Anomalous jumping in a double-well potential

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Noise-induced jumping between metastable states in a potential depends on the structure of the noise. For an α-stable noise, jumping triggered by single extreme events contributes to the transition probability. This is also called Levy flights and might be of importance in triggering sudden changes in geophysical flow and perhaps even climatic changes. The steady-state statistics is also influenced by the noise structure leading to a non-Gibbs distribution for an α-stable noise. [S1063-651X(99)04707-8]

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I. INTRODUCTION

Noise-induced jumping between metastable states separated by potential barriers is common in physical systems. The time scale for the barrier penetration depends on the structure of the noise. Most often the noise is Gaussian. However, non-Gaussian noises distributed with power-function tails, Levy flights, are observed in many different physical systems [1] such as turbulent diffusion [2,3] and vortex dynamics [4]. Levy flights also seem to be a common feature in dynamical models [5] and critical phenomena [6].

The Levy flights can result from a Langevin equation driven by α-stable noise and give rise to anomalous diffusion of a random walker with position $r(t)$ such that $\langle|r(t)-r(0)|^2\rangle \propto Dt^{2/\alpha}$ where D is a constant and $0<\alpha<2$ [16]. The case $\alpha=2$ corresponds to normal diffusion where D is the diffusion constant. The exponent α is related to the scaling of the tail of the probability distribution for the increments of the random walker, $P(X>r)\propto r^{-\alpha}$. For $\alpha \geqslant 2$ the second moment exists and by the central limit theorem the random walker reduces in the continuum limit to a Gaussian random walker unless the diffusion takes place on a fractal set like in a quenched random medium [7]. In this case the random walk can be subdiffusive. Another example of a process which can be subdiffusive is the Levy walk where a random walker has a constant speed in between discrete stochastic time points (a renewal process) with a power-function tail distribution. Note that since the time process is discrete for a Levy walk it cannot result from a Langevin equation.

Anomalous diffusion was first observed in hydrological time series [9]. Recently evidence for α-stable statistics in atmospheric circulation data has been reported [10]. In a long paleoclimatic time series an α-stable noise-induced jumping in a double-well potential was found [11]. In both cases α was found to be around 1.7. The latter describes a jumping, in glacial times, between two climatic states governed by the oceanic flow forced by random fluctuations from the atmosphere. Understanding the role of extreme events and the time scales for these climatic shifts is the main motivation for this study.

In this paper we will interchangeably use the physics jargon $\langle x\rangle$ and the mathematics jargon $E[x]$ for the expectation value for x. The latter will be used in the case of conditional expectations. We use the usual convention that probability distribution functions P are capitalized and probability density functions, $p=dP/dx$, are in small letters.

II. α-STABLE DISTRIBUTIONS

For distributions with power-function tails, $P(X>x)\propto x^{-\gamma}$, only moments of order less than γ exist ($\langle|x|^\beta\rangle=\infty$ for $\beta\geqslant\gamma$). For $0<\gamma<2$ a generalized version of the central limit theorem applies, namely, that the average of n-independent stochastic variables from the distribution P asymptotically will have an α-stable distribution as $n\rightarrow\infty$ with $\alpha=\gamma$. The α-stable distributions are defined by their characteristic functions, $\langle \exp(ikX)\rangle = \exp(-\sigma^\alpha|k|^\alpha/\alpha)$. The α-stable distributions are stable with respect to averaging, $Y_n = n^{-1/\alpha}\Sigma_{i=1}^n X_i$, meaning that Y_n has the same distribution as X_i where the X_i's are independent identically distributed (i.i.d.) α stable, thus the phrase ''α stable.'' As for the case of Gaussian noise, the dynamics of a noise-driven system with power-function tail distributions for the noise increments, $P(X>x)\propto x^{-\alpha}, 0<\alpha<2$, will reduce to a system with an α-stable noise in the continuum limit, described by a Langevin equation [8],

$$dX = f(X)dt + \sigma(X)dL_\alpha. \tag{2.1}$$

A random walker with α-stable noise increments will be superdiffusive due to the large jumps from the tails of the distribution surviving the averaging in the continuum limit. See Appendix A for a further short description.

III. FOKKER-PLANCK EQUATION

The probability density for X in Eq. (2.1) is determined from the Fokker-Planck equation (FPE), see Appendix B for a derivation,

$$\partial_t p(x) = -\partial_x[f(x)p(x)] - \frac{1}{\alpha}\int\int e^{-ikx}\widehat{\sigma^\alpha}(k-k_1)|k|^\alpha \hat{p}(k_1)dkdk_1. \tag{3.1}$$

The second term on the right-hand side is expressed in terms of the Fourier-transformed probability density $\hat{p}(k)$. This term reduces to the ordinary diffusion term

$\partial_{x^2}[\sigma^2(x)p(x)]/2$ when $\alpha=2$. In this case the solution for the stationary probability density function can be expressed explicitly in the well-known form,

$$p(x)\propto\frac{1}{\sigma^2(x)}\exp\left\{2\int_0^x\frac{f(y)}{\sigma^2(y)}dy\right\}. \tag{3.2}$$

For $\alpha<2$ the FPE (3.1) is nonlocal in spectral space. This is a reflection of the superdiffusivity of the process (2.1). Besides the Gaussian case we can only solve the FPE explicitly for $\alpha=1$. This is the case of a system driven by Cauchy-distributed noise having the probability density, $q(x)=1/[\pi(1+x^2)]$, see Appendix C for further details on the Cauchy distribution. We are using $p(x)$ for the probability density for X in Eq. (2.1) and $q(x)$ for the probability density of the noise. Then the stationary FPE becomes

$$i\int\hat{f}(k_1-k)\hat{p}(k_1)dk_1=\mathrm{sgn}(k)\sigma\hat{p}(k), \tag{3.3}$$

where the noise intensity σ is taken to be constant. From taking the derivative with respect to k on both sides of Eq. (3.3) and performing a partial integration on the left-hand side, it follows that

$$i\int\hat{f}(k_1-k)\hat{p}^{(m)}(k_1)dk_1=\mathrm{sgn}(k)\sigma\hat{p}^{(m)}(k) \tag{3.4}$$

for any m. The solution is $\hat{p}(k)=e^{-\lambda|k|}$, where λ is determined by

$$i\int\hat{f}(k_1-k)e^{-\lambda(k_1-k)}dk_1=if(i\lambda)=\mathrm{sgn}(k)\sigma. \tag{3.5}$$

Thus the solution is determined by the analytic continuation of $f(x)$ into the complex plane, provided it exists. Note that the solution also applies for $k=0$ where the right-hand side of Eq. (3.4) jumps, since from the definition of the Fourier transform of the probability density we have $\hat{p}(0)=E[1]=1$. By complex conjugation of Eq. (3.5) we get $if(-i\lambda^*)=\mathrm{sgn}(-k)\sigma$, so for $k<0$ the solution is given as $-\lambda^*$, where λ solves Eq. (3.5) for $k>0$. With $\lambda=\beta+i\delta$ the characteristic function is given as $\hat{p}(k)=e^{-\beta|k|}e^{-i\delta k}$. For $\hat{p}(k)$ to be a characteristic function we must have $\beta>0$, and the stationary distribution is

$$p(x)=\sum_{i=1}^{N}p_i\frac{1}{\pi}\frac{\beta_i}{\beta_i^2+(x+\delta_i)^2}, \tag{3.6}$$

where the sum is over the N zero points of the complex function $if(i\lambda)-\mathrm{sgn}(k)\sigma$ in the upper half-plane $(\beta>0)$. In this solution of the stationary FPE there is an indeterminacy since any $p(x)$ with $\Sigma_i p_i=1$ is a probability density that satisfies Eq. (3.1).

The indeterminacy might be related to the problem of conservation of probability. If there is a finite probability for the random walker to escape to infinity, it must be reinserted into the system for a stationary probability density to be conserved. Then the indeterminacy in the reinsertion could result in the indeterminacy in the coefficients p_i in Eq. (3.6).

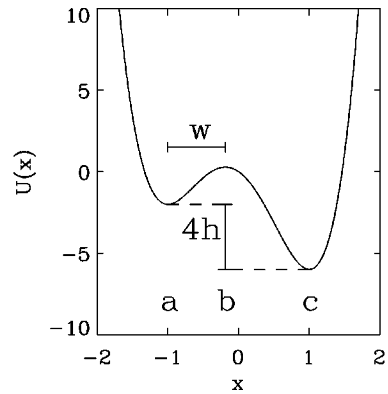

FIG. 1. The potential (4.1). $4h=U(a)-U(c)$ is the potential difference between the two minima, $w=b-a$ is the ''left half-width.'' Units are arbitrary.

However, the indeterminacy can be lifted in the limit $\sigma\rightarrow 0$. When the intensity of the noise becomes small the Cauchy distribution approaches a δ distribution (when acting on functions that are bounded by $|x|^\beta$ for some $\beta<\alpha$ as $x\rightarrow\infty$). Then we can approximate the system by a system with discrete states and the stationary Fokker-Planck equation Eq. (3.3) is approximated by N transition (Master) equations for the weights $p_i, i=1,\ldots,N$,

$$p_i=\sum_j p_j p(j\rightarrow i), \tag{3.7}$$

where i,j represents the N minima defined in Eq. (3.6). The transition probabilities $p(j\rightarrow i)$ are related to the transition waiting times, which will be defined in the following.

IV. POTENTIAL

Before proceeding we will define the drift term as resulting from a potential. The governing equation then describes a massless, viscous particle moving in a potential, $f(x)=-dU/dx$. As an example for study we define the potential as

$$U(x)=4(x/\Delta)^4+h(x/\Delta)^3-8(x/\Delta)^2-3h(x/\Delta). \tag{4.1}$$

$U(x)$ is a double-well potential for $-16/3<h<16/3$. $4h$ is the level difference between two potential minima at $x=-\Delta\equiv a$ and $x=\Delta\equiv c$. The local potential maximum between the two minima is at $x=-3h\Delta/16\equiv b$, and the potential values are $[U(a),U(b),U(c)]=[-4(1-h/2),(3h/16)^2(8-3h^2/64),-4(1+h/2)]$. See Fig. 1. The results are readily generalized to other forms of the potential $U(x)$.

V. WAITING TIME

The waiting time for jumping between the two potential minima (from a to c) of $U(x)$ defined above is exponentially distributed. With $p_{ac}(\tau>t)$ being the probability of staying in minimum a longer than t we have $p_{ac}(\tau>t)=\exp(-t/T_{ac})$ with a mean waiting time T_{ac}. This follows from the Markov property of the Langevin equation in the discrete state limit, since we have

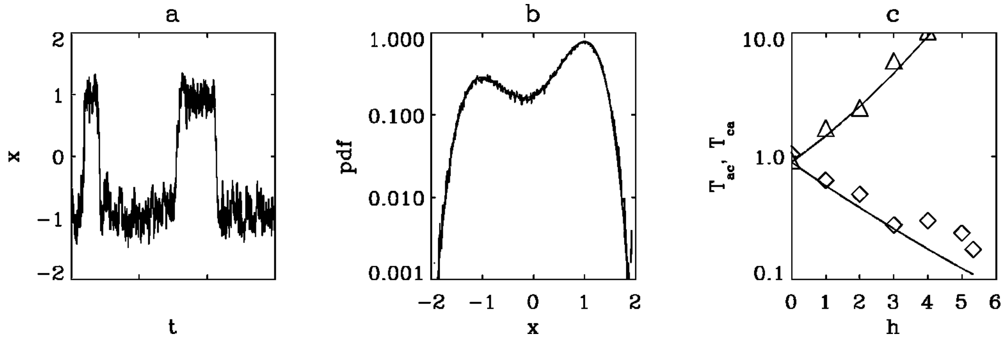

FIG. 2. A simulation of Eq. (2.1) with Gaussian noise and the potential shown in Fig. 1. (a) shows a realization and (b) the probability density function. The actual simulation is 1000 times longer than what is shown in (a). The smooth curve in (b) is the probability density function calculated from Eq. (3.2). (c) shows the mean waiting times, T_{ac} (diamonds) and T_{ca} (triangles), for seven simulations with varying *h*. The curves are the waiting times calculated from Eq. (5.2). Units are arbitrary.

$$P(t<\tau<t+\Delta t)/\Delta t=(1-\lambda_{ac}\Delta t)^{t/\Delta t}\lambda_{ac}\to\lambda_{ac}\exp(-\lambda_{ac}t), \tag{5.1}$$

as $\Delta t\to 0$, where $\lambda_{ac}=1/T_{ac}$ is the transition probability intensity. In the nondiscrete case, a little more rigorous treatment is needed [12]. However, the result holds, if the potential wells are substituted for the minima, and the waiting time is defined as the time between consecutive crossings of *a* and *c*.

A. Gaussian noise and Arrhenius formula

In the case of Gaussian noise in Eq. (2.1) T_{ac} can be calculated from the backward Fokker-Planck equation [12],

$$T_{ac}\approx\frac{2}{\sigma^2}\int_{-\infty}^{b}dx e^{-2U(x)/\sigma^2}\int_a^c dy e^{2U(y)/\sigma^2}, \tag{5.2}$$

and correspondingly for T_{ca}. By using the saddle-point approximation in Eq. (5.2) we obtain the Arrhenius formula,

$$T_{ac}\propto \exp\{2[U(b)-U(a)]/\sigma^2\}. \tag{5.3}$$

For comparison with the case of α-stable noise, Fig. 2 displays the standard result of a numerical simulation in the case of Gaussian noise. Figure 2(a) shows the simulated process with the potential in Fig. 1. Figure 2(b) shows the simulated probability density function and the right-hand side of Eq. (3.2). Figure 2(c) shows the time scale for jumping as a function of the parameter *h*. The time scale is calculated from the exponential distribution of times between consecutive crossings of the levels *a* and *c*. Figure 3 shows the number of crossings (from *a* to *c* and from *c* to *a*, respectively) with a waiting time larger than each waiting time measured, normalized by the total number of crossings. These points are situated on straight lines in the semilogarithmic plot where T_{ac} and T_{ca} are the slopes of the lines. Figure 2(c) shows the time scales for seven simulations with different *h*. The curves are the time scales calculated from Eq. (5.2).

B. α-stable noise

In the case $\alpha<2$ the situation is radically different. The sample curves of the process are no longer continuous and the finite jumps or extreme events will contribute to the probability of jumping between the potential wells. The probability $[\lambda_{ac}\Delta t+o(\Delta t)]$ for jumping from the left well $x<b$ to any $y>b$ in a single jump in a time interval Δt is governed by the tail of the distribution, $p(x)\propto(x/\sigma)^{-(\alpha+1)}\Delta t/\sigma$. This is seen by observing that the process (2.1) can be obtained from the discrete process, $X(t+\Delta t)=X(t)+f(X(t))\Delta t+[\sigma\Delta t^{1/\alpha}]\eta(t)$, for $\Delta t\to 0$, where $\eta(t)$ has an α-stable distribution with unit intensity. Thus we have

$$\begin{aligned}\lambda_{ac}\Delta t&\approx P(X(t+\Delta t)>b|X(t)<b)/P(X(t)<b)\\&\propto\int_{-\infty}^{b}\left[\int_{b-x}^{\infty}p(u)du\right]\tilde{p}(x)dx\approx\int_{b-a}^{\infty}p(u)du\\&\approx[(b-a)/\sigma]^{-\alpha}\Delta t.\end{aligned} \tag{5.4}$$

The inner integral is the probability of jumping from $x<b$ to any $y>b$, and $\tilde{p}(x)$ is the stationary probability density. The outer integral is dominated by the central part of the probability distribution. This result is exact in the $\Delta t\to 0,\sigma\to 0$ limit where $p(x)\to\delta(x-a)$. Thus, we have

$$T_{ac}=c(\alpha)[(b-a)/\sigma]^{\alpha}, \tag{5.5}$$

where $c(\alpha)$ is some constant. So in this case we see that the waiting time scales with the ''left half-width'' of the barrier $b-a\equiv w$ to the power α. The height of the barrier has no influence on the transition probability. The results are confirmed by numerical simulation. Figure 4 displays the nu-

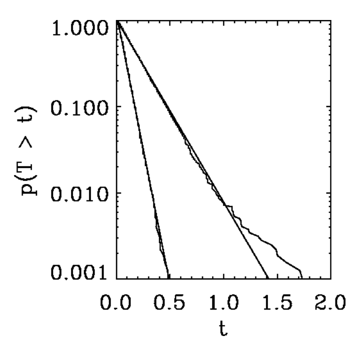

FIG. 3. The probability for waiting longer than *t* before jumping to the other well as a function of *t* obtained from the simulation. The slope of the upper curve gives T_{ca} and the slope of the lower curve gives T_{ac}. Units are arbitrary.

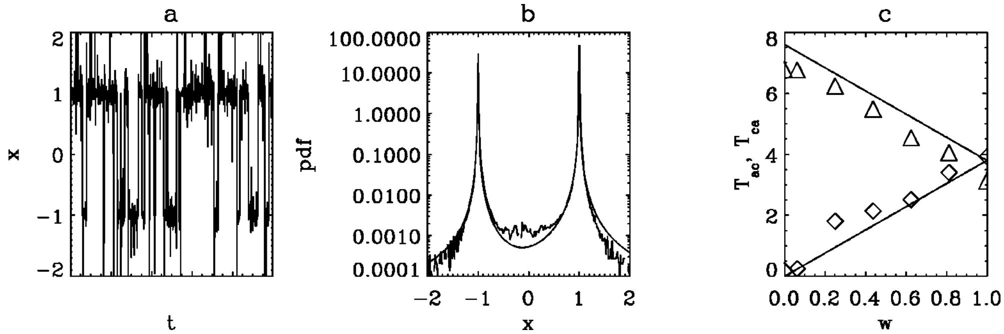

FIG. 4. The same as Fig. 2 but with Cauchy noise, $\alpha=1$. Note the linear axis in (c). The curves are obtained from Eq. (5.5).

merical simulation using Cauchy noise, $\alpha=1$, and the same potential as in the case displayed in Fig. 2. Note the linear scale in Fig. 4(c) showing the scaling of the time scale with w.

VI. STATIONARY DISTRIBUTION

For the Cauchy noise-driven system the indeterminacy in Eq. (3.6) can now be resolved by use of the master equation (3.7). In the limit $\sigma\to 0$ the system can be approximated as a discrete two-state system, with the two states corresponding to the two potential minima, at a and c. In this limit the system fulfills the stationary master equation,

$$0=p_a p(a\to c)-p_c p(c\to a). \tag{6.1}$$

The transition probabilities are now $p(a\to c)\propto 1/T_{ac}\propto(b-a)/\sigma$ and $p(c\to a)\propto(c-b)/\sigma$ and we get

$$p_a=1-p_c=(b-a)/(c-a). \tag{6.2}$$

Note that this is independent of $\exp\{-2[U(a)-U(c)]/\sigma^2\}$, which in the Gaussian case corresponds to the Gibbs distribution. Figure 5 shows the distribution p_a, which is different from the Gibbs distribution, as a function of w. Figure 4(b) shows the probability density function from the simulation plotted over the one calculated from Eqs. (3.6) and (6.2).

VII. BARRIER PENETRATION

When α is close to 2 we should expect the ''single jump penetration'' of the barrier to become more and more unlikely and the continuous penetration dominating. The Levy decomposition theorem [8] states that the α-stable process can be decomposed in a Brownian process and a compound Poisson process. The ''continuous'' barrier penetration can be estimated by considering the distribution to be truncated so that there are no jumps larger than the half-width of the barrier w. The truncated probability for the noise $p^t(x)$ is then defined by $p^t(x)\propto p(x)$ for $|x|<w$ and $p^t(x)=0$ for $|x|\geq w$. This part of the noise now has finite second-order moment and we can estimate the variance as $\sigma_{eff}^2 \propto\int x^2\tilde{p}(x)dx\propto w^{2-\alpha}$ asymptotically for large w or small noise intensity σ. The waiting time can be estimated as

$$T^c\propto\sigma_{eff}^{-2}\exp\{2[U(b)-U(a)]/\sigma_{eff}^2\}, \tag{7.1}$$

where c denotes ''continuous.'' Note that this part of the process is not strictly continuous, since it contains jumps smaller than w. The time scale for single jump penetration can be estimated from Eq. (5.5),

$$T^d\propto w^{\alpha}, \tag{7.2}$$

where d denotes ''discontinuous'' and we have

$$\frac{T^d}{T^c}\propto w^2\exp\{-\tilde{c}[U(b)-U(a)]w^{\alpha-2}\}, \tag{7.3}$$

where $\tilde{c}$ is a constant. So the relative importance of extremal jumping depends both on the height and the width of the barrier. To illustrate the relative importance of two jumping processes a simulation of Eq. (2.1) with an α-stable noise [13] with $\alpha=1.7$ and a potential (4.1) with $h=3$, was performed. Figure 6 shows part of a realization of this process. Here it is seen that the jumping from the deep to the shallow well is governed by the discontinuous part, $T^d(c\to a)\ll T^c(c\to a)$, while the jumping from the shallow to the deep well is dominated by the ''continuous'' part, $T^d(a\to c)\gg T^c(a\to c)$. For proportioning the continuous and discontinuous processes in a given situation the prefactor and the constant $\tilde{c}$ in Eq. (7.3) must be calculated or estimated.

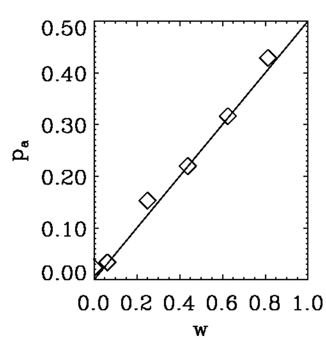

FIG. 5. The probability, $p_a=1-p_c$, for finding the particle in the left well as a function of w in the simulation with Cauchy noise. The curve is obtained using Eq. (6.2); the distribution is deviating strongly from a Gibbs distribution.

VIII. SUMMARY

We have seen that the statistics of noise-induced jumping between metastable states in a potential is different for α-stable noise from the usual Gaussian noise case. The stationary probability distribution deviates from the Gibbs distribution, and the waiting time for jumping depends in some

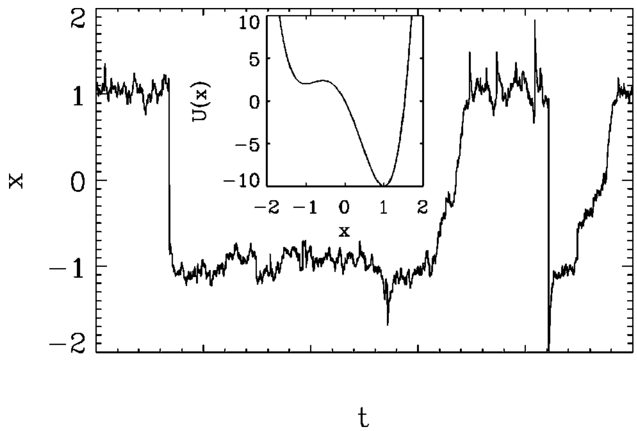

FIG. 6. A realization of the process with $\alpha=1.7$. The potential used is shown in the inset. The jumping from the left (shallow) well to the right (deep) well is triggered by the—almost—normal diffusion. The jumping the other way is driven by the tail of the α-stable distribution, the extreme events.

cases more on the width than on the height of the barrier. This is the case where a single extreme event triggers the jumping. These observations might be of importance for understanding the triggering mechanisms of climatic changes, where the flow state of the ocean is trapped in a potential minimum, a stable climatic state. This flow is stochastically forced by the atmospheric flow. There is some evidence that this stochastic forcing is α-stable rather than Gaussian such that climatic shifts from one state to another could be triggered by single extreme events. This would perhaps explain why the climate models at present are not capable of reproducing the climatic changes observed in the geological records. The models are too coarse-grained and contain too much diffusive smoothening to allow for extreme events.

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APPENDIX A: ADDITION OF α-STABLE RANDOM VARIABLES

Textbooks on α-stable processes are now available [14,13], but for those readers not familiar with the α-stable distributions and processes a few notes are added in the following.

When $\{X_i, i=1,\ldots,n\}$ is a series of i.i.d. random variables, the distribution of the variable $Y=c(n)\Sigma_{j=1}^{n}X_j$ can be determined from the characteristic function,

$$\langle \exp[ikY]\rangle = \left\langle \exp\left[ikc(n)\sum_{j=1}^{n} X_j\right]\right\rangle = \left\langle \prod_{j=1}^{n} \exp[ikc(n)X_j]\right\rangle = \langle e^{ikc(n)X}\rangle^n. \tag{A1}$$

If the distribution for Y is the same as for X Eq. (A1), for the characteristic function, $f(k)=\langle \exp[ikY]\rangle$, is

$$f(k)=f(c(n)k)^n \tag{A2}$$

with the solution

$$f(k)=\exp[-\sigma^\alpha |k|^\alpha/\alpha], \tag{A3}$$

$$c(n)=n^{-1/\alpha}. \tag{A4}$$

The constant σ^α/α is chosen so that it coincides with the usual notation in the Gaussian case $\alpha=2$. Only for $\alpha>0$ does Eq. (A3) represent a characteristic function. It can be shown that the characteristic function (A3) corresponds to distributions with power-function tails, $P(X>x)\sim x^{-\alpha}$ [15,14]. For $\alpha>2$ the second moment of the distribution exists and sums of i.i.d. variables converge by the central limit theorem to the Gaussian distribution $\alpha=2$. For $0<\alpha<2$ the distributions have a domain of attraction in the sense that sums of i.i.d. random variables with tail distributions, $P(X>x)\sim x^{-\gamma}$, under rather general conditions, converge to an α-stable distribution with $\alpha=\gamma$. This is the generalization of the central limit theorem for α-stable distributions. The proof of this is similar to the proof of the central limit theorem for the normal distribution. It basically substitutes a limit, $\tilde{f}(c(n)k)^n \to f(k)$ for Eq. (A2). The proof can be found in Feller's book [15], pp. 574–581.

Now we can intuitively understand the noise term dL_α in the Langevin equation (2.1) as the continuum limit of addition of small increments,

$$\Delta L_\alpha(\Delta t)=\frac{1}{m^{1/\alpha}}\sum_{j=1}^{m} X(j\Delta t/m), \tag{A5}$$

where $X(t)$ is a random process with power-function tails $P(X(t)>x)\sim x^{-\alpha}$ and unit intensity. In the limit $m\to\infty$, ΔL_α will be an α-stable noise. It follows from Eq. (A4) that $dL_\alpha = dt^{1/\alpha}$, which in the Gaussian case is the well-known relation $dB^2=dt$.

For $\alpha<2$ the α-stable variables have infinite variance. This concept can be difficult to comprehend when considering measurements from a given physical system. In the case a sample is taken, say of n measurements of the variable X, where X has an α-stable distribution with stability index α, then, of course, any of the measurements $x_1,\ldots,x_n$ of X is finite so that the sample variance, $(x_1^2+\cdots+x_n^2)/n$, is some finite number. The variable $Y=X^2$ will have a tail distribution given by $P(Y>x^2)=P(X>x)\sim x^{-\alpha}=y^{-\alpha/2}$, so that, asymptotically for large n, $Z_n \equiv n^{-2/\alpha}(Y_1+\cdots+Y_n)$ will have an α-stable distribution with stability index $\alpha/2$. Imagine now that we estimate the (infinite) variance of variable X by taking samples of length n, estimating the variance as $(X_1^2+\cdots+X_n^2)/n = n^{2/\alpha-1}Z_n$. Then the estimate itself will be an α-stable process with stability index $\alpha/2$ and intensity $n^{2/\alpha-1}$. This estimate will be fluctuating with an intensity growing with n for $\alpha<2$.

APPENDIX B: FOKKER-PLANCK EQUATION

In the following the Fokker-Planck equation (3.1) corresponding to the Langevin equation (2.1) will be derived. The Fokker-Planck equation will be derived in spectral form using that the α-stable processes are defined by their characteristics functions. Following the lines of Stratonovich [17], we define the functional

$$I=\int R(y)\partial_t p(x_0|y,t)dy=\lim_{\Delta t\to 0} I_{\Delta t}, \quad \text{(B1)}$$

where

$$I_{\Delta t}=\frac{1}{\Delta t}\int R(y)[p(x_0|y,t+\Delta t)-p(x_0|y,t)]dy. \quad \text{(B2)}$$

$R(y)$ is an arbitrary generator function, and $p(x_0|x_1,t)$ is the conditional probability density at x_1 corresponding to passing from x_0 to x_1 during time t. Assuming stationarity we suppress the first temporal index, $p(x_0|x_1,t)\equiv p(x_0,0|x_1,t)=p(x_0,\tau|x_1,\tau+t)$. For simplicity of writing we make the convention that $\int$ is to be read as $(1/\sqrt{2\pi})\int_{-\infty}^{\infty}$.

With $p(x_0|x,t)$ being a probability density in x we trivially have

$$\int_{-\infty}^{\infty} p(x_0|x,t)dx=1 \quad \text{(B3)}$$

and the Chapman-Kolmogorov equation

$$p(x_0|x_1,t)=\int_{-\infty}^{\infty} p(x_0|x,\tau)p(x|x_1,t-\tau)dx. \quad \text{(B4)}$$

For the functional we then get

$$I_{\Delta t}=\frac{1}{\Delta t}\int R(y)\left[\int_{-\infty}^{\infty} p(x|y,\Delta t)p(x_0|x,t)dx-p(x_0|y,t)\right]dy$$
$$=\frac{\sqrt{2\pi}}{\Delta t}\int p(x_0|x,t)\left\{\int p(x|y,\Delta t)[R(y)-R(x)]dy\right\}dx. \quad \text{(B5)}$$

We now define the Fourier transforms

$$R(x)=\int \hat{R}(k)e^{ikx}dk, \quad \hat{R}(k)=\int R(x)e^{-ikx}dx, \quad \text{(B6)}$$

similarly for $f(x)$ in Eq. (2.1), and for $\sigma^\alpha(x)$ to be introduced below. However, for the probability density $p(x_0|x,t)$, we define

$$p(x_0|x,t)=\int \hat{p}(x_0|k,t)e^{-ikx}dk, \quad \text{(B7)}$$

$$\hat{p}(x_0|k,t)=\int p(x_0|x,t)e^{ikx}dx, \quad \text{(B8)}$$

consistent with the standard definition of characteristic function except for the factor $\sqrt{2\pi}$. With these definitions it is easy to derive the formula

$$\int f(x)p(x_0|x,t)e^{ikx}dx=\int \hat{f}(k_1-k)\hat{p}(x_0|k_1,t)dk_1, \quad \text{(B9)}$$

from which it directly follows that

$$f(x)p(x_0|x,t)=\int\left[\int \hat{f}(k_1-k)\hat{p}(x_0|k_1,t)dk_1\right]e^{-ikx}dk. \quad \text{(B10)}$$

Using the spectral representation for the generator function we get

$$\frac{1}{\sqrt{2\pi}}I_{\Delta t}=\int p(x_0|x,t)\left\{\int p(x|y,\Delta t)\frac{1}{\Delta t}\int \hat{R}(k)\times(e^{iky}-e^{ikx})dk\right\}dx$$
$$=\int\int e^{ikx}\hat{R}(k)p(x_0|x,t)\frac{1}{\Delta t}\times E[e^{ik[X(t+\Delta t)-x]}-1|X(t)=x]dkdx. \quad \text{(B11)}$$

The conditional expectation is evaluated using the Langevin equation (2.1) and the characteristic function of the α-stable Levy noise increment dL_α,

$$E\{\exp[ikdL_\alpha]\}=\exp[-\sigma^\alpha\Delta t|k|^\alpha/\alpha]. \quad \text{(B12)}$$

We get

$$\frac{1}{\Delta t}E[e^{ik[X(t+\Delta t)-x]}-1|X(t)=x]$$
$$=\frac{1}{\Delta t}E[e^{ik[f(x)\Delta t+o(\Delta t)+dL_\alpha]}-1]$$
$$=\frac{1}{\Delta t}e^{ik[f(x)\Delta t+o(\Delta t)]-\sigma^\alpha\Delta t|k|^\alpha/\alpha}-1$$
$$\rightarrow ikf(x)-\sigma^\alpha|k|^\alpha/\alpha, \quad \text{(B13)}$$

as $\Delta t\rightarrow 0$. Substitution of Eq. (B13) in Eq. (B11) and combining with Eqs. (B1) and (B2) then gives

$$\int\int \hat{R}(k)e^{ikx}\partial_t p(x_0|x,t)dxdk$$
$$=\int\int \hat{R}(k)e^{ikx}[ikf(x)-\sigma^\alpha(x)|k|^\alpha/\alpha]p(x_0|x,t)dxdk. \quad \text{(B14)}$$

Here we have permitted the scaling factor σ^α/α, corresponding to the variance $\sigma^2/2$ of the noise increment in the case $\alpha=2$, to depend on the variable x. By eliminating the x by use of Eq. (B9) we get, suppressing x_0,

$$\int \hat{R}(k)\partial_t\hat{p}(k,t)dk=\int\int \hat{R}(k)[ik\hat{f}(k_1-k)-\widehat{\sigma^\alpha}(k_1-k)|k|^\alpha/\alpha]\hat{p}(k_1,t)dkdk_1, \quad \text{(B15)}$$

and finally since $\hat{R}(k)$ is arbitrary we get the spectral Fokker-Planck equation for the integrand, suppressing the t index,

$$\partial_t \hat{p}(k) = \int [ik\hat{f}(k_1-k) - \widehat{\sigma^\alpha}(k_1-k)|k|^\alpha/\alpha]\hat{p}(k_1)dk_1. \tag{B16}$$

Multiplying by e^{-ikx} and using Eq. (B10) gives the Fokker-Planck equation in the usual form

$$\partial_t p(x) = -\partial_x[f(x)p(x)] - \frac{1}{\alpha}\int\int e^{-ikx}\hat{\sigma}^\alpha(k-k_1)|k|^\alpha \hat{p}(k_1)dkdk_1. \tag{B17}$$

For the stationary Fokker-Planck equation the left-hand side of (B17) vanishes and the partial derivatives become total derivatives. The last term on the right-hand side is a generalized diffusion, which formally can be written

$$\frac{1}{\alpha}\frac{d^\alpha}{dx^\alpha}[\sigma^\alpha(x)p(x)] \equiv -\frac{1}{\alpha}\int\int e^{-ikx} \times \hat{\sigma}^\alpha(k-k_1)|k|^\alpha \hat{p}(k_1)dkdk_1. \tag{B18}$$

In the case $\alpha=2$ this is the usual diffusion term corresponding to Gaussian white noise excitation of intensity $\sigma^2(x)$. For $\alpha<2$ the diffusion is nonlocal. The physical meaning of this term is that for α-stable processes there will, due to the fat tails of the distributions, be finite size jumps in the process.

APPENDIX C: CAUCHY DISTRIBUTION

The probability density can only be expressed explicitly for $\alpha=1$, and $\alpha=1/2$. For $\alpha=1$, Cauchy noise, the characteristic function is $c(k)=\exp(-\sigma|k|)$ and its Fourier transform is

$$p(x) = \frac{1}{\pi\sigma[1+(x/\sigma)^2]}. \tag{C1}$$

For this distribution even the mean does not exist. Note that even though the density distribution (C1) is symmetric, $p(-x)=p(x)$; this does not imply that $\langle x\rangle=0$. For a data sampling this manifests itself in the fact that the average of n data points $Z_n=(X_1+\dots+X_n)/n$ is Cauchy distributed with the same intensity as X_i, so that there is no convergence for the series $Z_n, n=1,2,\dots$; it fluctuates exactly as the data X_i itself.

A classical example of this characteristic of the Cauchy distribution is seen by considering the distribution of light on a line L from a point source, see Fig. 7. Since the light is uniformly distributed over the angles, $p(\theta)=1/\pi, \theta\in[0,\pi]$, the distribution on the line will be $p(x)=p(\theta)(d\theta/dx) =1/\{\delta\pi[1+(x/\delta)]^2\}$, where $X=\delta\tan(\theta)$ is a stochastic variable representing the point where a photon released from S at the (stochastic) angle θ crosses L. Now inserting $n-1$ lines, L_i, parallel to L, between the light source, S and L, we can apply Huygens principle, saying that L_i will act as a line of point sources, where the light follows the path $S\rightarrow X_1 \rightarrow X_1+X_2\rightarrow \dots \rightarrow X$, where $X=X_1+\dots+X_n$. The variables X_i are independent and Cauchy distributed with scale parameter δ/n. Thus Huygens principle is consistent with the fact that $X=(\Sigma_{i=1}^n X_i)/n$ has the same distribution as X_i.

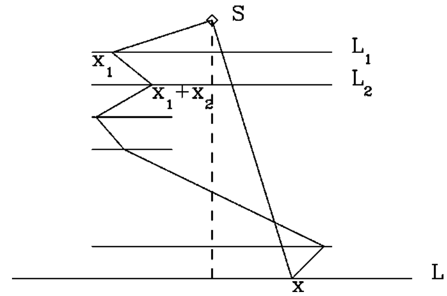

FIG. 7. Huygens principle applied to the light from a point source on a line. This illustrates the behavior of the averaging of Cauchy-distributed stochastic variables.

APPENDIX D: SIMULATIONS

The simulations performed in this paper only involves Cauchy noise, which is easily obtained from a random variable X uniformly distributed in the interval $[-\pi/2,\pi/2]$, as $Y=\tan(X)$. More generally, a random variable with an α-stable distribution [13] is obtained from

$$Y=[\sin(\alpha X)/\cos(X)^{(1/\alpha)}]\{-\cos([1-\alpha]X)/\ln(W)\}^{(1-\alpha)/\alpha}, \tag{D1}$$

where X is defined as above and W is another random variable uniformly distributed on the interval $[0,1]$. When simulating Eq. (2.1) by a discrete numerical time stepping the (fixed size) time steps usually need to be much smaller than would be expected from numerical integration of the drift term alone. This is due to the large excursions from the tails of the distribution of the noise. It is thus important to use a stable integration routine for the drift term. A simple durable routine, which is the one used in these simulations, is Heun's integration scheme. The simulation is performed as

$$x(t+\Delta t)=x(t)+(f[x(t)]+f\{x(t)+f[x(t)]\Delta t\})\Delta t/2 + \sigma\Delta t^{1/\alpha}\eta(t), \tag{D2}$$

where $\eta(t)$ is generated by Eq. (D1).

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