Ann. Phys. (Leipzig) **15**, No. 7 – 8, 508 – 519 (2006) / **DOI** 10.1002/andp.200510206

Anomalous Hall effect in ferromagnetic disordered metals

P. Wölfle[1,*] and **K.A. Muttalib**[2,**]

[1] Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, 76128 Karlsruhe, Germany
[2] Department of Physics, University of Florida, Gainesville, FL 32611, USA

Received 13 October 2005, accepted 11 November 2005
Published online 26 May 2006

Key words Anomalous Hall effect, ferromagnets, skew scattering, side-jump scattering.
PACS 73.50.Jt, 72.15Gd, 72.25.Ba, 72.10Bg

In commemoration of Paul Drude (1863–1906)

The anomalous Hall effect in disordered band ferromagnets is considered in the framework of quantum transport theory. A microscopic model of electrons in a random potential of identical impurities including spin-orbit coupling is used. The Hall conductivity is calculated from the Kubo formula for both, the skew scattering and the side-jump mechanisms. The recently discussed Berry phase induced Hall current is also evaluated within the model. The effect of strong impurity scattering is analyzed and it is found to affect the ratio of the non-diagonal (Hall) and diagonal components of the conductivity as well as the relative importance of different mechanisms.

1 Introduction

The Hall effect is known to occur in conductors subject to a magnetic field. On a classical level it is explained by the Lorentz force acting on the charge carriers in a magnetic field. However, an external (or internal) magnetic field is not necessary for a Hall response to exist. It follows from basic principles of statistical mechanics that the Hall effect can appear whenever time reversal invariance is broken. Indeed it was recognized in the 1950's that a Hall effect should exist in ferromagnetic metals even in the absence of an external magnetic field, if the magnetic polarization of the spin system is coupled to the orbital motion by spin-orbit coupling. An anomalous Hall effect in ferromagnets has been observed in many systems (for example see [1, 2]). The direct coupling of the moving charge carriers to the magnetic field generated by the spins is much too small to explain the experimental observations.

In a pioneering paper, Karplus and Luttinger [3] worked out a theory of this effect, in which they pointed out the existence of an additional term in the velocity operator proportional to the gradient of any electrical potential acting on the carriers and to a term acting like a magnetic field (not the dipolar field). The latter has been identified recently as a Berry phase term [4–7]. It generates a Hall current in equilibrium, and should exist even in the absence of any impurity scattering. The Hall conductivity derived from this mechanism is proportional to the Berry phase curvature averaged over all occupied conduction band states. The precise dependence of this quantity on the ferromagnetic polarization and on the spin-orbit interaction depends on the details of the system considered.

At about the same time, in 1955, Smit [8] described a different mechanism known as skew scattering. It is based on the fact that electrons in a plane perpendicular to the magnetic polarization scatter from an impurity potential in an asymmetric fashion, if they feel the polarization via the spin-orbit interaction. This mechanism has been worked out in great detail. It yields a Hall conductivity approximately proportional

* Corresponding author E-mail: woelfle@tkm.physik.uni-karlsruhe.de
** E-mail: muttalib@phys.ufl.edu

to the longitudinal conductivity (which is governed by impurity scattering at low temperatures), to the ferromagnetic polarization and to the spin-orbit coupling (assumed to be weak). The relation of this to the earlier works was discussed in [9].

Yet another mechanism for the anomalous Hall effect was proposed by Berger [10], the "side-jump" mechanism. It is based on the observation that the trajectory of an electron scattering off an impurity is shifted sidewise by the action of the spin-orbit coupling in the presence of a ferromagnetic (or antiferromagnetic) spin polarization. The effect gives rise to a Hall conductivity independent of the density of impurities, i.e. of the mean free path. The characteristic length replacing the mean free path is the shift of the trajectory, which may be estimated to be of the order of the lattice spacing. Hence this contribution is small compared to the skew scattering term, except in the case of short mean free path. Although this mechanism is similar to the Berry phase mechanism, and may be shown to originate from the extra term in the velocity operator, it involves the nonequilibrium quasiparticle distribution.

Quantum corrections to the anomalous Hall conductivity have not received much attention so far. The weak localization correction is cut off by the spin-orbit as well as the phase relaxation rates. Nonetheless, it has been found to be of order unity in the disorder parameter $(k_F l)^{-1}$, within the skew scattering mechanism [11–13] in accordance with experimental observation [14]. In the case of the side-jump mechanism, weak localization corrections have been found to be negligibly small [12]. Interaction corrections have been shown to be absent within the skew scattering model in [11, 13].

In this paper we will review these different mechanisms of the anomalous Hall effect from a common perspective, such that their dependences on parameters are displayed and their relative magnitudes are estimated. We will limit our discussion to two-dimensional or quasi two-dimensional disordered metallic band ferromagnets. As a convenient and not unrealistic model of disorder we assume identical short range impurity potentials at random positions, including spin-orbit interaction induced by the impurity potential. The strengths of the impurity potential and of the spin-orbit coupling will be left as free parameters. In particular we will be interested in strong impurity scattering, which has not been considered previously in this context within such a model, to our knowledge.

2 The model

We consider ferromagnetic metallic films with conduction electrons occupying a spin-split band. Transport at low temperatures is governed by impurity scattering. We will model the disorder by assuming identical impurities of density $n_{\rm imp}$ at random positions $\mathbf{R}_i$. Electron-electron interaction effects will be neglected. Spin-orbit interaction at the impurities will give rise to an anomalous Hall effect, as pointed out in the early papers by Smit [8] and Luttinger [9]. This so-called skew scattering arises because the finite magnetization $\mathbf{M}$ of the conduction electrons introduces a sense of rotation about the direction of $\mathbf{M}$. In addition there is a "side-jump " contribution ([10]), caused by a sideways shift of the scattering wave packet due to the spin orbit interaction. The spin-orbit interaction is a relativistic effect and therefore rather small for transition metal atoms. As pointed out in [10], the mixing of different d-orbitals provides a renormalization of the coupling constant leading to an enhancement by a factor of 10^4. We will take this effect into account by employing a phenomenological coupling constant g_σ of order unity.

The single particle Hamiltonian of a conduction electron in a ferromagnetic disordered metal, including spin-orbit interaction induced by the disorder potential $V_{\rm dis}(\mathbf{r})$, is given in its simplest form by

$$H_1 = \left[-\frac{\nabla^2}{2m} + V_{\rm dis}(\mathbf{r}) \right] \delta_{\sigma\sigma'} - M_z \tau^z_{\sigma\sigma'} - i(g_\sigma/4\pi n_\sigma)[\tau_{\sigma\sigma'} \cdot (\nabla V_{\rm dis} \times \nabla)], \tag{1}$$

where $n_\sigma = (k_{F\sigma}^2/4\pi)$ and $k_{F\sigma}$ are the density of conduction electrons and the Fermi wave vector of spin σ, $\tau_{\sigma\sigma'}$ is the vector of Pauli matrices, and g_σ is a dimensionless spin-orbit coupling constant. The bare coupling constant is given by $g_\sigma^{(0)} = (4\pi n_\sigma)(2mc)^{-2}$ with m the electron mass and c the velocity of light. Note that g_σ is in general spin dependent. We use units such that M_z is half the Zeeman energy splitting of the conduction electron energies caused by the ferromagnetic polarization $\mathbf{M}$. Here $\mathbf{M}$ is assumed to be oriented along z, perpendicular to the layer. The disordered potential will be modelled as $V_{\text{dis}}(\mathbf{r}) = \Sigma_j V(\mathbf{r}-\mathbf{R}_j)$. We will later average over the impurity positions $\mathbf{R}_j$.

The matrix elements of H_1 in the plane wave representation are given by

$$\langle \mathbf{k}'\sigma'|H_1|\mathbf{k}\sigma\rangle = \left(\frac{k^2}{2m} - M\sigma\right)\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} + \sum_j V(\mathbf{k}-\mathbf{k}')\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_j]\{\delta_{\sigma\sigma'} - ig_\sigma\tau_{\sigma\sigma'}\cdot(\widehat{\mathbf{k}}\times\widehat{\mathbf{k}}')\} \tag{2}$$

where $V(\mathbf{k}-\mathbf{k}')$ is the Fourier transform of the single impurity potential, and $\widehat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$.

The many-body Hamiltonian is given in terms of electron creation and annihilation operators $c^+_{\mathbf{k}\sigma}, c_{\mathbf{k}\sigma}$ for Bloch states $|\,\mathbf{k}\sigma>$ as

$$H = \sum_{\mathbf{k}\sigma}(\varepsilon_\mathbf{k} - M_z\sigma)c^+_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\sigma,\mathbf{k}'\sigma'}\sum_j V(\mathbf{k}-\mathbf{k}')\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_j]\{\delta_{\sigma\sigma'} - ig_\sigma\tau_{\sigma\sigma'}\cdot(\widehat{\mathbf{k}}\times\widehat{\mathbf{k}}')\}c^+_{\mathbf{k}'\sigma'}c_{\mathbf{k}\sigma}, \tag{3}$$

3 Normal and anomalous conductivity in the limit of weak impurity scattering

3.1 Kubo formula and single particle Green's function

The conductivity $\sigma_{\alpha\beta}$ will be calculated from the current response functions $L_{\alpha\beta}(\Omega_m)$ by employing the Kubo formula

$$\sigma_{\alpha\beta} = e^2 \lim_{\Omega_m\to 0}\frac{1}{\Omega_m}L_{\alpha\beta}(i\Omega_m)\ ,\quad L_{\alpha\beta}(i\Omega_m) = \int_0^\beta d\tau e^{i\Omega_m\tau}\langle T_\tau[j_\alpha(\tau)j_\beta(0)]\rangle \tag{4}$$

where $j_\alpha(\tau)$ are the components of the current density in the Heisenberg representation, $\Omega_m = 2\pi Tm$ are bosonic Matsubara frequencies and the angular brackets denote the thermal average. We will calculate the current correlation function within diagrammatic perturbation theory in the impurity potential, averaging over the random impurity positions. To simplify the calculations, we will drop the momentum dependence of the single impurity potential, $V(\mathbf{k}) = V_0$. The momentum dependence of the spin-orbit term in the potential will be kept,of course, as it is the source of the anomalous Hall effect.

The single particle Green's function $G_{\mathbf{k}\sigma}(i\omega_n)$ is defined in terms of the self-energy $\Sigma_{\mathbf{k}\sigma}(i\omega_n)$ as

$$G_{\mathbf{k}\sigma}(i\omega_n) = [i\omega_n - \varepsilon_{\mathbf{k}\sigma} - \Sigma_{\mathbf{k}\sigma}(i\omega_n)]^{-1} \tag{5}$$

Here $\omega_n = \pi T(2n+1)$ are fermionic Matsubara frequencies, and $\varepsilon_{\mathbf{k}\sigma} = \varepsilon_\mathbf{k} - M_z\sigma$ are the Bloch energies. The Bloch states are filled up to the Fermi energy $\varepsilon_{F\sigma}$ for each subband.

In lowest order in the impurity potential, assuming all momenta to lie in the x-y-plane and after averaging over the random positions of the impurities we have

$$\Sigma_{\mathbf{k}\sigma}(i\omega_n) = n_{\text{imp}}V_0^2\sum_{\mathbf{k}'}\{1 - ig_\sigma\tau^z_{\sigma\sigma}\cdot(\widehat{\mathbf{k}}\times\widehat{\mathbf{k}}')_z\}^2\ G_{\mathbf{k}'\sigma}(i\omega_n) \tag{6}$$

Since the Green's function as a function of $\varepsilon_{\mathbf{k}\sigma}$ is strongly peaked at the Fermi energy, one may separate the $\mathbf{k}'$-summation into an integral over energy, taking the momenta, in particular that of $\Sigma_{\mathbf{k}\sigma}(i\omega_n)$ to be at the Fermi surface, and an integral over the angle φ' formed by $\mathbf{k}'$ and the x-axis (note $\widehat{\mathbf{k}}' = (\cos\varphi', \sin\varphi')$. As a result, one finds

$$\Sigma_{\mathbf{k}\sigma}(i\omega_n) = -isign(\omega_n)n_{\text{imp}}(\pi N_\sigma)^{-1}[w_\sigma + 2u_\sigma] = -isign(\omega_n)(2\tau_\sigma)^{-1} \tag{7}$$

where N_σ is the conduction electron density of states at the Fermi level for spin subband σ and τ_σ are the single particle relaxation times. We have defined the dimensionless coupling constants for potential scattering and spin-orbit scattering

$$w_\sigma = (\pi N_\sigma V_0)^2 \quad \text{and} \qquad u_\sigma = (g_\sigma/2)^2\, w_\sigma\,. \tag{8}$$

3.2 Longitudinal conductivity

The diagonal elements of the conductivity tensor are given in lowest order within our model by the bubble diagram. Assuming isotropic scattering potential, there are no vertex corrections in lowest order. We have

$$L_{\alpha\alpha}(i\Omega_m) = T\sum_{\omega_n}\sum_{\mathbf{k},\sigma} v^2_{\mathbf{k}\sigma,\alpha} G_{\mathbf{k}\sigma}(i\omega_n) G_{\mathbf{k}\sigma}(i\omega_n - i\Omega_m) \tag{9}$$

where $v_{\mathbf{k}\sigma,\alpha} = \partial\varepsilon_{\mathbf{k}\sigma}/\partial k_\alpha = k_\alpha/m_\sigma$ is the particle velocity, where m_σ is the effective mass of quasi-particles with spin σ (we assume parabolic bands). The integration over energy ε_k is only finite and equal to $2\pi\tau_\sigma N_\sigma$ if the two poles of the G's are on opposite side of the real axis, yielding the restriction on the frequency summation $0 \le \omega_n \le \Omega_m$, and hence $(\Omega_m/2\pi T)$ (identical) terms. The angular average over the velocity factors yields $\langle v^2_{\mathbf{k}\sigma,\alpha}\rangle = \frac{1}{2}v^2_{F\sigma}$. As a result one finds the sum of Drude conductivities for both spin orientations

$$\sigma^{(0)}_{\alpha\alpha} = e^2\sum_\sigma \frac{n_\sigma\tau_\sigma}{m_\sigma} = e^2\sum_\sigma D^{(0)}_\sigma N_\sigma = e^2[4\pi n_{\text{imp}}V_0^2]^{-1}\sum_\sigma v^2_{F\sigma}/\left(1+\frac{1}{2}g^2_\sigma\right) \tag{10}$$

where $D^{(0)}_\sigma = \frac{1}{2}v^2_{F\sigma}\tau_\sigma$ is the diffusion constant. Thus, for weak scattering the conductivity is found to be inversely proportional to (1) the density of impurities, (2) the potential scattering cross section V_0^2, and (3) the factor $(1+\frac{1}{2}g^2_\sigma)$. Note that $D^{(0)}_\sigma N_\sigma = \varepsilon_{F\sigma}\tau_\sigma/2\pi \gg 1$, considering that in the parabolic band approximation $N_\sigma = m_\sigma/2\pi$.

3.3 Anomalous Hall effect: Skew scattering contribution

The lowest order diagram contributing to the Hall conductivity is a bubble with three scattering processes at the same impurity, denoted by lines running across (vertex correction needed to give finite angular averages), shown in Fig. 1a,

$$L_{xy}(i\Omega_m) = n_{\text{imp}}T\sum_{\omega_n}\sum_{\mathbf{k},\mathbf{k}',\sigma} v_{\mathbf{k}\sigma,x}v_{\mathbf{k}'\sigma,y}G_{\mathbf{k}\sigma}(i\omega_n)G_{\mathbf{k}\sigma}(i\omega_n - i\Omega_m)$$
$$\sum_{\mathbf{k}''} V_{\mathbf{k}\mathbf{k}''}G_{\mathbf{k}''\sigma}(i\omega_n)V_{\mathbf{k}''k'}G_{\mathbf{k}'\sigma}(i\omega_n)G_{\mathbf{k}'\sigma}(i\omega_n - i\Omega_m)V_{\mathbf{k}'\mathbf{k}} \tag{11}$$

and a contribution with upper and lower line interchanged. Here we use $V_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} = V_0\{1 - ig_\sigma\tau^z_{\sigma\sigma}\cdot(\widehat{\mathbf{k}}\times\widehat{\mathbf{k}}')_z\}$.

Again the energy integrations on $\varepsilon_k, \varepsilon_{k'}$ provide the restriction $0 \le \omega_n \le \Omega_m$ and yield a factor $(2\pi\tau_\sigma N_\sigma)^2$ while the energy integration on the single $G_{\mathbf{k}''\sigma}(i\omega_n)$ gives $-isign(\omega_n)\pi N_\sigma$ and the two

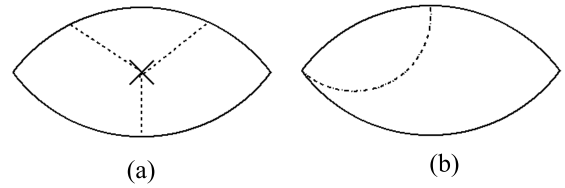

Fig. 1 lowest order diagrams for (a) skew scattering and (b) side-jump contributions. As described in the text, there are two diagrams of type (a) and four of type (b).

angular integrations are finite only for the cross terms in the product, using $\langle \widehat{k}_\alpha \widehat{k}_\beta \rangle = \frac{1}{2}\delta_{\alpha\beta}$. The skew scattering contribution to the Hall conductivity is then given by

$$\sigma_{xy}^{ss(0)} = e^2 \sum_\sigma \tau^z_{\sigma\sigma} D^{(0)}_\sigma N_\sigma \sqrt{w_\sigma} g_\sigma/(1+\frac{1}{2}g_\sigma^2) = \frac{e^2}{4}\sum_\sigma \tau^z_{\sigma\sigma}\Big(\frac{n_\sigma}{n_{\rm imp}}\Big)\frac{1}{\sqrt{w_\sigma}}\frac{g_\sigma}{(1+g_\sigma^2/2)^2} \quad (12)$$

where in the last equality we used the definition of $D^{(0)}_\sigma$ and of N_σ in the parabolic band approximation. It is interesting to note that the Hall conductivity is proportional to the spin-orbit coupling for small g_σ, while for large spin-orbit coupling, $g_\sigma \gg 1$, it is seen to decrease as $1/g_\sigma^3$. This regime may be reached only for sufficiently weak scattering, such that the conditions for the validity of the above weak coupling calculation, $w_\sigma, u_\sigma \ll 1$, are satisfied. In the absence of spin splitting, i.e. without spontaneous ferromagnetic polarization, the factor $\tau^{\,z}_{\sigma\sigma}$ causes $\sigma_{xy}^{ss(0)}$ to vanish. In the limit of weak polarization $\mathbf{M}$ (along the z-axis, perpendicular to the ferromagnetic layer), $\sigma_{xy}^{ss(0)}$ is proportional to M_z. The sign of $\sigma_{xy}^{ss(0)}$ is given by the sign of $(-M_z)$, provided $D^{(0)}_\sigma, N_\sigma, g_\sigma$ are increasing functions of the chemical potential, which would be the case for a parabolic energy spectrum. The ratio of anomalous Hall conductivity and longitudinal conductivity is small, proportional to (1) the ferromagnetic Zeeman energy in units of the Fermi energy (M_z/ε_F) (2) the spin-orbit coupling constant g_σ (for small coupling) (3) the dimensionless impurity potential $(V_0 N_0)$ where N_0 is the spin averaged density of states at the Fermi energy:

$$\sigma_{xy}^{ss(0)}/\sigma^{(0)}_{\alpha\alpha} \simeq \Big(\frac{M_z}{\epsilon_F}\Big)\; V_0 N_0 \;\langle g_0\rangle_\sigma \quad (13)$$

3.4 Anomalous Hall effect: Side-jump contribution

This effect may be calculated in a straightforward way [15,16] by observing that the side-jump leads to an additional term in the particle velocity due to the spin-orbit interaction. Indeed, the quantum mechanical velocity obtained from the Heisenberg equation of motion for the position operator has two terms,

$$\mathbf{v} = \frac{d}{dt}\mathbf{r} = -i[\mathbf{r}, H_1] = \frac{\mathbf{p}}{m} + (g_\sigma/4\pi n_\sigma)(\tau_{\sigma\sigma}\times\nabla V_{\rm dis})\,. \quad (14)$$

The matrix elements of $\mathbf{v}$ are given by

$$\langle \mathbf{k}'\sigma'|v\,|\,\mathbf{k}\sigma\rangle = \frac{\mathbf{k}}{m}\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} - i(g_\sigma/4\pi n_\sigma)\sum V(\mathbf{k}-\mathbf{k}')e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_j}\{\tau_{\sigma\sigma'}\times(\mathbf{k}-\mathbf{k}')\} \quad (15)$$

In lowest order in the impurity scattering, the side-jump contribution to the Hall conductivity σ_{xy} is calculated from four diagrams with an impurity correlator line ended at one of the current vertices and the upper or lower line, respectively (see Fig. 1b). Since they give identical contributions, we have to consider only the first one. In 2d systems confined to the x-y plane, and assuming the magnetization oriented in the z-direction, and the external frequency Ω_m to be positive,

$$L_{xy}^{sj,a} = n_{\rm imp}T\sum_{\omega_n}\sum_{\mathbf{k},\mathbf{k}',\sigma}\sum G_{\mathbf{k}'\sigma}(\omega_n)G_{\mathbf{k}\sigma}(\omega_n)G_{\mathbf{k}\sigma}(\omega_n-\Omega_m)\times$$

$$\times\,(-ig_\sigma V_0^2/\varepsilon_{F\sigma})[\tau_{\sigma\sigma'}\times\frac{(\mathbf{k}-\mathbf{k}'}{)}2m]_x\frac{k_y}{m} \tag{16}$$

Performing the energy integrations and the angular integrations as in the above, using the definition of τ_σ and multiplying the above by a factor of four to account for all the diagrams, it follows that the Hall conductivity is obtained as

$$\sigma_{xy}^{sj} = (e^2/2\pi)\sum_\sigma[g_\sigma/(1+\frac{1}{2}g_\sigma^2)]\;\tau\,^z_{\sigma\sigma} \tag{17}$$

Remarkably, σ_{xy}^{sj} is independent of the impurity concentration. Since the effective mean free path characteristic of the side-jump contribution is rather short, of order g_σ/k_F, it will be important only for dirty samples, when the skew scattering contribution is also small. A change in sign of the skew scattering contribution has been found for high impurity concentration, in the framework of the Coherent Potential Approximation [15].

3.5 Hall current in the clean limit

The extra term in the velocity operator derived in Eq. (14) involves any potential $V(\mathbf{r})$ acting on the conduction electrons. In Sect. 3.4 the potential considered was that due to impurities, $V_{\rm dis}(\mathbf{r})$. In the pesence of an applied electric field $\mathbf{E}$, putting ∇V in Eq. (14) equal to $-e\mathbf{E}$, the extra term in the velocity is directed orthogonal to $\mathbf{E}$ and to the magnetization $\mathbf{M}$, and hence may give rise to a Hall current. Since the velocity operator in this case is itself proportional to the applied field, to obtain the linear response current it is sufficient to average it with the equilibrium statistical operator resulting in

$$j_x = e^2\sum_{\mathbf{k}\sigma}\frac{g_\sigma}{4\pi n_\sigma}\;\tau^z_{\sigma\sigma}E_y f_{k\sigma} \tag{18}$$

where $f_{\mathbf{k}\sigma}$ is the Fermi function. With the help of $\sum_{\mathbf{k}} f_{\mathbf{k}\sigma} = n_\sigma$, one finds the contribution to the Hall conductivity even in the absence of impurities ("clean limit"):

$$\sigma_{xy}^c = \frac{e^2}{4\pi}\sum_\sigma g_\sigma\;\tau^z_{\sigma\sigma}\;. \tag{19}$$

This current flows in equilibrium and does not require the redistribution of particles from excited states into the equilibrium state. It is therefore similar to a ballistic current.

It is a single particle current and it is therefore not stabilized or proteced by collective effects. For that reason one may wonder how strongly this current decays as a consequence of inelastic or dephasing processes. This will be subject of future investigations.

Recently a Hall current in a perfect crystal lattice of rather similar form has been discussed [4–7] It arises due to a Berry phase acquired by electrons moving in the periodic potential of the crystal. A Bloch electron moving in reciprocal space under the influence of the combined effect of spin-orbit interaction and a ferromagnetic polarization along a path C acquires a Berry phase

$$\chi(\mathbf{k}) = -\int_c^{\mathbf{k}} d\mathbf{k}'\cdot\mathbf{X}(\mathbf{k}')\;\;. \tag{20}$$

Here the Berry vector potential $\mathbf{X}(\mathbf{k})$ (of dimension length, and thus describing a shift of the Wannier coordinate of the Bloch states within the unit cell), is given by

$$\mathbf{X}(\mathbf{k}) = \int_{\rm cell} d^2r\;\;u^*_{n\mathbf{k}}(\mathbf{r})\;i\nabla_k\;\;u_{n\mathbf{k}}(\mathbf{r})\;\;. \tag{21}$$

Associated with $\mathbf{X}(\mathbf{k})$ is a "magnetic field"

$$\boldsymbol{\Omega}(\mathbf{k}) = \nabla_k \times \mathbf{X}(\mathbf{k}) \tag{22}$$

acting in $\mathbf{k}$-space.

The quasi-classical dynamics of Bloch electrons including the Berry phase may be derived from the Bloch Hamiltonian

$$H_{\mathbf{k}} = V(i\nabla_k + \mathbf{X}_k) + \epsilon_{n\sigma}(\mathbf{k}) \ , \tag{23}$$

where $\epsilon_n(\mathbf{k})$ are the Bloch energies (including the spin-orbit interaction and the ferromagnetic polarization), and $V(\mathbf{r})$ is the applied external potential. Putting $\nabla V(\mathbf{r}) = -e\mathbf{E}$, the quasiclassical equations of motion derived from this effective Hamiltonian are

$$\begin{aligned} \dot{\mathbf{k}} &= e\mathbf{E} + e\dot{\mathbf{r}} \times \mathbf{B} \\ \dot{\mathbf{r}} &= \nabla_{\mathbf{k}}\epsilon_{n\mathbf{k}} - e\mathbf{E} \times \boldsymbol{\Omega} \end{aligned} \tag{24}$$

The additional term in the velocity $\dot{\mathbf{r}}$ leads to a Hall current

$$\mathbf{j}_H = -e^2 n \, \langle \boldsymbol{\Omega} \rangle \times \mathbf{E} \rightarrow \qquad \sigma_{xy}^B = e^2 n \quad \langle \Omega_z \rangle \tag{25}$$

where $\langle \boldsymbol{\Omega} \rangle = n^{-1} \sum_{\mathbf{k}\sigma} \ \boldsymbol{\Omega}_\sigma(\mathbf{k}) f(\epsilon_{\mathbf{k}\sigma})$ is the average of the Berry magnetic field over all occupied states in $\mathbf{k}$-space and $n = \Sigma_\sigma n_\sigma$. This average is zero unless time reversal symmetry is broken, e.g. in a ferromagnet, and the spin polarization is coupled to the orbital motion. This Hall current has been computed for ferromagnetic semiconductors [6], and found to be of the magnitude observed in such systems.

It would be instructive to compare the two types of Hall currents discussed in this subsection within the same model. The Berry phase mechanism is derived from the topological properties of Bloch states in a priodic potential, moving in $\mathbf{k}$-space under the influence of an external field. The mechanism discussed first does not require the presence of a periodic potential, but derives solely from the extra term in the velocity operator due to spin-orbit interaction.

4 Strong impurity scattering

The scattering potential of a single impurity is not necessarily small compared to the Fermi energy. If the product $N_\sigma V_0$ is of order unity, or even larger than unity, or, in other words, if the coupling constants w_σ, u_σ are not small compared to unity, repeated multiple scattering off the same impurity becomes important.In technical terms, the low order (Born) transition amplitudes we have used in the above must be replaced by the full scattering amplitude. As may be expected, this results in a substantial renormalization of the conductivities.

4.1 Single particle scattering amplitude

The repeated scattering of an electron off a single impurity may be described in terms of the dimensionless scattering amplitude

$$f_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} = \pi N_\sigma \delta_{\sigma\sigma'} [V_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} + VGV + VGVGV + ...] \tag{26}$$

where G is the single particle Green's function.V is the bare interaction with one impurity at $\mathbf{R} = \mathbf{0}$. In the following we assume short-ranged interaction and again drop the k-dependence of $V(\mathbf{k} - \mathbf{k}')$. The dependence on energy is governed by the sharply peaked G's, so that we may neglect the dependence on the magnitude of momentum and may put $|\mathbf{k}| = k_{F\sigma}$, the Fermi momentum for spin orientation σ. Then the energy integration in intermediate states may be done, The remaining integrations on angle may be done using (in 2d and neglecting the angular dependence of $V(\mathbf{k}-\mathbf{k}')$), $\langle\widehat{\mathbf{k}}\rangle = 0, \langle\widehat{\mathbf{k}}^2\rangle = \frac{1}{2}$, $\langle(\widehat{\mathbf{k}}\times\widehat{\mathbf{k}}_1)_z(\widehat{\mathbf{k}}_1\times\widehat{\mathbf{k}}')_z\rangle_{k_1} = -\frac{1}{2}(\widehat{\mathbf{k}}\cdot\widehat{\mathbf{k}}')$, $\langle(\widehat{\mathbf{k}}\cdot\widehat{\mathbf{k}}_1)(\widehat{\mathbf{k}}_1\times\widehat{\mathbf{k}}')_z\rangle_{\mathbf{k}_1} = \frac{1}{2}(\widehat{\mathbf{k}}\times\widehat{\mathbf{k}}')_z$, where we defined $\langle O\rangle = \int\frac{d\varphi}{2\pi}O$. Within these approximations the potential scattering and spin-orbit scattering terms do not mix. Odd terms in V do not depend on s_{ω_n} whereas even terms do. The terms of perturbation theory may be summed up to infinity as a geometric series, with the result

$$f_{\mathbf{k}\sigma,\mathbf{k}'\sigma} = \frac{\widetilde{w}_\sigma}{\sqrt{w_\sigma}} - i\tau^z_{\sigma\sigma}(\widehat{\mathbf{k}}\times\widehat{\mathbf{k}}')\frac{2\widetilde{u}_\sigma}{\sqrt{u_\sigma}} - is_{\omega_n}[\widetilde{w}_\sigma + 2\widetilde{u}_\sigma(\widehat{\mathbf{k}}\cdot\widehat{\mathbf{k}}')]. \tag{27}$$

Here we defined $\widetilde{w}_\sigma = w_\sigma/(1+w_\sigma)$, and $\widetilde{u}_\sigma = u_\sigma/(1+u_\sigma)$, where $w_\sigma = (\pi N_\sigma V_0)^2$ and $u_\sigma = (g_\sigma/2)^2 w_\sigma$, and all quantities depend on the spin orientation σ (suppressed in the following, except in the final expressions involving spin summation).

4.2 Single particle relaxation rate

The single particle relaxation rate τ_σ^{-1} is obtained from the imaginary part of the self energy

$$\frac{1}{\tau_\sigma} = 2s_{\omega_n}\mathrm{Im}\Sigma_{\mathbf{k}\sigma}(\omega_n) = 2s_{\omega_n}n_{\mathrm{imp}}\mathrm{Im}f_{\mathbf{k}\sigma,\mathbf{k}\sigma} = 2\frac{n_{\mathrm{imp}}}{\pi N_\sigma}(\widetilde{w}_\sigma + 2\widetilde{u}_\sigma) \tag{28}$$

where n_{imp} is the density of impurities (number of impurities per volume). One observes that $\frac{1}{\tau_\sigma}$ is proportional to the Fermi energy, the average number of impurities per electron and the dimensionless factor $(\widetilde{w}+2\widetilde{u})$ expressing the effective scattering strength per impurity.

4.3 Particle-hole propagator

The particle-hole propagator $\Gamma_{\mathbf{k}\mathbf{k}'}(\mathbf{q};\epsilon_n,\epsilon_n-\Omega_m)$ is an important ingredient of vertex corrections of any kind. Here $\mathbf{k}+\mathbf{q/2},\mathbf{k}-\mathbf{q/2}$ are the initial, $\mathbf{k}'+\mathbf{q/2},\mathbf{k}'-\mathbf{q/2}$ the final momenta and $\epsilon_n,\epsilon_n-\Omega_m$ are the Matsubara frequencies of the particle and the hole line, respectively. Γ satisfies the following Bethe-Salpeter equation (we have defined dimensionless quantities Γ, t by multiplying both with a factor $(2\pi N_\sigma\tau_\sigma)$)

$$\begin{aligned}\Gamma_{\mathbf{k}\mathbf{k}'}(\mathbf{q};i\epsilon_n,i\epsilon_n-i\Omega_m) = {} & t_{\mathbf{k}\mathbf{k}'}(\mathbf{q};i\epsilon_n,i\epsilon_n-i\Omega_m) + (2\pi N_\sigma\tau_\sigma)^{-1} \\ & \times\sum_{\mathbf{k}_1}t_{\mathbf{k}\mathbf{k}_1}(\mathbf{q};i\epsilon_n,i\epsilon_n-i\Omega_m)G_{\mathbf{k}_1+\mathbf{q}/2,\sigma}(i\epsilon_n) \\ & G_{\mathbf{k}_1-\mathbf{q}/2,\sigma}(i\epsilon_n-i\Omega_m)\Gamma_{\mathbf{k}_1\mathbf{k}'}(\mathbf{q};i\epsilon_n,i\epsilon_n-i\Omega_m)\end{aligned} \tag{29}$$

The kernel of this equation is the impurity averaged particle-hole scattering amplitude (we consider only the case of equal spin of particle and hole)

$$t_{\mathbf{k}\mathbf{k}'}(\mathbf{q};i\epsilon_n,i\epsilon_n-i\Omega_m) = n_{\mathrm{imp}}f_{\mathbf{k}+\mathbf{q}/2,\sigma;\mathbf{k}'+\mathbf{q}/2,\sigma}(i\epsilon_n)f_{\mathbf{k}'-\mathbf{q}/2,\sigma;\mathbf{k}-\mathbf{q}/2,\sigma}(i\epsilon_n-i\Omega_m) \tag{30}$$

It is useful to represent the operator $t_{\mathbf{k}\mathbf{k}'}(\mathbf{q}=0)$ in terms of its eigenvalues λ_m. Assuming isotropic band structure, the eigenfunctions $\chi_m(\widehat{\mathbf{k}}) = \exp(im\varphi)$ are those of the angular momentum operator component L_z. The operator $t^{+-}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}=0)$ may be represented as

$$t^{+-}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}=0) = \sum_m\lambda_m\chi_m(\widehat{\mathbf{k}})\chi^*_m(\widehat{\mathbf{k}}') \quad \text{and} \quad t^{-+}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}=0) = [t^{+-}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}=0)]^* \tag{31}$$

The eigenvalues for $s_p = \mathrm{sign}(\epsilon_\mathrm{n}) = 1$, $s_h = \mathrm{sign}(\epsilon_\mathrm{n}-\Omega_\mathrm{m}) = -1$ are given by

$$\lambda_0 = 1 \quad , \qquad \lambda_1 = 2\widetilde{w}\widetilde{u}(\widetilde{w}+2\widetilde{u})^{-1}(1+is_p\frac{1}{\sqrt{u}}\tau^z_{\sigma\sigma}) \tag{32}$$

$$\lambda_2 = \frac{\widetilde{u}^2}{u}(\widetilde{w} + 2\widetilde{u})^{-1}(u - 1 + 2is_p\sqrt{u}\tau^z_{\sigma\sigma}) \quad and \quad \lambda_{-m} = \lambda^*_m. \tag{33}$$

The energy integral over the product of Green's functions in the integral equation for $\Gamma_{\mathbf{kk}'}$ may be done first, after expanding the G's in Ω_m and q, where $\mathbf{q} \cdot \mathbf{v}_k = qv_F(\widehat{\mathbf{q}} \cdot \widehat{\mathbf{k}})$. Expanding $\Gamma_{\mathbf{kk}'}$ and $t_{\mathbf{kk}'}$ in terms of eigenfunctions $\chi_m(\widehat{\mathbf{k}})$, $\Gamma_{\mathbf{kk}'} = \sum_m \Gamma_{m\mathbf{k}'}\chi_m(\widehat{\mathbf{k}})$, one obtains

$$\Gamma_{m\mathbf{k}'} = \lambda_m\chi_m(\widehat{\mathbf{k}}') + \lambda_m\Big\{\Big[1 - \tau(|\Omega_m| + D_0q^2)\Big] - \frac{i}{2}v_Fq\tau\Big[\Gamma_{m-1\mathbf{k}'}\chi^*_1(\widehat{\mathbf{q}}) + \\ + \Gamma_{m+1\mathbf{k}'}\chi_1(\widehat{\mathbf{q}})\Big] - \frac{1}{4}(v_Fq\tau)^2\Big[\Gamma_{m-2\mathbf{k}'}\chi^*_2(\widehat{\mathbf{q}}) + \Gamma_{m+2\mathbf{k}'}\chi_2(\widehat{\mathbf{q}})\Big]\Big\} \tag{34}$$

The case $m = 0$ needs special consideration, because particle number conservation causes $\Gamma_{0\mathbf{k}'}$ to have a pole in the limit $\Omega_m, q \to 0$, here expressed by $\lambda_0 = 1$.

The complete particle-hole propagator in the regime $v_Fq\tau < 1$ is given by

$$\Gamma_{\mathbf{kk}'} = \frac{1/\tau}{|\Omega_m| + Dq^2}\gamma_{\mathbf{k}}\widetilde{\gamma}_{\mathbf{k}'} + \sum_{m\neq 0}\widetilde{\lambda}_m\chi_m(\widehat{\mathbf{k}})\chi^*_m(\widehat{\mathbf{k}}') \quad ; \quad \widetilde{\lambda}_m = \frac{\lambda_m}{1 - \lambda_m} \tag{35}$$

$$\gamma_{\mathbf{k}} = 1 - \frac{i}{2}v_Fq\tau\sum_{m=\pm 1}\widetilde{\lambda}_m\chi_m(\widehat{\mathbf{k}})\chi^*_m(\widehat{\mathbf{q}}') = 1 - i\tau\sum_{m=\pm 1}\widetilde{\lambda}_m\chi_m(\widehat{\mathbf{k}})\langle\mathbf{q} \cdot \mathbf{v}_{k'}\chi^*_m(\widehat{\mathbf{k}}')\rangle_{\mathbf{k}'} \tag{36}$$

$$\widetilde{\gamma}_{\mathbf{k}} = 1 - i\tau\sum_{m=\pm 1}\widetilde{\lambda}_m\chi^*_m(\widehat{\mathbf{k}})\langle\mathbf{q} \cdot \mathbf{v}_{k'}\chi_m(\widehat{\mathbf{k}}')\rangle_{\mathbf{k}'} \tag{37}$$

where the renormalized diffusion constant is defined as

$$D_\sigma = D^{(0)}_\sigma\frac{1 - \lambda'_1}{|1 - \lambda_1|^2}, \qquad \text{where} \qquad \lambda'_1 = \text{Re} \quad \lambda_1. \tag{38}$$

The vertex corrections of the current vertices $j_{\mathbf{k}\alpha}$ and $\widetilde{j}_{\mathbf{k}\alpha}$ (for the incoming and outgoing current) are obtained by

$$j_{\mathbf{k}\sigma,\alpha}(q) = \mathbf{v}_{\mathbf{k}\alpha} + \langle\mathbf{v}_{\mathbf{k}'\alpha}\Gamma_{\mathbf{k}'\mathbf{k}}\rangle_{\mathbf{k}'} \\ \widetilde{j}_{\mathbf{k}\sigma,\alpha}(q) = \mathbf{v}_{\mathbf{k}\alpha} + \langle\mathbf{v}_{\mathbf{k}'\alpha}\Gamma_{\mathbf{kk}'}\rangle_{\mathbf{k}'} \tag{39}$$

Note that $\widetilde{j}_{\mathbf{k}\alpha} \neq (j_{\mathbf{k}\alpha})^*$ as the eigenvalues $\widetilde{\lambda}_m$ are complex valued, in general.

4.4 Conductivity tensor: Skew scattering

Multiple scattering off the same impurity generates momentum dependence of the effective scattering amplitude and gives rise to vertex corrections to the current vertex as calculated above. The current-current correlator is now given by

$$L_{\alpha\beta}(i\Omega_m) = T\sum_{\omega_n}\sum_{\mathbf{k},\sigma}v_{\mathbf{k}\sigma,\alpha}\widetilde{j}_{\mathbf{k}\sigma,\beta}(q = 0)G_{\mathbf{k}\sigma}(i\omega_n)G_{\mathbf{k}\sigma}(i\omega_n - i\Omega_m) \tag{40}$$

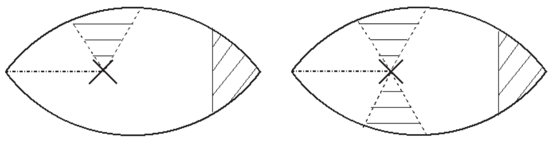

Fig. 2 Two types of diagrams for the side-jump contribution in the strong scattering regime. There are four diagrams of type (a) and two diagrams of type (b).

Using the explicit expressions for the components of the current vertex

$$\widetilde{j}_{\mathbf{k}\sigma,x}(q=0) = v_{F\sigma}[(1+\widetilde{\lambda}_1')\widehat{k}_x - \widetilde{\lambda}_1''\widehat{k}_y] \ ; \quad \widetilde{j}_{\mathbf{k}\sigma,y}(q=0) = v_{F\sigma}[(1+\widetilde{\lambda}_1')\widehat{k}_y + \widetilde{\lambda}_1''\widehat{k}_x] \tag{41}$$

where the terms involving the diffusion pole drop out (they involve an angular average $\langle q_x \ q_y \rangle = 0$), one finds for the conductivity tensor

$$\sigma^{ss}_{\alpha\beta} = e^2 \sum_\sigma D^{(0)}_\sigma N_\sigma \begin{pmatrix} 1+\widetilde{\lambda}_\sigma' & \widetilde{\lambda}_\sigma'' \\ -\widetilde{\lambda}_\sigma'' & 1+\widetilde{\lambda}_\sigma' \end{pmatrix} \tag{42}$$

where we have defined $\widetilde{\lambda}_\sigma' = \mathrm{Re}\widetilde{\lambda}_1^\sigma$ and $\widetilde{\lambda}_\sigma'' = \mathrm{Im}\widetilde{\lambda}_1^\sigma$. We recall that $\widetilde{\lambda}_\sigma = \lambda_{1\sigma}/(1-\lambda_{1\sigma})$ and $\lambda_{1\sigma}' = 2\widetilde{w}\widetilde{u}(\widetilde{w}+2\widetilde{u})^{-1}$, $\lambda_{1\sigma}'' = \lambda_{1\sigma}' \frac{1}{\sqrt{u}} \tau^z_{\sigma\sigma}$. Defining the tensor of diffusion coefficients $D^\sigma_{\alpha\beta}$ as

$$D^\sigma_{\alpha\alpha} = \frac{1}{2} v^2_{F\sigma} \tau^{tr}_\sigma \ , \quad D^\sigma_{xy} = D^\sigma_{\alpha\alpha}[\widetilde{\lambda}''/(1+\widetilde{\lambda}')] = -D^\sigma_{yx} \tag{43}$$

where $\tau^{tr}_\sigma = \tau_\sigma(1+\widetilde{\lambda}')$ is the momentum relaxation time, we may write

$$\sigma^{ss}_{\alpha\beta} = \sum_\sigma N_\sigma D^\sigma_{\alpha\beta} \ . \tag{44}$$

4.5 Side-jump mechanism

In the strong scattering regime the bare impurity scattering potential needs to be replaced by the scattering amplitude. In addition the vertex corrections to the current density operator have to be applied. There are two new diagrams involving the scattering line denoting the spin-orbit term in the velocity operator, Eq. (14), and ending at one of the current vertices, framed by two (instead of only one) scattering amplitude lines (see Fig. 2).

The result of adding the six diagrams is

$$\sigma^{sj}_{xy} = \frac{e^2}{2\pi} \sum_\sigma \tau^z_{\sigma\sigma} g_\sigma \frac{\tilde{w}_\sigma}{\tilde{w}_\sigma + 2\tilde{u}_\sigma} \frac{1+\tilde{\lambda}_{1\sigma}'}{(1+u_\sigma)} \tag{45}$$

4.6 Limiting cases of Anomalous Hall conductivity and comparison of contributions

In the limit $N_0 V_o \gg 1$, the results simplify considerably. The sum of the skew scattering and side jump contributions is then given by

$$\sigma^{ss}_{xy} + \sigma^{sj}_{xy} = e^2 \sum_\sigma \tau^z_{\sigma\sigma} \Big[\frac{1}{2} \frac{1}{g_\sigma \sqrt{w_\sigma}} \Big(\frac{n_\sigma}{n_{\mathrm{imp}}} \Big) + \frac{6}{\pi} \frac{1}{g_\sigma w_\sigma} \Big] \tag{46}$$

It is seen that in this case the skew scattering term dominates even in the limit of large impurity concentration.

In the case of strong potential scattering, but weak spin-orbit interaction, $w \gg 1$, but $u \ll 1$, we find

$$\sigma_{xy}^{ss} + \sigma_{xy}^{sj} = e^2 \sum_{\sigma} \tau_{\sigma\sigma}^{z} \Big[\frac{1}{2} \sqrt{u_\sigma} \Big(\frac{n_\sigma}{n_{\rm imp}} \Big) + \frac{1}{2\pi} g_\sigma \Big] \, . \tag{47}$$

By comparison, the clean limit Hall conductivity σ_{xy}^{c} depends only on the spin-orbit coupling g_σ (see Eq. (19)) or in the case of the Berry phase contribution, on the Berry magnetic field $\mathbf{\Omega}$ (see Eq. (25)). The former is not dependent on impurity scattering, as well as the latter. One observes that the signs of σ_{xy}^{c} and σ_{xy}^{sj} are the same. In the weak scattering limit σ_{xy}^{sj} is a factor of two larger than σ_{xy}^{c}, but with increasing scattering strength it drops to values much less than σ_{xy}^{c}. The sign and magnitude of σ_{xy}^{B} have not been calculated for the model under consideration here.

Within the model considered here the skew scattering contribution will dominate all other contributions in the limit $n_{\rm imp} \to 0$. In a more refined model, however, assuming scattering centers with weak (or no) spin orbit interaction of density n_n in addition to the skew scattering centers with density $n_{\rm imp}$, the mean free path in the limit $n_{\rm imp} \to 0$ will be limited by the normal scattering processes, and consequently $\sigma_{xy}^{ss} \propto n_{\rm imp}/n_n^2$ and $\sigma_{xy}^{sj} \propto n_{\rm imp}/n_n$ will tend to zero for $n_{\rm imp} \to 0$. In this case the clean limit contributions σ_{xy}^{c} and σ_{xy}^{B} will survive.

5 Conclusion

The anomalous Hall effect is a surprisingly rich phenomenon with many interesting facets. It requires broken time reversal symmetry as realized in magnetically ordered states and, as far as the symmetry breaking occurs in spin space, it requires sufficiently strong spin-orbit coupling. The quantum nature of electron scattering by any impurity potential including spin-orbit coupling leads to a right-left asymmetry of the average scattering probability in third order of the potential ("skew scattering") and to an extra contribution to the velocity operator ("side-jump" effect). Both contribute to the Hall conductivity in zero (or low) magnetic field. In addition, an extra term in the velocity arises directly from the applied electric field, for a uniform system as well as in the periodic potential of the crystal. The latter contribution has been shown to be a consequence of a Berry phase associated with the motion of Bloch electrons in momentum space. The extra term in the velocity proportional to the modulus of the applied electric field and directed perpendicular to it has a finite equilibrium expectation value, yielding a Hall current for the clean system. A systematic experimental investigation showing the existence of all these contributions in a controlled fashion does not exist yet. Further experimental work on well-characterized systems as a function of magnetization, impurity concentration, for different strengths of spin-orbit interaction is necessary. Theoretical models for these systems need to be refined, using band structure calculations, to model impurity scattering and spin-orbit interaction in a more realistic way.

Acknowledgements We thank Art Hebard for stimulating our interest in the anomalous Hall effect and for useful discussions. KAM thanks U. Karlsruhe for support and hospitality during his visit. PW acknowledges partial support through a Max-Planck Research Award. This project has also been supported by the DFG Center for Functional Nanostructures under subproject B2.9.

References

[1] M. Pugh, N. Rostoker, and A. Schindler, Phys. Rev. **80**, 688 (1950).
[2] I.A. Campbell and A. Fert, Ferromagnetic Materials, vol. 3, edited by E.P. Wohlfarth (North-Holland, Amsterdam, 1982).
[3] R. Karplus and J.M. Luttinger, Phys. Rev. **95**, 1154 (1954).
[4] G. Sundaram and Q. Niu, Phys. Rev. B **59**, 14915 (1999).
[5] M. Onada and N. Nagaosa, J. Phys. Soc. Jpn **71**, 19 (2002).

[6] T. Jungwirth, Q. Niu, and A. H. MacDonald, Phys. Rev. Lett. **88**, 207208 (2002).
[7] F. D. M. Haldane, Phys. Rev. Lett. **93**, 206602 (2004).
[8] J. Smit, Physica **21**, 877 (1955); Phys. Rev. B **8**, 2349 (1973).
[9] J. M. Luttinger, Phys. Rev. **112**, 739 (1958).
[10] L. Berger, Phys. Rev. B **2**, 4559 (1970).
[11] A. Langenfeld and P. Wölfle, Phys. Rev. Lett. **67**, 739 (1991).
[12] V. K. Dugaev, A. Crepieux, and P. Bruno, J. Mag. Mag. Mat. **240**, 159 (2002); Phys. Rev. B **64**, 104411 (2001).
[13] K. A. Muttalib and P. Wölfle, unpublished.
[14] P. Mitra and A. F. Hebard, unpublished.
[15] A. Crepieux, J. Wunderlich, V. K. Dugaev, and P. Bruno, J. Mag. Mag. Mat. **242–245**, 464 (2002).
[16] A. Crepieux, P. Bruno, Phys. Rev. B **64**, 014416 (2001).