

Anomalous Hall Effect and Magnetoresistance of Ferromagnetic Metals

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The anomalous Hall effect and the magnetoresistance of ferromagnetic metals are investigated on the basis of the localized d -electron (or f -electron) model. First the Hamiltonian of the interaction between conduction electrons and localized electrons is given, which is valid when the orbital angular momenta of localized electrons are not quenched. Then we find that the scattering matrix of a conduction electron depends in a complicated way on the directions of the initial and final wave vectors and of the magnetization. The transition probability calculated to the first Born approximation by using the above matrix elements leads to the electrical resistivity of the form expressed by Eq. (1. 1) in the text. To the second Born approximation we find that the transition probability from a state \mathbf{k} to another state \mathbf{k}' is not equal to that from \mathbf{k}' to \mathbf{k} and leads to the anomalous Hall effect. The magnitude and the temperature dependence of both effects are reasonable for iron and nickel when compared with experiment.

In rare-earth metals, both effects can also be obtained and, besides, the electric quadrupole moments which are associated with the orbital angular momenta cause an additional scattering. When we calculate the normal resistivity by adding the exchange and quadrupole scatterings, we obtain a good agreement with experiment on the magnetic contribution to the resistivity of rare-earth metals with more than half-filled $4f$ shells.

§ 1. Introduction

The electrical resistivity of ferromagnetic metals shows an anomalous temperature dependence below the Curie temperature. This fact has been explained by Mott¹⁾ on the basis of the band model and by Kasuya²⁾ on the basis of the localized d -electron model. Ferromagnetic metals also show the anomalous Hall effect and the magnetoresistance, which have been given theoretical explanations by Karplus and Luttinger³⁾ and by Smit⁴⁾ on the basis of the band model. On the other hand, the usual s - d interaction is isotropic and does not give rise to these effects. It should be noted, however, that the orbital angular momenta of localized electrons are not quenched in rare-earth metals and even in iron group metals they are remaining because of the spin orbit coupling. So the interaction between conduction electrons and the orbital angular momenta of localized electrons must be taken into account as well as that between conduction electrons and the spin angular momenta.

We shall show in this paper that the above phenomena can be accounted for by this interaction on the basis of the localized d -electron model. In § 2

we give the Hamiltonian of the interaction between conduction electrons and localized electrons which is valid when the orbital angular momenta of localized electrons are not quenched. In § 3 we calculate the transition probability to the first Born approximation by using the above Hamiltonian and show that the resistivity of cubic transition metals can be expressed by

$$\rho = \rho_0 + \rho_1 (\boldsymbol{\sigma} \cdot \boldsymbol{\zeta})^2 + \rho_2 \sum_{i>j} \sigma_i \sigma_j \zeta_i \zeta_j, \tag{1.1}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\zeta}$ are unit vectors in the directions of the applied electric field and of the magnetization, respectively. In § 4 the transition probability is calculated to the second Born approximation, where it is found that the probability from a state \mathbf{k} to another state \mathbf{k}' is not equal to that from \mathbf{k}' to \mathbf{k} . This fact leads to the anomalous Hall effect and it is found that the electric current of transition metals is expressed by

$$\mathbf{j} = \sigma_{||} \mathcal{E} (\boldsymbol{\sigma} + \alpha \boldsymbol{\zeta} \times \boldsymbol{\sigma}), \tag{1.2}$$

where \mathcal{E} is the magnitude of the applied field, $\sigma_{||}$ is the normal electrical conductivity and α is a constant which depends on temperature. In § 5 we discuss the anomalous Hall effect of rare-earth metals. We also note there that, when we calculate the resistivity of rare-earth metals above the Curie temperature by adding the exchange and quadrupole scatterings, we obtain a good agreement with experiment on the magnetic contribution to the resistivity of rare-earth metals with more than half-filled 4*f* shells.

§ 2. The basis Hamiltonian

We start with the following Hamiltonian which represents the interaction between conduction electrons and localized electrons :

$$\begin{aligned} \mathcal{H} = & \sum_{n\mathbf{k}\mathbf{k}'} N^{-1} e^{i(\mathbf{k}'-\mathbf{k})\mathbf{R}_n} \times \\ & \sum_{mm'} \{ (b_{nm+}^* b_{nm'+} + b_{nm-}^* b_{nm'-}) (a_{k+}^* a_{k'+} + a_{k-}^* a_{k'-}) V_{mm'}(\mathbf{k}, \mathbf{k}') \\ & - (b_{nm+}^* b_{nm'+} a_{k+}^* a_{k'+} + b_{nm+}^* b_{nm'-} a_{k-}^* a_{k'+} \\ & + b_{nm-}^* b_{nm'+} a_{k+}^* a_{k'-} + b_{nm-}^* b_{nm'-} a_{k-}^* a_{k'-}) J_{mm'}(\mathbf{k}, \mathbf{k}') \}, \end{aligned} \tag{2.1}$$

where N is the total number of atoms in the crystal, $a_{k\pm}$ and $b_{nm\pm}$ are the destruction operators of the conduction electron with wave number \mathbf{k} and of the localized electron in the m -th orbital state of the n -th ion, respectively, and V and J are defined by

$$V_{mm'}(\mathbf{k}, \mathbf{k}') = N e^{-i(\mathbf{k}'-\mathbf{k})\mathbf{R}_n} \iint \phi_{nm}^*(1) \phi_{\mathbf{k}}^*(2) (e^2/r_{12}) \phi_{nm'}(1) \phi_{\mathbf{k}'}(2) dv_1 dv_2, \tag{2.2}$$

$$J_{mm'}(\mathbf{k}, \mathbf{k}') = N e^{-i(\mathbf{k}'-\mathbf{k})\mathbf{R}_n} \iint \phi_{nm}^*(1) \phi_{\mathbf{k}}^*(2) (e^2/r_{12}) \phi_{\mathbf{k}'}(1) \phi_{nm'}(2) dv_1 dv_2, \tag{2.3}$$

where $\phi_{\mathbf{k}}$ and ϕ_{nm} are the Bloch function with a wave vector \mathbf{k} and the m -th

localized wave function of the n -th site with $3d$ or $4f$ character, respectively. Usually, m was restricted to be equal to m' .

If the total orbital and spin angular momenta, \mathbf{L}_n and \mathbf{S}_n , of the n -th ion are assumed to be good quantum numbers, Eq. (2.1) can be represented in terms of \mathbf{L}_n and \mathbf{S}_n . For this purpose we have to do two things. Firstly, we rewrite the product of the operators b and b^* in the usual coordinate space and put

$$b_{nm\pm}^* b_{nm'\pm} = \sum_{i=1}^{n_0} p_{mm'}(i) (\pm s_{i\xi} + 1/2), \quad (2.4)$$

$$b_{nm\mp}^* b_{nm'\pm} = \sum_{i=1}^{n_0} p_{mm'}(i) s_{i\mp}, \quad (2.5)$$

where n_0 is the number of $3d$ (or $4f$) electrons in the n -th ion and $p_{mm'}(i)$ is a function of the orbital angular momentum of the i -th electron, which has the following property :

$$\int \psi_{nm_1}^*(i) p_{mm'}(i) \psi_{nm_2}(i) d\mathbf{v}_i = \delta_{mm_1} \delta_{m'm_2}, \quad (2.6)$$

\mathbf{s}_i is the spin operator for the i -th electron, $\boldsymbol{\xi}$ is a unit vector in the direction of the magnetization and $s_{i\mp}$ are defined by

$$s_{i\mp} = s_{i\xi} \mp i s_{i\eta}, \quad (2.7)$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are unit vectors perpendicular to $\boldsymbol{\xi}$ and to each other. Secondly, we calculate the explicit dependence of $V_{mm'}(\mathbf{k}, \mathbf{k}')$ and $J_{mm'}(\mathbf{k}, \mathbf{k}')$ on \mathbf{k} and \mathbf{k}' by expanding the Bloch function $\psi_{\mathbf{k}}(\mathbf{r})$ as follows :

$$\psi_{\mathbf{k}}(\mathbf{r}) = V^{-1/2} \sum_s (2s+1) i^s j_s(kr) P_s(\cos \theta_{kr}), \quad (2.8)$$

which represents a plane wave, when j_s is the spherical Bessel function. V is the total volume of the crystal. In (2.8) we restrict ourselves to the terms of $s=0$ and 1, that is, to the lowest order in which we can obtain the relevant effects. The localized wave function has the form

$$\psi_{nm}(\mathbf{r}) = P(|\mathbf{r}-\mathbf{R}_n|) / |\mathbf{r}-\mathbf{R}_n| Y_{lm}(\theta, \varphi). \quad (2.9)$$

Then, by calculating $V_{mm'}$ and $J_{mm'}$ for all combinations of m and m' , we obtain the following results :

$$\sum_{mm'} p_{mm'}(i) V_{mm'}(\mathbf{k}, \mathbf{k}') = R_{00} + R_0(\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}') - 2R_1 \langle \mathbf{l}_i \mathbf{l}_i \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle, \quad (2.10)$$

$$\begin{aligned} \sum_{mm'} p_{mm'}(i) J_{mm'}(\mathbf{k}, \mathbf{k}') \\ = F_0 + 2F_1(\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}') + (3i/2) F_2 \mathbf{l}_i \cdot (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) - F_3 \langle \mathbf{l}_i \mathbf{l}_i \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle, \quad (l=2) \quad (2.11) \\ = F_0 + 2F_1(\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}') + iF_2 \mathbf{l}_i \cdot (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) - (2/5) F_3 \langle \mathbf{l}_i \mathbf{l}_i \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle, \quad (l=3) \end{aligned}$$

where $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ are the unit vectors in the directions of \mathbf{k} and \mathbf{k}' and

$$R_{00} = (Ne^2/V) 4\pi R_0^c(0, 0), \quad (2.12)$$

$$R_0 = (Ne^2/V) 12\pi R_0^c(1, 1), \tag{2.13}$$

$$R_1 = (Ne^2/V) (12\pi/35) R_2^c(1, 1), \tag{l=2}$$

$$= (Ne^2/V) (4\pi/25) R_2^c(1, 1), \tag{l=3} \tag{2.14}$$

$$F_0 = (Ne^2/V) (4\pi/5) R_2^{ex}(0, 0), \tag{l=2}$$

$$= (Ne^2/V) (4\pi/7) R_3^{ex}(0, 0), \tag{l=3} \tag{2.15}$$

$$F_1 = (Ne^2/V) (4\pi/5) \{R_1^{ex}(1, 1) + (9/14) R_3^{ex}(1, 1)\}, \tag{l=2}$$

$$= (Ne^2/V) (18\pi/35) \{R_2^{ex}(1, 1) + (20/27) R_4^{ex}(1, 1)\}, \tag{l=3} \tag{2.16}$$

$$F_2 = (Ne^2/V) (4\pi/5) \{R_1^{ex}(1, 1) - (3/7) R_3^{ex}(1, 1)\}, \tag{l=2}$$

$$= (Ne^2/V) (18\pi/35) \{R_2^{ex}(1, 1) - (5/9) R_4^{ex}(1, 1)\}, \tag{l=3} \tag{2.17}$$

$$F_3 = (Ne^2/V) (4\pi/5) \{R_1^{ex}(1, 1) + (9/49) R_3^{ex}(1, 1)\}, \tag{l=2}$$

$$= (Ne^2/V) (18\pi/35) \{R_2^{ex}(1, 1) + (25/81) R_4^{ex}(1, 1)\}, \tag{l=3} \tag{2.18}$$

$$R_h^c(s, t) = \int_0^\infty \int_0^\infty j_s(k_F r_1) j_t(k_F r_1) (r_<^h/r_>^{h+1}) P(r_2)^2 r_1^2 dr_1 dr_2, \tag{2.19}$$

$$R_h^{ex}(s, t) = \int_0^\infty \int_0^\infty j_s(k_F r_1) j_t(k_F r_2) (r_<^h/r_>^{h+1}) P(r_1) P(r_2) r_1 r_2 dr_1 dr_2. \tag{2.20}$$

The notation $\langle \mathbf{AB} \rangle$ means a tensor of second order, which is obtained from the product of two vectors \mathbf{A} and \mathbf{B} and whose components are defined by

$$\langle \mathbf{AB} \rangle_{\pm 2} = A_{\pm 1} B_{\pm 1},$$

$$\langle \mathbf{AB} \rangle_{\pm 1} = (1/2)^{1/2} (A_{\pm 1} B_0 + A_0 B_{\pm 1}),$$

$$\langle \mathbf{AB} \rangle_0 = (1/6)^{1/2} (A_1 B_{-1} + 2A_0 B_0 + A_{-1} B_1), \tag{2.21}$$

where

$$A_{\pm 1} = \mp (1/2)^{1/2} (A_x \pm iA_y), \quad A_0 = A_z. \tag{2.22}$$

The scalar products of two tensors is defined by

$$\langle \mathbf{LL} \rangle \cdot \langle \mathbf{AB} \rangle \equiv \sum_{i=-2}^2 \langle \mathbf{AB} \rangle_i^* \langle \mathbf{LL} \rangle_i$$

$$\equiv \{(\mathbf{L} \cdot \mathbf{A})(\mathbf{L} \cdot \mathbf{B}) + (\mathbf{L} \cdot \mathbf{B})(\mathbf{L} \cdot \mathbf{A})\} / 2 - L(L+1) (\mathbf{A} \cdot \mathbf{B}) / 3. \tag{2.23}$$

In (2.19) and (2.20), we have assumed $|\mathbf{k}| = |\mathbf{k}'| = k_F$, where k_F is the magnitude of wave vectors at the Fermi surface, because, as will be seen later, only the wave vectors at the Fermi surface concern us, as long as we are concerned with the electric current.

From (2.4), (2.5), (2.10) and (2.11), the Hamiltonian (2.1) can be written as

$$\mathcal{H} = - \sum_{n\mathbf{k}\mathbf{k}'} N^{-1} e^{i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{R}_n} a_{\mathbf{k}}^* a_{\mathbf{k}'}$$

$$\times \sum_{i=1}^{n_0} [2R_1 \langle \mathbf{l}_i \mathbf{l}_i \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle + \{1/2 + 2(\mathbf{s}_i \cdot \mathbf{s}_c)\}]$$

$$\begin{aligned}
& \times \{F_0 + 2F_1(\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}') + (3i/2)F_2 \mathbf{l}_i \cdot (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) - F_3 \langle \mathbf{l}_i \mathbf{l}_i \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle\}], \quad (l=2) \\
= & - \sum_{nkk'} N^{-1} e^{i(k'-k)R_n} a_k^* a_{k'} \\
& \times \sum_{i=1}^{n_0} [2R_1 \langle \mathbf{l}_i \mathbf{l}_i \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle + \{1/2 + 2(\mathbf{s}_i \cdot \mathbf{s}_c)\} \\
& \times \{F_0 + 2F_1(\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}') + iF_2 \mathbf{l}_i \cdot (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) - (2/5)F_3 \langle \mathbf{l}_i \mathbf{l}_i \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle\}], \quad (l=3) \\
& \quad \quad \quad (2 \cdot 24)
\end{aligned}$$

where we have dropped the spin part from the suffixes of a^* and a , which is represented by the spin operator \mathbf{s}_c for a conduction electron.

If we consider the matrix elements of (2·24) between states with the same $|\mathbf{L}_n\rangle$ and $|\mathbf{S}_n\rangle$, Eq. (2·24) can be represented in terms of \mathbf{L}_n and \mathbf{S}_n as follows:

$$\begin{aligned}
\mathcal{H} = & - \sum_{nkk'} N^{-1} e^{i(k'-k)R_n} a_k^* a_{k'} \\
& \times [c_1(2R_1 - F_3/2) \langle \mathbf{L}_n \mathbf{L}_n \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle + (3i/4)F_2 \mathbf{L}_n \cdot (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) \\
& + 2(\mathbf{S}_n \cdot \mathbf{s}_c) \{F_0 + 2F_1(\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}') + (3i/2)c_2 F_2 \mathbf{L}_n \cdot (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) - c_3 F_3 \langle \mathbf{L}_n \mathbf{L}_n \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle\}], \quad (l=2) \\
= & - \sum_{nkk'} N^{-1} e^{i(k'-k)R_n} a_k^* a_{k'} \\
& \times [c_1(2R_1 - F_3/5) \langle \mathbf{L}_n \mathbf{L}_n \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle + (i/2)F_2 \mathbf{L}_n \cdot (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) \\
& + 2(\mathbf{S}_n \cdot \mathbf{s}_c) \{F_0 + 2F_1(\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}') + ic_2 F_2 \mathbf{L}_n \cdot (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) - (2/5)c_3 F_3 \langle \mathbf{L}_n \mathbf{L}_n \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle\}], \quad (l=3) \\
& \quad \quad \quad (2 \cdot 25)
\end{aligned}$$

where c_i are defined by the relations

$$\sum_{i=1}^{n_0} \langle \mathbf{l}_i \mathbf{l}_i \rangle = c_1 \langle \mathbf{L} \mathbf{L} \rangle, \quad (2 \cdot 26)$$

$$\sum_{i=1}^{n_0} (\mathbf{l}_i \cdot \mathbf{A})(\mathbf{s}_i \cdot \mathbf{B}) = c_2 (\mathbf{L} \cdot \mathbf{A})(\mathbf{S} \cdot \mathbf{B}), \quad (2 \cdot 27)$$

$$\sum_{i=1}^{n_0} \langle (\mathbf{l}_i \mathbf{l}_i) \cdot \langle \mathbf{A} \mathbf{B} \rangle \rangle (\mathbf{s}_i \cdot \mathbf{C}) = c_3 \langle \langle \mathbf{L} \mathbf{L} \rangle \cdot \langle \mathbf{A} \mathbf{B} \rangle \rangle (\mathbf{S} \cdot \mathbf{C}), \quad (2 \cdot 28)$$

The values of c_i for various ions are given in Table I.

Table I

	L	S	c_1	c_2	c_3
d^1	2	1/2	1	1	1
d^2	3	1	1/5	1/2	1/10
d^3	3	3/2	-1/5	1/3	-1/15
d^4	2	2	-1	1/4	-1/4
d^6	2	2	1	-1/4	-1/4
d^7	3	3/2	1/5	-1/3	-1/15
d^8	3	1	-1/5	-1/2	1/10
d^9	2	1/2	-1	-1	1
f^1	3	1/2	1	1	1
f^2	5	1	1/3	1/2	1/6
f^3	6	3/2	1/11	1/3	1/33
f^4	6	2	-1/11	1/4	-1/44

	L	S	c_1	c_2	c_3
f^5	5	5/2	-1/3	1/5	-1/15
f^6	3	3	-1	1/6	-1/6
f^8	3	3	1	-1/6	-1/6
f^9	5	5/2	1/3	-1/5	-1/15
f^{10}	6	2	1/11	-1/4	-1/44
f^{11}	6	3/2	-1/11	-1/3	1/33
f^{12}	5	1	-1/3	-1/2	1/6
f^{13}	3	1/2	-1	-1	1

There are two cases in which (2.25) is further simplified.

(1) When the orbital ground state is non-degenerate, we have $\langle 0|\mathbf{L}|0\rangle = \langle 0|\langle \mathbf{LL}\rangle|0\rangle = 0$. Then (2.25) reduces to the usual s - d Hamiltonian

$$\mathcal{H}_0 = -\sum_{n\mathbf{k}\mathbf{k}'} N^{-1} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_n} a_{\mathbf{k}}^* a_{\mathbf{k}'} 2(\mathbf{S}_n \cdot \mathbf{s}_c) J(\mathbf{k}, \mathbf{k}'), \tag{2.29}$$

where

$$J(\mathbf{k}, \mathbf{k}') = F_0 + 2F_1(\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}'). \tag{2.30}$$

If we introduce the spin-orbit coupling

$$\mathcal{H}_{LS} = \lambda \sum_n \mathbf{L}_n \cdot \mathbf{S}_n, \tag{2.31}$$

some amounts of the orbital angular momentum are associated with the spin angular momentum through non-diagonal matrix elements of \mathbf{L}_n and we have corrections to the zeroth-order Hamiltonian \mathcal{H}_0 through a perturbation procedure. When we retain terms up to second order of λ , we have the following expression for the corrected Hamiltonian:

$$\begin{aligned} H = & \mathcal{H}_0 + \sum_i \left\{ \frac{\langle 0|\mathcal{H} - \mathcal{H}_0|i\rangle \langle i|\mathcal{H}_{LS}|0\rangle}{E_0 - E_i} + \text{comp. conj.} \right\} \\ & + \sum_{ii'} \frac{\langle 0|\mathcal{H}_{LS}|i\rangle \langle i|\mathcal{H} - \mathcal{H}_0|i'\rangle \langle i'|\mathcal{H}_{LS}|0\rangle}{(E_i - E_0)(E_{i'} - E_0)} \\ & + \sum_{ii'} \left\{ \frac{\langle 0|\mathcal{H}_{LS}|i\rangle \langle i|\mathcal{H}_{LS}|i'\rangle \langle i'|\mathcal{H} - \mathcal{H}_0|0\rangle}{(E_i - E_0)(E_{i'} - E_0)} + \text{comp. conj.} \right\} \end{aligned} \tag{2.32}$$

When we assume the cubic symmetry around an ion, (2.32) becomes in the case of $l=2$

$$\begin{aligned} H = & -\sum_{n\mathbf{k}\mathbf{k}'} N^{-1} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}_n} a_{\mathbf{k}}^* a_{\mathbf{k}'} \left[2(\mathbf{S}_n \cdot \mathbf{s}_c) J(\mathbf{k}, \mathbf{k}') \right. \\ & + iA_1 F_2 \{ \mathbf{S}_n \cdot (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) + 4c_2 \langle \mathbf{S}_n \mathbf{S}_n \rangle \cdot \langle (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) \mathbf{s}_c \rangle \} \\ & + \sum_{(xy)} (\boldsymbol{\kappa}_x \boldsymbol{\kappa}_y' + \boldsymbol{\kappa}_y \boldsymbol{\kappa}_x') [c_1 (2R_1 - F_3/2) (A_2 + A_4) (S_x S_y + S_y S_x)_n \\ & \quad \left. - 2c_3 A_2 F_3 \{ S_x (\mathbf{S} \cdot \mathbf{s}_c) S_y + S_y (\mathbf{S} \cdot \mathbf{s}_c) S_x \}_n \right] \end{aligned}$$

$$\begin{aligned}
& -c_3 A_4 F_3 \{ (\mathbf{S} \cdot \mathbf{s}_c) (S_x S_y + S_y S_x) + (S_x S_y + S_y S_x) (\mathbf{S} \cdot \mathbf{s}_c) \}_n \Big] \\
& + c_1 (2R_1 - F_3/2) (A_3 + A_5) \\
& \quad \times \{ (S_x^2 - S_y^2)_n (\kappa_x \kappa_x' - \kappa_y \kappa_y') + (2S_z^2 - S_x^2 - S_y^2)_n (\kappa_z \kappa_z' - (\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}')/3) \} \\
& - 2c_3 A_3 F_3 \{ (S_x (\mathbf{S} \cdot \mathbf{s}_c) S_x - S_y (\mathbf{S} \cdot \mathbf{s}_c) S_y)_n (\kappa_x \kappa_x' - \kappa_y \kappa_y') \\
& \quad + (2S_z (\mathbf{S} \cdot \mathbf{s}_c) S_z - S_x (\mathbf{S} \cdot \mathbf{s}_c) S_x - S_y (\mathbf{S} \cdot \mathbf{s}_c) S_y)_n (\kappa_z \kappa_z' - (\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}')/3) \} \\
& - c_3 A_5 F_3 \{ ((\mathbf{S} \cdot \mathbf{s}_c) (S_x^2 - S_y^2) + (S_x^2 - S_y^2) (\mathbf{S} \cdot \mathbf{s}_c))_n (\kappa_x \kappa_x' - \kappa_y \kappa_y') \\
& \quad + ((\mathbf{S} \cdot \mathbf{s}_c) (2S_z^2 - S_x^2 - S_y^2) + (2S_z^2 - S_x^2 - S_y^2) (\mathbf{S} \cdot \mathbf{s}_c))_n \\
& \quad \quad \quad \times (\kappa_z \kappa_z' - (\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}')/3) \} \Big], \tag{2.33}
\end{aligned}$$

where $\sum_{(xy)}$ means the summation over cyclic permutation of x, y and z . A_i are defined by

$$A_1 = \frac{3\lambda}{2} \cdot \frac{\langle 0 | L_x | F_{2x} \rangle \langle F_{2x} | L_x | 0 \rangle}{E_0 - E_{F_2}}, \tag{2.34}$$

$$A_2 = \frac{\lambda^2}{2} \cdot \frac{\langle 0 | L_x | F_{2x} \rangle \langle F_{2x} | L_x L_y + L_y L_x | F_{2y} \rangle \langle F_{2y} | L_y | 0 \rangle}{(E_{F_2} - E_0)^2} \tag{2.35}$$

$$A_3 = \frac{\lambda^2}{2} \cdot \frac{\langle 0 | L_x | F_{2x} \rangle \langle F_{2x} | L_x^2 - L_y^2 | F_{2x} \rangle \langle F_{2x} | L_x | 0 \rangle}{(E_{F_2} - E_0)^2}, \tag{2.36}$$

$$A_4 = \lambda^2 \cdot \frac{\langle 0 | L_z | F_{2z} \rangle \langle F_{2z} | L_y | F_{1x} \rangle \langle F_{1x} | L_y L_z + L_z L_y | 0 \rangle}{(E_{F_2} - E_0) (E_{F_1} - E_0)}, \tag{2.37}$$

$$A_5 = \lambda^2 \cdot \frac{\langle 0 | L_z | F_{2z} \rangle \langle F_{2z} | L_z | E_{x^2-y^2} \rangle \langle E_{x^2-y^2} | (3L_z^2 - L(L+1)) / 2 | 0 \rangle}{(E_{F_2} - E_0) (E_E - E_0)}, \tag{2.38}$$

when the orbital ground state has the A_2 symmetry and by

$$A_1 = \frac{3\lambda}{2} \cdot \frac{\langle 0 | L_x | F_{1x} \rangle \langle F_{1x} | L_x | 0 \rangle}{E_0 - E_{F_1}}, \tag{2.39}$$

$$A_2 = \frac{\lambda^2}{2} \cdot \frac{\langle 0 | L_x | F_{1x} \rangle \langle F_{1x} | L_y L_x + L_x L_y | F_{1y} \rangle \langle F_{1y} | L_y | 0 \rangle}{(E_{F_1} - E_0)^2}, \tag{2.40}$$

$$A_3 = \frac{\lambda^2}{2} \cdot \frac{\langle 0 | L_x | F_{1x} \rangle \langle F_{1x} | L_x^2 - L_y^2 | F_{1x} \rangle \langle F_{1x} | L_x | 0 \rangle}{(E_{F_1} - E_0)^2}, \tag{2.41}$$

$$A_4 = \lambda^2 \cdot \frac{\langle 0 | L_z | F_{1z} \rangle \langle F_{1z} | L_y | F_{2x} \rangle \langle F_{2x} | L_y L_z + L_z L_y | 0 \rangle}{(E_{F_1} - E_0) (E_{F_2} - E_0)}, \tag{2.42}$$

$$A_5 = \lambda^2 \cdot \frac{\langle 0 | L_z | F_{1z} \rangle \langle F_{1z} | L_z | E_{3z^2-r^2} \rangle \langle E_{3z^2-r^2} | (3L_z^2 - L(L+1)) / 2 | 0 \rangle}{(E_{F_1} - E_0) (E_E - E_0)}, \tag{2.43}$$

when the orbital ground state has the A_1 symmetry. Here F_{2x} , F_{2y} and F_{2z} , for example, denote the wave functions of the excited states with the F_2 symmetry, which transform like $x(y^2 - z^2)$, $y(z^2 - x^2)$ and $z(x^2 - y^2)$ under the cubic trans-

formation. For $L=3$, A_i become

$$A_1 = 6\lambda / (E_0 - E_{F_2}), \quad A_2 = 15\lambda^2 / (E_{F_2} - E_0)^2, \\ A_4 = 30\lambda^2 / (E_{F_2} - E_0)(E_{F_1} - E_0), \quad A_3 = A_5 = 0. \quad (2.44)$$

The first term in rectangular brackets of (2.33) is the usual s - d interaction, while the second term comes from the second-order perturbation and is linear in the spin-orbit coupling constant λ and is proportional to $(\mathbf{k}' \times \mathbf{k})$. The other terms come from the third-order perturbation and are proportional to λ^2 .

(2) In rare-earth metals except Sm and Eu, $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is a constant of motion and (2.25) can be represented in terms of \mathbf{J} as follows:

$$H = - \sum_{n\mathbf{k}\mathbf{k}'} N^{-1} e^{i(\mathbf{k}' - \mathbf{k})\mathbf{R}_n} a_{\mathbf{k}}^* a_{\mathbf{k}'} \left[d_1 (2R_1 - F_3/5) \langle \mathbf{J}_n \mathbf{J}_n \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle \right. \\ + 2(g_J - 1) (\mathbf{J}_n \cdot \mathbf{s}_c) J(\mathbf{k}, \mathbf{k}') + (i/2) (2 - g_J) F_2 \mathbf{J}_n \cdot (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) \\ + 2id_3 F_3 \langle \mathbf{J}_n \mathbf{J}_n \rangle \cdot \langle (\boldsymbol{\kappa}' \times \boldsymbol{\kappa}) \mathbf{s}_c \rangle - (4/75) d_2 F_3 (\mathbf{J}_n \cdot \mathbf{s}_c) (\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}') \\ - (1/25) d_2 F_3 \{ (\mathbf{s}_c \times \boldsymbol{\kappa}) \cdot (\boldsymbol{\kappa}' \times \mathbf{J}_n) + (\mathbf{s}_c \times \boldsymbol{\kappa}') \cdot (\boldsymbol{\kappa} \times \mathbf{J}_n) \} \\ \left. - (4/5) d_4 F_3 [\boldsymbol{\kappa} \boldsymbol{\kappa}' \mathbf{s}_c] \cdot [\mathbf{J} \mathbf{J} \mathbf{J}] \right], \quad (2.45)$$

where g_J is the Landé g -factor and $[ABC]$ denotes a tensor of third order, whose components are defined by

$$[ABC]_{\pm 3} = A_{\pm 1} B_{\pm 1} C_{\pm 1}, \\ [ABC]_{\pm 2} = (1/3)^{1/2} (A_{\pm 1} B_{\pm 1} C_0 + A_{\pm 1} B_0 C_{\pm 1} + A_0 B_{\pm 1} C_{\pm 1}), \\ [ABC]_{\pm 1} = (1/15)^{1/2} \{ A_{\pm 1} B_{\pm 1} C_{\mp 1} + A_{\pm 1} B_{\mp 1} C_{\pm 1} + A_{\mp 1} B_{\pm 1} C_{\pm 1} \\ + 2(A_{\pm 1} B_0 C_0 + A_0 B_{\pm 1} C_0 + A_0 B_0 C_{\pm 1}) \}, \\ [ABC]_0 = (1/10)^{1/2} (A_1 B_0 C_{-1} + A_0 B_1 C_{-1} + A_1 B_{-1} C_0 + A_{-1} B_1 C_0 + A_{-1} B_0 C_1 \\ + A_0 B_{-1} C_1 + 2A_0 B_0 C_0). \quad (2.46)$$

The scalar product of two tensors $[\boldsymbol{\kappa} \boldsymbol{\kappa}' \mathbf{s}_c]$ and $[\mathbf{J} \mathbf{J} \mathbf{J}]$ is defined by

$$[\boldsymbol{\kappa} \boldsymbol{\kappa}' \mathbf{s}_c] \cdot [\mathbf{J} \mathbf{J} \mathbf{J}] = \sum_{i=-3}^3 [\boldsymbol{\kappa} \boldsymbol{\kappa}' \mathbf{s}_c]_i^* [\mathbf{J} \mathbf{J} \mathbf{J}]_i. \quad (2.47)$$

In (2.45), d_i are defined by the equations

$$\langle \mathbf{L} \mathbf{L} \rangle = (d_1/c_1) \langle \mathbf{J} \mathbf{J} \rangle, \\ 6(\mathbf{S} \times \mathbf{L}) \times \mathbf{L} + 4L(L+1)\mathbf{S} = (d_2/c_3) \mathbf{J}, \\ \langle \mathbf{L} \mathbf{S} \rangle = (d_3/c_2) \langle \mathbf{J} \mathbf{J} \rangle, \\ [\mathbf{L} \mathbf{L} \mathbf{S}] = (d_4/c_3) [\mathbf{J} \mathbf{J} \mathbf{J}]. \quad (2.48)$$

§ 3. Magnetoresistance

We shall calculate the electrical resistivity, regarding the Hamiltonian (2.33) or (2.45) as a perturbation on the motion of conduction electrons which causes

the deviation from the periodic potential. In this section we take (2.33) and calculate the transition probabilities to the first Born approximation. In calculating the resistivity, we make the following assumptions.

(1) The energy of a conduction electron is given by $E_k = \hbar^2 \mathbf{k}^2 / 2m$, m representing an effective mass.

(2) There is no correlation between two quantities which are related to two different localized spins. This enables us to apply the molecular field approximation to the localized spin system.

(3) The localized spin system is in thermal equilibrium.

(4) The number of conduction electrons with up spin is equal to that with down spin. The last two assumptions are not quite essential. We expect that the errors due to these assumptions may be small.

The rate of change of the distribution function f_k^\pm of the conduction electron with +spin due to collision with localized spins is given by

$$\begin{aligned} (\partial f_k^\pm / \partial t)_{coll} = & \sum_{k'} W(\mathbf{k}' +, \mathbf{k} +) (f_{k'}^\pm - f_k^\pm) + \sum_{k'} W(\mathbf{k}' -, \mathbf{k} +) (f_{k'}^\mp - f_k^\pm) \\ & + \sum_{k'} \sum_M N W(\mathbf{k}' - M + 1, \mathbf{k} + M) \{ \omega_{M+1} f_{k'}^\mp (1 - f_k^\pm) - \omega_M f_k^\pm (1 - f_{k'}^\mp) \} \\ & + \sum_{k'} \sum_M N W(\mathbf{k}' + M + 1, \mathbf{k} + M) \{ \omega_{M+1} f_{k'}^\pm (1 - f_k^\pm) - \omega_M f_k^\pm (1 - f_{k'}^\pm) \} \\ & + \sum_{k'} \sum_M N W(\mathbf{k}' + M - 1, \mathbf{k} + M) \{ \omega_{M-1} f_{k'}^\pm (1 - f_k^\pm) - \omega_M f_k^\pm (1 - f_{k'}^\pm) \} \\ & + \sum_{k'} \sum_M N W(\mathbf{k}' - M + 2, \mathbf{k} + M) \{ \omega_{M+2} f_{k'}^\mp (1 - f_k^\pm) - \omega_M f_k^\pm (1 - f_{k'}^\mp) \}, \end{aligned} \quad (3.1)$$

where ω_M represents the probability with which S_z takes a value M and $W(\mathbf{k}' +, \mathbf{k} +)$ represents the transition probability from a state $\mathbf{k}' +$ to another state $\mathbf{k} +$, which is also equal to that of the reverse process. The other probabilities have similar meaning. $W(\mathbf{k}' - M + 1, \mathbf{k} + M)$, for example, denotes the probability of the process in which an electron in a state $\mathbf{k}' -$ is scattered into another state $\mathbf{k} +$, while S_z of a localized spin changes from $M + 1$ to M . This probability is equal to that of the reverse process. When there is no external electric field, this expression vanishes, if

$$f_k^\pm = f_k^0 \equiv 1 / \{ \exp(E_k - E_F) / kT + 1 \} \quad (3.2)$$

and

$$\omega_M = \frac{\exp(g\beta H_{eff} M / kT)}{\sum_M \exp(g\beta H_{eff} M / kT)}, \quad (3.3)$$

where β is the Bohr magneton and H_{eff} is the effective field on the localized spin.

When the external field is applied, f_k^\pm deviate from f_k^0 as follows:

$$f_k^\pm = f_k^0 - \mathcal{C} g_k^\pm (\partial f_k^0 / \partial E_k), \quad (3.4)$$

while, from the assumption (3), we take the original value (3.3) for ω_M .

Using the relation (3.4) and the conservation relation of energy, which is implicit in the expressions of the transition probabilities, we can write (3.1) as follows:

$$\begin{aligned}
 (\partial f_k^+ / \partial t)_{col} = & \mathcal{E}(\partial f_k^0 / \partial E_k) \sum_{k'} \{ [W(\mathbf{k}' +, \mathbf{k} +) (g_k^+ - g_{k'}^+) + W(\mathbf{k}' -, \mathbf{k} +) (g_k^+ - g_{k'}^-) \\
 & + \sum_M N W(\mathbf{k}' - M + 1, \mathbf{k} + M) \omega_M \{ (1 - f_{k'}^0) / (1 - f_k^0) \} (g_k^+ - g_{k'}^-) \\
 & + \sum_M N W(\mathbf{k}' + M + 1, \mathbf{k} + M) \omega_M \{ (1 - f_{k'}^0) / (1 - f_k^0) \} (g_k^+ - g_{k'}^+) \\
 & + \sum_M N W(\mathbf{k}' + M, \mathbf{k} + M + 1) \omega_M (f_{k'}^0 / f_k^0) (g_k^+ - g_{k'}^+) \\
 & + \sum_M N W(\mathbf{k}' - M + 2, \mathbf{k} + M) \omega_M \{ (1 - f_{k'}^0) / (1 - f_k^0) \} (g_k^+ - g_{k'}^-) \}]. \quad (3.5)
 \end{aligned}$$

We note that $(\partial f_k^- / \partial t)_{col}$ can be expressed by a similar expression, which we shall not present here.

From (2.33) and the assumption (2), the transition probabilities are given up to the second order of λ by

$$W(\mathbf{k}' +, \mathbf{k} +) = (2\pi/\hbar) \delta(E_k - E_{k'}) |H_{k+, k'}|^2 = \delta(E_k - E_{k'}) (U_1 + V_1 + V') \quad (3.6)$$

$$W(\mathbf{k}' -, \mathbf{k} +) = (2\pi/\hbar) \delta(E_k - E_{k'}) |H_{k', -, k}|^2 = \delta(E_k - E_{k'}) V_2, \quad (3.7)$$

$$\begin{aligned}
 \sum_M N W(\mathbf{k}' - M + 1, \mathbf{k} + M) \omega_M = & (2\pi/\hbar) \delta(E_k - E_{k'} + g\beta H_{eff}) \sum_M N |H_{k' - M + 1, k + M}|^2 \omega_M \\
 = & \delta(E_k - E_{k'} + g\beta H_{eff}) (U_2 + V_3), \quad (3.8)
 \end{aligned}$$

$$\begin{aligned}
 \sum_M N W(\mathbf{k}' + M + 1, \mathbf{k} + M) \omega_M = & (2\pi/\hbar) \delta(E_k - E_{k'} + g\beta H_{eff}) \sum_M N |H_{k' + M + 1, k + M}|^2 \omega_M \\
 = & \delta(E_k - E_{k'} + g\beta H_{eff}) (V_5 + V''), \quad (3.9)
 \end{aligned}$$

$$\begin{aligned}
 \sum_M N W(\mathbf{k}' - M + 2, \mathbf{k} + M) \omega_M = & (2\pi/\hbar) \delta(E_k - E_{k'} + 2g\beta H_{eff}) \sum_M N |H_{k' - M + 2, k + M}|^2 \omega_M \\
 = & \delta(E_k - E_{k'} + 2g\beta H_{eff}) V_4, \quad (3.10)
 \end{aligned}$$

where

$$U_1 = (2\pi/\hbar N) |J(\mathbf{k}, \mathbf{k}')|^2 p_1, \quad (3.11)$$

$$U_2 = (2\pi/\hbar N) |J(\mathbf{k}, \mathbf{k}')|^2 p_5, \quad (3.12)$$

$$\begin{aligned}
 V_1 = & (2\pi/\hbar N) [A_1^2 F_2^2 (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_\xi^2 (4c_2^2 p_2 + p_1) \\
 & - (c_3 F_3 / 2) J(\mathbf{k}, \mathbf{k}') \{ G_1(\mathbf{k}, \mathbf{k}') (3p_4 + (1 - S^2 - S) p_1) + G_2(\mathbf{k}, \mathbf{k}') (3p_4 - S(S + 1) p_1) \}], \quad (3.13)
 \end{aligned}$$

$$V_2 = (2\pi/\hbar N) A_1^2 F_2^2 c_2^2 \{ (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})^2 - (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_\xi^2 \} p_2, \quad (3.14)$$

$$\begin{aligned}
 V_3 = & (2\pi/\hbar N) [A_1^2 F_2^2 c_2^2 (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_\xi^2 p_6 - (c_3 F_3 / 8) J(\mathbf{k}, \mathbf{k}') \\
 & \times \{ G_1(\mathbf{k}, \mathbf{k}') (3p_6 + (1 - 4S - 4S^2) p_5) + G_2(\mathbf{k}, \mathbf{k}') (3p_6 + (3 - 4S - 4S^2) p_5) \}], \quad (3.15)
 \end{aligned}$$

$$V_4 = (2\pi/\hbar N) A_1^2 F_2^2 c_2^2 \{ (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})^2 - (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_\xi^2 \} p_8, \quad (3.16)$$

$$V_5 = (2\pi/\hbar N) (A_1^2 F_2^2 / 4) \{ (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})^2 - (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_\xi^2 \} (p_5 + c_2^2 p_6), \quad (3.17)$$

$$V' = (2\pi/\hbar N) [A_1^2 F_2^2 (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})^2 4c_2 p_3$$

$$+ (3c_1/2) (2R_1 - F_3/2) J(\mathbf{k}, \mathbf{k}') \{G_1(\mathbf{k}, \mathbf{k}') + G_2(\mathbf{k}, \mathbf{k}')\} p_3], \quad (3.18)$$

$$V'' = (2\pi/\hbar N) (A_1^2 F_2^2/2) \{(\boldsymbol{\kappa}' \times \boldsymbol{\kappa})^2 - (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_z^2\} c_2 p_7, \quad (3.19)$$

$$p_1 = \langle (M - \langle M \rangle)^2 \rangle, \quad (3.20)$$

$$p_2 = \langle (M^2 - \langle M^2 \rangle)^2 \rangle, \quad (3.21)$$

$$p_3 = \langle (M - \langle M \rangle) (M^2 - \langle M^2 \rangle) \rangle, \quad (3.22)$$

$$p_4 = \langle (M - \langle M \rangle) (M^3 - \langle M^3 \rangle) \rangle, \quad (3.23)$$

$$p_5 = \langle S^2 + S - M^2 - M \rangle, \quad (3.24)$$

$$p_6 = \langle (2M+1)^2 (S^2 + S - M^2 - M) \rangle, \quad (3.25)$$

$$p_7 = \langle (2M+1) (S^2 + S - M^2 - M) \rangle, \quad (3.26)$$

$$p_8 = \langle (S-M) (S+M+1) (S-M-1) (S+M+2) \rangle, \quad (3.27)$$

$$G_1(\mathbf{k}, \mathbf{k}') = A_2 \sum_{(xy)} 2\xi_x \xi_y (\kappa_x \kappa'_y + \kappa_y \kappa'_x) \\ + A_3 \{(\xi_x^2 - \xi_y^2) (\kappa_x \kappa'_x - \kappa_y \kappa'_y) + (3\xi_z^2 - 1) (\kappa_z \kappa'_z - (\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}')/3)\}, \quad (3.28)$$

$$G_2(\mathbf{k}, \mathbf{k}') = A_4 \sum_{(xy)} 2\xi_x \xi_y (\kappa_x \kappa'_y + \kappa_y \kappa'_x) \\ + A_5 \{(\xi_x^2 - \xi_y^2) (\kappa_x \kappa'_x - \kappa_y \kappa'_y) + (3\xi_z^2 - 1) (\kappa_z \kappa'_z - (\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}')/3)\}. \quad (3.28)'$$

Here $\langle f(M) \rangle$ denotes $\sum_M f(M) \nu_M$. U_1 and U_2 come from the first term in rectangular brackets of Eq. (2.33) and represent the usual isotropic scattering. V_2, V_4, V_5, V'' and the first terms of V_1, V_3 and V' come from the absolute square of the second term of (2.33) and are proportional to A_1^2 and thus to λ^2 . The other terms of V_1, V_3 and V' come from the cross product of the first and the third terms of (2.33) and are proportional to one of the quantities A_2, A_3, A_4 and A_5 and thus to λ^2 .

So far we have neglected the phonon scattering. We assume that this effect can be taken into account by adding a transition probability of the form

$$W'(\mathbf{k}' +, \mathbf{k} +) = \delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) U_3 \equiv (2\pi/\hbar N) \delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) P(\mathbf{k}, \mathbf{k}') \quad (3.29)$$

to $W(\mathbf{k}' +, \mathbf{k} +)$, (3.6), where $P(\mathbf{k}, \mathbf{k}')$ is assumed to depend only on the angle between \mathbf{k} and \mathbf{k}' .

Substituting (3.6), (3.7), (3.8), (3.9), (3.10) and (3.29) in (3.5) and neglecting $g\beta H_{eff}$ compared with $E_{\mathbf{k}}$, because we need only the $g_{\mathbf{k}}^{\pm}$ at the Fermi surface, the rate of change of $f_{\mathbf{k}}^{\pm}$ at $|\mathbf{k}| = k_F$ becomes

$$(\partial f_{\mathbf{k}}^{\pm} / \partial t)_{coll} = \mathcal{E} (\partial f_{\mathbf{k}}^0 / \partial E_{\mathbf{k}}) \sum_{\mathbf{k}'} \delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) \{ (U_1 + U_3 + V_1 + 2pV_5 + V' + 2pV'') \\ \times (g_{\mathbf{k}}^{\pm} - g_{\mathbf{k}'}^{\pm}) + (pU_2 + V_2 + pV_3 + p'V_4) (g_{\mathbf{k}}^{\pm} - g_{\bar{\mathbf{k}}}^{\pm}) \}, \quad (3.30)$$

where

$$p = 2 \exp(g\beta H_{eff}/kT) / \{1 + \exp(g\beta H_{eff})\}, \quad (3.31)$$

$$p' = 2 \exp(2g\beta H_{eff}/kT) / \{1 + \exp(2g\beta H_{eff})\}. \quad (3.32)$$

The rate of change of $f_{\bar{k}}$ of the electron with $-$ spin can be obtained in a similar way and is given by

$$(\partial f_{\bar{k}}/\partial t)_{col} = \mathcal{E}(\partial f_{\bar{k}}^0/\partial E_k) \sum_{k'} \delta(E_k - E_{k'}) \{ (U_1 + U_3 + V_1 + 2pV_5 - V' - 2p'V'')(g_{\bar{k}} - g_{\bar{k}'}) + (pU_2 + V_2 + pV_3 + p'V_4)(g_{\bar{k}} - g_{\bar{k}'}) \}. \quad (3.33)$$

We solve the Boltzmann equation

$$(\partial f_{\bar{k}}^\pm/\partial t)_{col} + \mathcal{E}(\partial f_{\bar{k}}^0/\partial E_k) (\hbar e/m) (\boldsymbol{\sigma} \cdot \mathbf{k}) = 0 \quad (3.34)$$

by assuming U_i are much larger than V_i, V' and V'' , the latter being of the order of λ^2 , and by expanding $g_{\bar{k}}^\pm$ as a power series in λ ,

$$g_{\bar{k}}^\pm = (g_{\bar{k}}^\pm)_0 + (g_{\bar{k}}^\pm)_2 + \dots \quad (3.35)$$

Then the zeroth-order part of (3.34) becomes

$$\sum_{k'} \delta(E_k - E_{k'}) [(U_1 + U_3) \{ (g_{\bar{k}}^\pm)_0 - (g_{\bar{k}'}^\pm)_0 \} + pU_2 \{ (g_{\bar{k}}^\pm)_0 - (g_{\bar{k}'}^\mp)_0 \}] + (\hbar e/m) (\boldsymbol{\sigma} \cdot \mathbf{k}) = 0, \quad (|\mathbf{k}| = k_F) \quad (3.36)$$

which can be satisfied by putting

$$(g_{\bar{k}}^\pm)_0 = (g_{\bar{k}}^\mp)_0 = \Phi(\boldsymbol{\sigma} \cdot \mathbf{k}), \quad (|\mathbf{k}| = k_F) \quad (3.37)$$

where

$$\frac{1}{\Phi} = -\frac{m}{2\pi e \hbar^2} \frac{V}{N} \left(\frac{2m}{\hbar^2} \right)^{3/2} (E_F)^{1/3} \cdot A, \quad (3.38)$$

$$A = (1/2) \int_0^\pi \{ |J(\mathbf{k}, \mathbf{k}')|^2 (p_1 + pp_5) + P(\mathbf{k}, \mathbf{k}') \} (1 - \cos \theta) \sin \theta d\theta, \quad (3.39)$$

where θ is the angle between \mathbf{k} and \mathbf{k}' .

By adding the second-order parts of (3.34) for $+$ spin and $-$ spin, we obtain the equation for $(g_k)_2 \equiv (1/2) \{ (g_k^+)_2 + (g_k^-)_2 \}$ as follows:

$$\sum_{k'} \delta(E_k - E_{k'}) (V_1 + V_2 + pV_3 + p'V_4 + 2pV_5) \{ (g_k)_0 - (g_{k'})_0 \} + \sum_{k'} \delta(E_k - E_{k'}) (U_1 + pU_2 + U_3) \{ (g_k)_2 - (g_{k'})_2 \} = 0, \quad (3.40)$$

where (3.37) should be substituted for $(g_k)_0$. The solution for $(g_k)_2$ can be expressed as a sum of spherical harmonics, $Y_{lm}(\theta_k, \varphi_k)$. In order to calculate the current density, we need only the terms with $l=1$, which are given by

$$(g_k)_2 = \Psi_1 \{ 2(\boldsymbol{\sigma} \cdot \mathbf{k}) - (\boldsymbol{\sigma} \cdot \boldsymbol{\zeta})(\boldsymbol{\zeta} \cdot \mathbf{k}) \} + \Psi_2 \sum_{(xy)} 2\zeta_x \zeta_y (\sigma_x k_y + \sigma_y k_x) + \Psi_3 \{ (\zeta_x^2 - \zeta_y^2)(\sigma_x k_x - \sigma_y k_y) + (3\zeta_z^2 - 1)(\sigma_z k_z - (\boldsymbol{\sigma} \cdot \mathbf{k})/3) \}, \quad (3.41)$$

where

$$\Psi_1/\Phi = -(2/15A) A_1^2 F_2^2 q_1, \quad (3.42)$$

$$\Psi_2/\Phi = (1/3A) c_3 F_3 (4F_1/5 - F_0) (q_2 A_2 + q_3 A_4), \quad (3.43)$$

$$\Psi_3/\Phi = (1/3A) c_3 F_3 (4F_1/5 - F_0) (q_2 A_3 + q_3 A_5), \quad (3.44)$$

$$q_1 = p_1 - pp_5/2 + c_2^2 (3p_2 + pp_6/2 - p'p_8), \quad (3.45)$$

$$q_2 = (1/8) \{12p_4 + 4(1 - S - S^2)p_1 + 3pp_6 + (1 - 4S - 4S^2)pp_6\}, \quad (3.46)$$

$$q_3 = (1/8) \{12p_4 - 4(S + S^2)p_1 + 3pp_6 + (3 - 4S - 4S^2)pp_6\}. \quad (3.47)$$

The average current density is expressed from the assumption (1) by

$$\begin{aligned} \langle \mathbf{j}^\pm \rangle &= - (e\hbar/mV) \sum_{\mathbf{k}} \mathbf{k} (f_{\mathbf{k}}^\pm - f_{\mathbf{k}}^0) \\ &= (e\hbar/mV) \mathcal{E} \sum_{\mathbf{k}} \mathbf{k} (\partial f_{\mathbf{k}}^0 / \partial E_{\mathbf{k}}) g_{\mathbf{k}}^\pm. \end{aligned} \quad (3.48)$$

The current density in the direction of the applied field is given from (3.37) and (3.41) by

$$\begin{aligned} j_x &= - (e/3\pi^2\hbar) (2m/\hbar^3)^{3/2} E_F^{3/2} \Phi \mathcal{E} [1 + (\Psi_1/\Phi) \{2 - (\boldsymbol{\sigma} \cdot \boldsymbol{\xi})^2\} \\ &\quad + (\Psi_2/\Phi) \sum_{(xy)} 4\xi_x \xi_y \sigma_x \sigma_y \\ &\quad + (\Psi_3/\Phi) \{(\xi_x^2 - \xi_y^2) (\sigma_x^2 - \sigma_y^2) + (3\xi_z^2 - 1) (\sigma_z^2 - 1/3)\}]. \end{aligned} \quad (3.49)$$

Then the resistivity becomes

$$\begin{aligned} \rho &= \frac{3\pi m}{2\hbar e^2} \frac{V}{N} \frac{A}{E_F} \\ &\quad - \frac{3\pi m}{2\hbar e^2} \frac{V}{N} \frac{1}{E_F} \frac{4}{3} c_3 F_3 \left(\frac{4}{5} F_1 - F_0 \right) \{q_2 (A_2 - A_3) + q_3 (A_4 - A_5)\} \sum_{(xy)} \xi_x \xi_y \sigma_x \sigma_y \\ &\quad - \frac{3\pi m}{2\hbar e^2} \frac{V}{N} \frac{1}{E_F} \left\{ \frac{2}{15} A_1^2 F_2^2 q_1 + \frac{2}{3} c_3 F_3 \left(\frac{4}{5} F_1 - F_0 \right) (q_2 A_3 + q_3 A_5) \right\} (\boldsymbol{\sigma} \cdot \boldsymbol{\xi})^2, \end{aligned} \quad (3.50)$$

which is just of the form of Eq. (1.1).

The first term of (3.50) is an isotropic one, which comes from exchange scattering as well as from phonon scattering, while the other terms are anisotropic ones which do not depend on phonon scattering. The experimental result on the electrical resistivity of cubic ferromagnetic metals has been analysed with the use of the formula of the form (1.1), which we have deduced by assuming a non-degenerate orbital ground state. However, the one weak point in the argument is that actually more than one orbital state surely does come into play. We do not want to enter into details as to how the orbital angular momentum is quenched, because the mechanism of quenching is probably not the same as the crystal-line potential operative in paramagnetic salts. Therefore, we do not expect a quantitative agreement between (3.50) and observation. In a rough estimation, the ratios of the anisotropic terms to the isotropic one arising from exchange scattering is of the order of A_1^2 , A_2 , A_3 , A_4 and A_5 , corresponding to various contributions to the anisotropic scattering. A reasonable estimate may be $(\lambda/\Delta E)^2 = 10^{-3}$, then these quantities are of the order of 10^{-2} according to (2.34) ~ (2.43). This order of magnitude is just what is observed for iron and nickel.⁵⁾⁻⁷⁾ The temperature dependence of the anisotropic terms comes from q_1 , q_2 and q_3 , which are zero both at the absolute zero and at the Curie point and show maxima at an intermediate temperature. This behavior is also observed,⁷⁾ although the

actual temperature dependence is more complicated than that of q_i . It should be noted, however, that, when we take $L=3$, A_3 and A_5 vanish and then the sign of ρ_1 is definitely negative. This contradicts most of the experimental results.⁶⁾ In our treatment we have assumed the periodic lattice. When impurity atoms are introduced, we have many additional contributions to the scattering mechanism other than those considered in this section. This might correspond to a marked anisotropy which is shown by ferromagnetic alloys. But we shall not enter into the calculation of this effect any more.

§ 4. Anomalous Hall effect

In the previous section, we have seen that the transition probability from a state \mathbf{k} to another state \mathbf{k}' is equal to that from \mathbf{k}' to \mathbf{k} in the first Born approximation. This is because the probability is given by the absolute square of the matrix element in that approximation. It should be noted, however, that the matrix element itself is not symmetric between \mathbf{k} and \mathbf{k}' . The second term of (2.33) is proportional to $\mathbf{k}' \times \mathbf{k}$ and changes its sign when \mathbf{k} and \mathbf{k}' are interchanged. Thus we expect that the transition probability calculated to the higher Born approximation might contain a part which is proportional to $\mathbf{k}' \times \mathbf{k}$. When the probability contains an asymmetric term, the direction of the current does not necessarily coincide with that of the applied field and thus we have the anomalous Hall effect.

In calculating the current to the higher Born approximation, we shall make the three assumptions of the previous section and, besides, we shall take account only of the elastic scattering, in which the ζ -component of any localized spin does not change. In doing so, we expect that we can obtain a qualitatively correct result. The transition probability per unit time from $\mathbf{k}' \pm$ to $\mathbf{k} \pm$ is given to the second Born approximation by

$$W(\mathbf{k}' \pm, \mathbf{k} \pm) = (2\pi/\hbar) \delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) [H_{\mathbf{k} \pm, \mathbf{k}' \pm} H_{\mathbf{k}' \pm, \mathbf{k} \pm} + \sum_{\mathbf{k}''} \{H_{\mathbf{k} \pm, \mathbf{k}'' \pm} H_{\mathbf{k}'' \pm, \mathbf{k}' \pm} H_{\mathbf{k}' \pm, \mathbf{k} \pm} / (E_{\mathbf{k}} - E_{\mathbf{k}''} + is) + \text{c.c.}\}], \quad (4.1)$$

where s is an infinitesimally small positive quantity. In the following we shall retain up to linear terms in λ . Then, from (2.33), we have

$$H_{\mathbf{k} \pm, \mathbf{k}' \pm} = \mp \sum_n N^{-1} e^{i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{R}_n} [J(\mathbf{k}, \mathbf{k}') (M_n - \langle M \rangle) + iA_1 F_2(\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_{\zeta} \{2c_2(M_n^2 - \langle M^2 \rangle) \pm (M_n - \langle M \rangle)\}]. \quad (4.2)$$

If we denote the part of $H_{\mathbf{k} \pm, \mathbf{k}' \pm} H_{\mathbf{k}' \pm, \mathbf{k} \pm} H_{\mathbf{k}' \pm, \mathbf{k} \pm}$, which is linear in λ , by Q^{\pm} , Q^{\pm} is expressed from the assumption (2) of the previous section by

$$Q^{\pm} = (iA_1 F_2/N^2) (r_1 \pm r_2) \{J(\mathbf{k}, \mathbf{k}'') J(\mathbf{k}'', \mathbf{k}') (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_{\zeta} + J(\mathbf{k}, \mathbf{k}'') (\boldsymbol{\kappa}'' \times \boldsymbol{\kappa}')_{\zeta} J(\mathbf{k}', \mathbf{k}) + (\boldsymbol{\kappa} \times \boldsymbol{\kappa}'')_{\zeta} J(\mathbf{k}'', \mathbf{k}') J(\mathbf{k}', \mathbf{k})\}, \quad (4.3)$$

where

$$r_1 = \langle (M - \langle M \rangle)^3 \rangle. \quad (4.4)$$

$$r_2 = 2c_2 \langle (M - \langle M \rangle)^2 (M^3 - \langle M^3 \rangle) \rangle, \quad (4.5)$$

Using the relation $1/(E_k - E_{k'} + is) = P(1/(E_k - E_{k'})) - i\pi\delta(E_k - E_{k'})$, we find

$$\begin{aligned} Q^\pm / (E_k - E_{k'} + is) + \text{c.c.} &= (2\pi A_1 F_2 / N^2) (r_1 \pm r_2) \delta(E_k - E_{k'}) \\ &\times \{J(\mathbf{k}, \mathbf{k}'') J(\mathbf{k}'', \mathbf{k}') (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_\zeta + J(\mathbf{k}, \mathbf{k}'') (\boldsymbol{\kappa}'' \times \boldsymbol{\kappa}')_\zeta J(\mathbf{k}', \mathbf{k}) \\ &+ (\boldsymbol{\kappa} \times \boldsymbol{\kappa}'')_\zeta J(\mathbf{k}'', \mathbf{k}') J(\mathbf{k}', \mathbf{k})\}. \end{aligned} \quad (4.6)$$

When summed over \mathbf{k}'' , this expression becomes

$$\begin{aligned} \sum_{k''} \{Q^\pm / (E_k - E_{k'} + is) + \text{c.c.}\} \\ &= (A_1 F_2 V / \pi N^2) (2m^3)^{1/2} \hbar^{-3} E_k^{1/2} (r_1 \pm r_2) (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_\zeta \\ &\times (1/3) \{3F_0^2 - 4F_0 F_1 - 4F_1^2 (\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}')\}. \end{aligned} \quad (4.7)$$

Neglecting other contributions than (4.7) to the second Born approximation and adding the phonon scattering, we find

$$W(\mathbf{k}' \pm, \mathbf{k} \pm) = \delta(E_k - E_{k'}) \{U_1 + U_3 + V^\pm(\mathbf{k}', \mathbf{k})\}, \quad (4.8)$$

where

$$\begin{aligned} V^\pm(\mathbf{k}', \mathbf{k}) &= (2\pi/\hbar) (A_1 F_2 V / 3\pi N^2) (2m^3)^{1/2} \hbar^{-3} E_k^{1/2} (r_1 \pm r_2) (\boldsymbol{\kappa}' \times \boldsymbol{\kappa})_\zeta \\ &\times \{3F_0^2 - 4F_0 F_1 - 4F_1^2 (\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}')\} = -V^\pm(\mathbf{k}, \mathbf{k}'). \end{aligned} \quad (4.9)$$

Now we shall solve the transport equation

$$\sum_{k'} W(\mathbf{k}' \pm, \mathbf{k} \pm) f_{k'}^\pm - \sum_{k'} W(\mathbf{k} \pm, \mathbf{k}' \pm) f_k^\pm + \mathcal{E}(\hbar e/m) (\partial f_k^0 / \partial E_k) (\boldsymbol{\sigma} \cdot \mathbf{k}) = 0, \quad (4.10)$$

which has been derived by Kohn and Luttinger.⁸⁾ If we put

$$f_k^\pm = f_k^0 - \mathcal{E}(\partial f_k^0 / \partial E_k) g_k^\pm, \quad (4.11)$$

the equations for g_k^\pm become

$$\sum_{k'} \delta(E_k - E_{k'}) \{(U_1 + U_3) (g_k^\pm - g_{k'}^\pm) - V^\pm(\mathbf{k}', \mathbf{k}) (g_k^\pm + g_{k'}^\pm)\} + (\hbar e/m) (\boldsymbol{\sigma} \cdot \mathbf{k}) = 0. \quad (4.12)$$

As in the previous section, we regard V^\pm as small quantities compared with U_1 and U_3 and solve (4.12) by putting

$$g_k^\pm = (g_k^\pm)_0 + (g_k^\pm)_1 + \dots, \quad (4.13)$$

where $(g_k^\pm)_0$ and $(g_k^\pm)_1$ are of the zero-th and first order of λ . Then the equations for $(g_k^\pm)_0$, which is expressed by

$$\sum_{k'} \delta(E_k - E_{k'}) (U_1 + U_3) \{(g_k^\pm)_0 - (g_{k'}^\pm)_0\} + (\hbar e/m) (\boldsymbol{\sigma} \cdot \mathbf{k}) = 0, \quad (4.14)$$

can be satisfied by

$$(g_k^\pm)_0 = \Phi(\boldsymbol{\sigma} \cdot \mathbf{k}), \quad (4.15)$$

where

$$\Phi^{-1} = - (m/2\pi e\hbar^3) (V/N) (2m/\hbar^3)^{3/2} E_F^{1/2} \cdot A', \tag{4.16}$$

$$A' = (1/2) \int_0^\pi \{ |J(\mathbf{k}, \mathbf{k}')|^2 p_1 + P(\mathbf{k}, \mathbf{k}') \} (1 - \cos \theta) \sin \theta d\theta. \tag{4.17}$$

By adding the equations for $(g_{\mathbf{k}}^+)_1$ and $(g_{\mathbf{k}}^-)_1$, we can obtain the equation for $(g_{\mathbf{k}})_1 \equiv (1/2) \{ (g_{\mathbf{k}}^+)_1 + (g_{\mathbf{k}}^-)_1 \}$ as follows:

$$\sum_{\mathbf{k}'} \delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) [(U_1 + U_3) \{ (g_{\mathbf{k}})_1 - (g_{\mathbf{k}'})_1 \} - V(\mathbf{k}', \mathbf{k}) \{ (g_{\mathbf{k}})_0 + (g_{\mathbf{k}'})_0 \}] = 0, \tag{4.18}$$

where

$$V(\mathbf{k}', \mathbf{k}) = (1/2) \{ V^+(\mathbf{k}', \mathbf{k}) + V^-(\mathbf{k}', \mathbf{k}) \}. \tag{4.19}$$

The solution of this equation can be expressed by a sum of spherical harmonics, $Y_{lm}(\theta_{\mathbf{k}}, \varphi_{\mathbf{k}})$, but we need only the terms of $l=1$, which are given by

$$(g_{\mathbf{k}})_1 = (\boldsymbol{\zeta} \times \boldsymbol{\sigma}) \cdot \mathbf{k}, \tag{4.20}$$

where

$$(\Psi/\Phi) = (A_1 F_2 V / 3\pi N) (2m^3)^{1/2} \hbar^{-3} E_k^{1/2} r_1 (F_0^2 - 4F_0 F_1 / 3) / A'. \tag{4.21}$$

Then the current density is given from (3.48) by

$$\mathbf{j} = \sigma_{||} \mathcal{E} \{ \boldsymbol{\sigma} + (\Psi/\Phi) \boldsymbol{\zeta} \times \boldsymbol{\sigma} \}, \tag{4.22}$$

where

$$\sigma_{||}^{-1} = \rho_{||} = (3\pi m / 2\hbar e^3) (V/N) (A' / E_F) \tag{4.23}$$

is the normal resistivity. The second term of (4.22) represents the component of the current which is perpendicular both to the electric field and to the magnetization and thus represents the anomalous Hall effect. According to (4.21), we see that the anomalous Hall coefficient is positive when λF_2 is positive, since r_1 is negative.

It is convenient to discuss the anomalous Hall effect in terms of the anomalous Hall resistivity, which is defined by

$$\rho_H = E_{\perp} / j_{||}, \tag{4.24}$$

where $j_{||}$ is the component of the current density along the direction of the applied field and E_{\perp} is the component of the electric field perpendicular both to the current and to the magnetization, when the electric field is applied in a plane perpendicular to the magnetization. Using (4.21), we obtain

$$\rho_H = A_1 (V/N)^2 (2m^5)^{1/2} r_1 F_2 (F_0^2 - 4F_0 F_1 / 3) / 2\hbar^4 e^2 E_F^{1/2}. \tag{4.25}$$

It is remarkable that the phonon scattering does not contribute to this result, although it does to the Hall current $\sigma_{||} \mathcal{E} (\Psi/\Phi) \boldsymbol{\zeta} \times \boldsymbol{\sigma}$.

We shall compare our result (4.25) with experiment^{9), 10)} on iron and nickel.

We assume that an atomic site of each crystal contributes one conduction electron. This assumption is arbitrary but the result does not appreciably depend on the number of conduction electrons, because ρ_H is proportional to its cube root. The temperature variation of ρ_H is represented by r_1 , which is calculated by assuming $S=1$ and $1/2$ for iron and nickel, respectively. Then, if we take

$$\begin{aligned} \{\lambda/(E_{F_2}-E_0)\} F_2(F_0^3-4F_0F_1/3) &= -0.03(\text{ev})^3 \text{ for nickel,} \\ &= +0.12(\text{ev})^3 \text{ for iron,} \end{aligned} \quad (4.26)$$

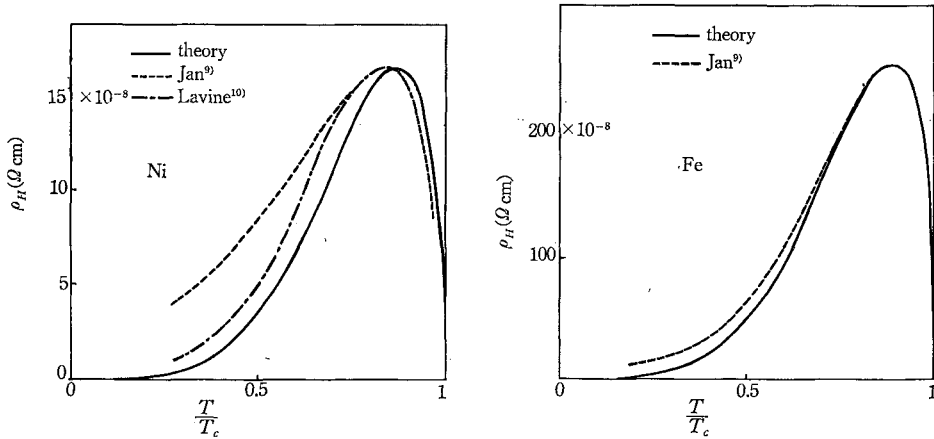


Fig. 1. Experimental and theoretical anomalous Hall resistivities for nickel and iron.

their agreement with experiment is such as shown in Fig. 1. The sharp maximum of ρ_H below the Curie point is well reproduced by our theory. It should be noted that, in Karplus and Luttinger's theory ρ_H is proportional to $\rho_{ij}^2 \langle M \rangle$, which incidentally has a temperature variation quite similar to r_1 .

§ 5. Rare-earth metals

In this section we shall discuss the anomalous Hall effect and the magnetic contribution to the normal resistivity in the paramagnetic region of rare-earth metals with more than half-filled $4f$ shell. The interaction between conduction electrons and $4f$ electrons is given by (2.45).

If we assume that the integrals F_1, F_2 and F_3 can be neglected compared with F_0 , we have from (2.45)

$$\begin{aligned} H &= -\sum_{nk k'} N^{-1} e^{i(k'-k)R_n} a_k^* a_{k'} \\ &\times \{2d_1 R_1 \langle \mathbf{J}_n \mathbf{J}_n \rangle \cdot \langle \boldsymbol{\kappa} \boldsymbol{\kappa}' \rangle + 2(g_J - 1) J(\mathbf{k}, \mathbf{k}') (\mathbf{J}_n \cdot \mathbf{s}_c)\}. \end{aligned} \quad (5.1)$$

The first term in brackets represents the Coulomb interaction between conduction electrons and the electric quadrupole moments of $4f$ electrons, while the second one represents the isotropic exchange interaction between spin angular

momenta of conduction electrons and $4f$ electrons.

We calculate the electrical resistivity in the paramagnetic region, regarding (5.1) as a perturbation on the motion of the conduction electrons which causes the deviation from the periodic potential. We shall also make the three assumptions of § 3 and moreover take the molecular field to be zero. If we restrict ourselves to the first Born approximation, the calculation can be carried out in the same manner as in § 3 and we shall present only the final result here, which is expressed by

$$\rho_{paramag} = (3\pi/2) (V/N) (m/e^2\hbar E_F) \times \{ (8/27) d_1^2 R_1^2 (2J-1)J(J+1)(2J+3) + (g_J-1)^2 J(J+1) F_0^2 \}. \quad (5.2)$$

Recently, Colvin, Legvold and Spedding¹¹⁾ have carried out an experiment on the magnetic contribution to the resistivity of rare-earth metals with more than half-filled $4f$ shell. According to their result, the contribution depends mainly on the spin, rather than on the magnetic moment of the rare-earth ion, and it is largest for gadolinium which has the largest spin, and smaller for holmium and dysprosium which have the largest magnetic moments. Their result should be compared with (5.2). We assume that the differences of the values of (5.2) for various elements come solely from the differences of the values of J , g_J and d_1 for these elements. Then, taking $F_0=0.25$ ev and $R_1=0.024$ ev for all elements, we obtain values of $\rho_{paramag}$, which we show in Fig. 2 along with the experimental ones.

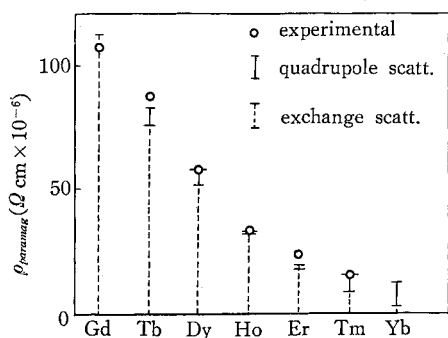


Fig. 2. $\rho_{paramag}$ for rare-earth metals with more than half-filled $4f$ shell.

As can be seen from the figure, the exchange contribution reproduces the rough dependence of $\rho_{paramag}$ on the elements. But the agreement between experimental and theoretical values is improved by including the quadrupole scattering into $\rho_{paramag}$. In thulium and ytterbium, in particular, the quadrupole scattering makes an appreciable contribution to $\rho_{paramag}$.

Next, we shall discuss the anomalous Hall effect of rare-earth metals. We may expect that the third term in rectangular brackets of (2.45), which contains $\mathbf{k}' \times \mathbf{k}$ as a factor, gives rise to the anomalous Hall effect. It may be a good approximation to calculate the anomalous Hall resistivity of those elements where the exchange scattering predominates over the quadrupole scattering by taking the isotropic exchange term and terms with a factor $\mathbf{k}' \times \mathbf{k}$, that is, the second, third and fourth terms of (2.45) into account. Then the calculation can be carried out in the same way as in § 4 and we can reach the final result of the Hall resistivity, which is expressed by

$$\rho_H = (2 - g_J)(g_J - 1)^2 (V/N)^2 (2m^5)^{1/2} r_1 F_2 F_0^2 / 4\hbar^4 e^2 E_F^{1/2}, \quad (5.3)$$

where r_1 is defined by (4.4). It should be noted that the sign of the anomalous Hall coefficient depends only on that of F_2 . It is positive when F_2 is negative, because r_1 is negative.

Measurements on the Hall effect of rare-earth metals were made in the paramagnetic region,¹²⁾ where r_1 becomes

$$r_1 = -(\chi H / 5g_J\beta) (2J^2 + 2J + 1), \quad (5.4)$$

where χ is the susceptibility per rare-earth ion and H is the magnitude of the magnetic field applied perpendicular to the electric field. Thus we see that the anomalous Hall coefficient is proportional to the susceptibility. The experimental results are rather complicated and indicate other contributions than that considered here. In particular it is remarkable that the largest effect is observed in gadolinium, which has no orbital angular momentum. Here we shall compare our result with the observations on dysprosium and erbium, where we find that the experimental results are represented by two contributions: One is temperature-independent and the other is proportional to the susceptibility. If the values of the susceptibility are substituted from the measurements, we can fit our result, (5.3), with the latter contribution by taking the value of F_2 as follows:

$$F_2 = +0.0050 \text{ ev for erbium} \\ +0.02 \text{ ev for dysprosium.}$$

The comparison between the theoretical and experimental temperature variations is shown in Fig. 3.

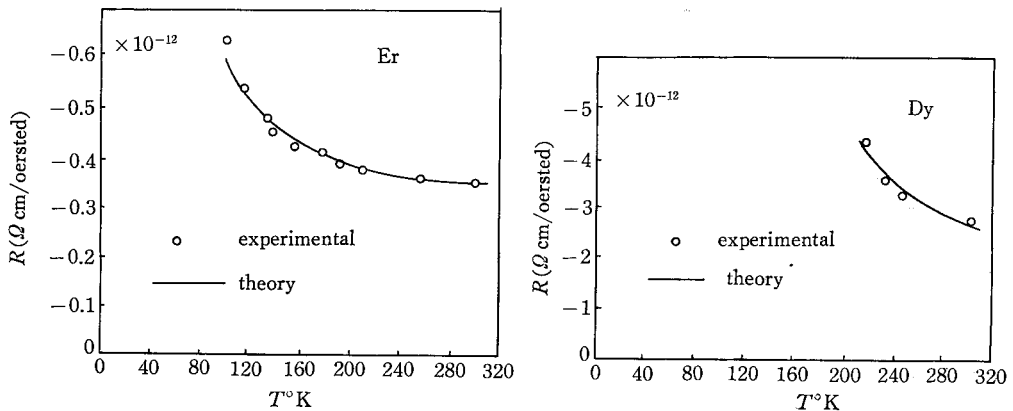


Fig. 3. Hall coefficients of erbium and dysprosium in the paramagnetic region. The theoretical curves are the sum of a constant and an anomalous term.

§ 6. Discussion

The electronic structure of the $3d$ state in iron group metals is not clearly understood at present. The itinerant model and the localized model are two extremely simple models for it. Both may not represent the true state of affairs. However, it may be interesting to see how far each of these two models can account for the observations on various properties of iron group metals. Although the predictions of both models are qualitatively similar in many respects, it has been considered that the anomalous Hall effect might distinguish between the two models. However, we have shown in this paper that the anomalous Hall effect and the magnetoresistance can also be accounted for by the localized model.

It should be noted, however, that these results were derived by assuming a non-degenerate orbital ground state. As was discussed in § 3, this assumption is not legitimate for iron group metals. The quenching of the orbital angular momenta in these metals may partly be due to the wandering of $3d$ electrons among lattice sites and then we should have the other contribution to the effect considered by Karplus and Luttinger. The values of parameters, Eq. (4.26), therefore, should not be taken too seriously.

On the other hand, the localized model may be a reasonable approximation for $4f$ electrons of rare-earth metals. Thus, the value of $F_0=0.25$ eV is a reasonable one judging from the optical data. As for R_1 , we can write it from (2.14) as

$$R_1 = (Ne^2/V) (\pi/25) \langle r^2 \rangle,$$

if the wavelength of the electron at the Fermi surface is much larger than the spatial extension of $4f$ electrons. Then, from $R_1=0.024$ eV, we have $\langle r^2 \rangle = 0.44 \text{Å}^2$. This magnitude may be a reasonable one. We could not make any reasonable estimate of the values of F_2 but this may be considered to be a little smaller than F_0 . We see that this is the case.

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