilon + T \ln K^{-1}}. \tag{6.10}$$

The exponent ν diverges at $T_c = \varepsilon/\ln K$ and thus signals the glass transition for this simple model.

Again we have the scaling relation (6.8) connecting thermally actived correlation decay and diffusion on the hierarchical structure

$$\theta_< = \frac{2}{D} \frac{T \ln z}{\varepsilon + T \ln K^{-1}}. \tag{6.11}$$

The scaling law $2\nu/\theta_< = D$ is *independent* of temperature unless the fractal dimension itself should vary with T. (6.8) is known to hold for constant range hopping and scale invariant transition rate. We have extended its domain of applicability for variable hopping *range* and *rate* on the fractal, which is another result of the present work.

Acknowledgments.

This work was supported in part by the Deutsche Forschungsgemeinschaft through Sonderforschungsbereich 123 Statistical Mathematical Models.

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