Anomalous localized resonance using a folded geometry in three dimensions*

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Abstract

If a body of dielectric material is coated by a plasmonic structure of negative dielectric material with nonzero loss parameter, then cloaking by anomalous localized resonance (CALR) may occur as the loss parameter tends to zero. It was proved in [1, 2] that if the coated structure is circular (2D) and dielectric constant of the shell is a negative constant (with loss parameter), then CALR occurs, and if the coated structure is spherical (3D), then CALR does not occur. The aim of this paper is to show that the CALR takes place if the spherical coated structure has a specially designed anisotropic dielectric tensor. The anisotropic dielectric tensor is designed by unfolding a folded geometry.

1 Introduction

If a body of dielectric material (core) is coated by a plasmonic structure of negative dielectric constant with nonzero loss parameter (shell), then anomalous localized resonance may occur as the loss parameter tends to zero. To be precise, let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, and D be a domain whose closure is contained in Ω. In other words, D is the core and $\Omega \setminus \overline{D}$ is the shell. For a given loss parameter $\delta > 0$, the permittivity distribution in $\mathbb{R}^d$ is given by

$$\epsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ \epsilon_s + i\delta & \text{in } \Omega \setminus \overline{D}, \\ \epsilon_c & \text{in } D. \end{cases} \tag{1.1}$$

Here ϵ_c is a positive constant, but ϵ_s is a negative constant representing the negative dielectric constant of the shell. For a given function f compactly supported in $\mathbb{R}^d \setminus \overline{\Omega}$ satisfying

$$\int_{\mathbb{R}^d} f \, d\mathbf{x} = 0 \tag{1.2}$$

(which is required by conservation of charge), we consider the following dielectric problem:

$$\nabla \cdot \epsilon_\delta \nabla V_\delta = f \quad \text{in } \mathbb{R}^d, \tag{1.3}$$

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with the decay condition $V_\delta(x) \to 0$ as $|x| \to \infty$. The equation (1.3) is known as the quasistatic equation and the real part of $-\nabla V_\delta(x)e^{-i\omega t}$, where ω is the frequency and t is the time, represents an approximation for the physical electric field in the vicinity of Ω, when the wavelength of the electromagnetic radiation is large compared to Ω.

Let

$$E_\delta := \Im \int_{\mathbb{R}^d} \epsilon_\delta |\nabla V_\delta|^2 \, d\mathbf{x} = \int_{\Omega\setminus D} \delta |\nabla V_\delta|^2 \, d\mathbf{x} \tag{1.4}$$

($\Im$ for the imaginary part), which, within a factor proportional to the frequency, approximately represents the time averaged electromagnetic power produced by the source dissipated into heat. Also for any region Υ let

$$E_\delta^0(\Upsilon) = \int_\Upsilon |\nabla V_\delta|^2 \, d\mathbf{x} \tag{1.5}$$

which when Υ is outside Ω approximately represents, within a proportionality constant, the time averaged electrical energy stored in the region Υ. Anomalous localized resonance is the phenomenon of field blow-up in a localized region. It may (and may not) occur depending upon the structure and the location of the source. Quantitatively, it is characterized by $E_\delta^0(\Upsilon) \to \infty$ as $\delta \to 0$ for all regions Υ that overlap the region of anomalous resonance, and this defines that region. Cloaking due to anomalous localized resonance (CALR) may occur when the support of the source, or part of it, lies in the anomalously resonant region. Physically the enormous fields in the anomalously resonant region interact with the source to create a sort of optical molasses, against which the source has to do a tremendous amount of work to maintain its amplitude, and this work tends to infinity as $\delta \to 0$. Quantitatively it is characterized by $E_\delta \to \infty$ as $\delta \to \infty$.

This phenomena of anomolous resonance was first discovered by Nicorovici, McPhedran and Milton [15] and is related to invisibility cloaking [11]: the localized resonant fields created by a source can act back on the source and mask it (assuming the source is normalized to produce fixed power). It is also related to superlenses [16, 17] since, as shown in [15], the anomalous resonance can create apparent point sources. For these connections and further developments tied to this form of invisibility cloaking, we refer to [1, 2, 3, 4, 10] and references therein. Anomalous resonance is also presumably responsible for cloaking due to complementary media [8, 18, 14], although we do not study this here.

The problem of cloaking by anomalous localized resonance (CALR) can be formulated as the problem of identifying the sources f such that first

$$E_\delta := \int_{\Omega\setminus D} \delta |\nabla V_\delta|^2 \, d\mathbf{x} \to \infty \quad \text{as } \delta \to 0, \tag{1.6}$$

and secondly, $V_\delta/\sqrt{E_\delta}$ goes to zero outside some radius a, as $\delta \to 0$:

$$|V_\delta(x)/\sqrt{E_\delta}| \to 0 \quad \text{as } \delta \to 0 \quad \text{when } |x| > a. \tag{1.7}$$

Since the quantity E_δ is proportional to the electromagnetic power dissipated into heat by the time harmonic electrical field averaged over time, (1.6) implies an infinite amount of energy dissipated per unit time in the limit $\delta \to 0$ which is unphysical. If we rescale the source f by a factor of $1/\sqrt{E_\delta}$ then the source will produce the same power independently of δ and the new associated potential $V_\delta/\sqrt{E_\delta}$ will, by (1.7), approach zero outside the radius a. Hence, cloaking due to anomalous localized resonance (CALR) occurs. The normalized source is essentially invisible from the outside, yet the fields inside are very large.

In the recent papers [1, 2] the authors developed a spectral approach to analyze the CALR phenomenon. In particular, they show that if D and Ω are concentric disks in $\mathbb{R}^2$ of radii r_i and r_e, respectively, and $\epsilon_s = -1$, then there is a critical radius r_* such that for any source f supported outside r_* CALR does not occur, and for sources f satisfying a mild (gap) condition CALR takes place. The critical radius r_* is given by $r_* = \sqrt{r_e^3/r_i}$ if $\epsilon_c = 1$, and by $r_* = r_e^2/r_i$ if $\epsilon_c \neq 1$. It is also proved that if $\epsilon_s \neq -1$, then CALR does not occur: E_δ is bounded regardless of δ and the location of the source. It is worth mentioning that these results (when $\epsilon_c = -\epsilon_s = 1$) were extended in [7] to the case when the core D is not radial by a different method based on a variational approach. There the source f is assumed to be supported on circles.

The situation in three dimensions is completely different. If D and Ω are concentric balls in $\mathbb{R}^3$, CALR does not occur whatever ϵ_s and ϵ_c are, as long as they are constants. We emphasize that this discrepancy comes from the convergence rate of the singular values of the Neumann-Poincaré-type operator associated with the structure. In 2D, they converge to 0 exponentially fast, but in 3D they converge only at the rate of $1/n$. See [2]. The absence of CALR in such coated sphere geometries is also linked with the absence of perfect plasmon waves: see the appendix in [7]. On the other hand, in a slab geometry CALR is known to occur in three dimensions with a single dipolar source [11]. (CALR is also known to occur for the full time-harmonic Maxwell equations with a single dipolar source outside the slab superlens [6, 11, 19].)

The purpose of this paper is to show that we are able to make CALR occur in three dimensions by using a shell with a specially designed anisotropic dielectric constant. In fact, let D and Ω be concentric balls in $\mathbb{R}^3$ of radii r_i and r_e, and choose r_0 so that $r_0 > r_e$. For a given loss parameter $\delta > 0$, define the dielectric constant $\boldsymbol{\epsilon}_\delta$ by

$$\boldsymbol{\epsilon}_\delta(\mathbf{x}) = \begin{cases} \mathbf{I}, & |\mathbf{x}| > r_e, \\ (\epsilon_s + i\delta)a^{-1}\left(\mathbf{I} + \dfrac{b(b-2|\mathbf{x}|)}{|\mathbf{x}|^2}\widehat{\mathbf{x}} \otimes \widehat{\mathbf{x}}\right), & r_i < |\mathbf{x}| < r_e, \\ \epsilon_c \sqrt{\dfrac{r_0}{r_i}}\mathbf{I}, & |\mathbf{x}| < r_i, \end{cases} \tag{1.8}$$

where $\mathbf{I}$ is the 3×3 identity matrix, ϵ_s and ϵ_c constants, $\widehat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$, and

$$a := \frac{r_e - r_i}{r_0 - r_e} > 0, \quad b := (1+a)r_e. \tag{1.9}$$

Note that $\boldsymbol{\epsilon}_\delta$ is anisotropic and variable in the shell. This dielectric constant is obtained by push-forwarding (unfolding) that of a folded geometry as in Figure 1. (See the next section for details.) It is worth mentioning that this idea of a folded geometry has been used in [12] to prove CALR in the analogous two-dimensional cylinder structure for a finite set of dipolar sources. Folded geometries were first introduced in [9] to explain the properties of superlenses, and their unfolding map was generalized in [12] to allow for three different fields, rather than a single one, in the overlapping regions. Folded cylinder structures were studied as superlenses in [20] and folded geometries using bipolar coordinates were introduced in [5] to obtain new complementary media cloaking structures. More general folded geometries were rigorously investigated in [14].

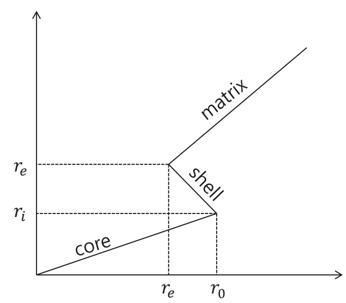

Figure 1: unfolding map

For a given source f supported outside $\overline{B_{r_e}}$ let V_δ be the solution to

$$\begin{cases} \nabla\cdot(\boldsymbol{\epsilon}_\delta \nabla V_\delta) = f & \text{in } \mathbb{R}^3, \\ V_\delta(\mathbf{x}) \to 0 & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{1.10}$$

and define

$$E_\delta = \Im \int_{\mathbb{R}^3} \boldsymbol{\epsilon}_\delta \nabla V_\delta \cdot \nabla \overline{V_\delta}\, d\mathbf{x}, \tag{1.11}$$

where $\overline{V_\delta}$ is the complex conjugate of V_δ. Let F be the Newtonian potential of the source f, *i.e.*,

$$F(\mathbf{x}) := \int_{\mathbb{R}^3} G(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \tag{1.12}$$

with $G(\mathbf{x}-\mathbf{y}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$. Since f is supported in $\mathbb{R}^3 \setminus \overline{B_{r_e}}$, F is harmonic in $|\mathbf{x}| < R$ for some $R > r_e$ and can be expressed there as

$$F(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} f_n^k |\mathbf{x}|^n Y_n^k(\widehat{\mathbf{x}}), \tag{1.13}$$

where $Y_n^k(\widehat{\mathbf{x}})$ is the (real) spherical harmonic of degree n and order k.

The following is the main result of this paper.

Theorem 1.1 *(i) If $\epsilon_c = -\epsilon_s = 1$, then weak CALR occurs and the critical radius is $r_* = \sqrt{r_e r_0}$, i.e., if the source function f is supported inside the sphere of radius r_* (and its Newtonian potential does not extend harmonically to $\mathbb{R}^3$), then*

$$\limsup_{\delta\to 0} E_\delta = \infty, \tag{1.14}$$

and there exists a constant C such that

$$|V_\delta(\mathbf{x})| < C \tag{1.15}$$

for all $\mathbf{x}$ with $|\mathbf{x}| > r_0^2 {r_e}^{-1}$. If, in addition, the Fourier coefficients f_n^k of F satisfy the following gap condition:

[GC1]*: There exists a sequence $\{n_j\}$ with $n_1 < n_2 < \cdots$ such that*

$$\lim_{j\to\infty} \rho^{n_{j+1}-n_j} \sum_{k=-n_j}^{n_j} n_j r_*^{2n_j} |f_{n_j}^k|^2 = \infty$$

where $\rho := r_e/r_0$, then CALR occurs, i.e.,

$$\lim_{\delta\to 0} E_\delta = \infty, \tag{1.16}$$

and $V_\delta/\sqrt{E_\delta}$ goes to zero outside the radius r_0^2/r_e, as implied by (1.15).

(ii) If $\epsilon_c \neq -\epsilon_s = 1$, then weak CALR occurs and the critical radius is $r_{**} = r_0$. If, in addition, the Fourier coefficients f_n^k of F satisfy

[GC2]*: There exists a sequence $\{n_j\}$ with $n_1 < n_2 < \cdots$ such that*

$$\lim_{j\to\infty} \rho^{2(n_{j+1}-n_j)} \sum_{k=-n_j}^{n_j} n_j r_0^{2n_j} |f_{n_j}^k|^2 = \infty,$$

then CALR occurs.

(iii) If $-\epsilon_s \neq 1$, then CALR does not occur.

We emphasize that [GC1] and [GC2] are mild conditions on the Fourier coefficients of the Newtonian potential of the source function. For example, if the source function is a dipole in $B_{r_*} \setminus \overline{B}_e$, *i.e.*, $f(\mathbf{x}) = \mathbf{a} \cdot \nabla \delta_{\mathbf{y}}(\mathbf{x})$ for a vector $\mathbf{a}$ and $\mathbf{y} \in B_{r_*} \setminus \overline{B}_e$ where $\delta_{\mathbf{y}}$ is the Dirac delta function at $\mathbf{y}$, [GC1] and [GC2] hold and CALR takes place. A proof of this fact is provided in the appendix. Similarly one can show that if f is a quadrupole, $f(x) = A : \nabla\nabla\delta_{\mathbf{y}}(\mathbf{x}) = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \delta_{\mathbf{y}}(\mathbf{x})$ for a 3×3 matrix $A = (a_{ij})$ and $\mathbf{y} \in B_{r_*} \setminus \overline{B}_e$, then [GC1] and [GC2] hold.

2 Proof of Theorem 1.1

Let r_i, r_e and r_0 be positive constants satisfying $r_i < r_e < r_0$, as before. For given constants κ_c, κ_s and κ_m, and a source f supported outside $\overline{B_{r_e}}$, let u_c, u_s and u_m be the functions satisfying

$$\begin{cases} \triangle u_c = 0 \quad \text{in } B_{r_0}, \\ \triangle u_s = 0 \quad \text{in } B_{r_0} \setminus \overline{B_{r_e}}, \\ \triangle u_m = f \quad \text{in } \mathbb{R}^3 \setminus \overline{B_{r_e}}, \\ u_c = u_s, \quad \kappa_c \dfrac{\partial u_c}{\partial r} = \kappa_s \dfrac{\partial u_s}{\partial r} \quad \text{on } \partial B_{r_0}, \\ u_s = u_m, \quad \kappa_s \dfrac{\partial u_s}{\partial r} = \kappa_m \dfrac{\partial u_m}{\partial r} \quad \text{on } \partial B_{r_e}, \\ u_m(\mathbf{x}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{2.1}$$

We emphasize that the domains of u_c, u_s, and u_m are overlapping on $r_e \leq |\mathbf{x}| \leq r_0$ so that the solutions combined may be considered as the solution of the transmission problem with dielectric constants κ_c, κ_s and κ_m in the folded geometry as shown in Figure 1. We unfold the solution into one whose domain is not overlapping, following the idea in [12].

In terms of spherical coordinates (r, θ, ϕ), the unfolding map $\Phi = \{\Phi_c, \Phi_s, \Phi_m\}$ is given by

$$\begin{cases} \Phi_m(r,\theta,\phi) = (r,\theta,\phi), & r \geq r_e, \\ \Phi_s(r,\theta,\phi) = \left(r_e - \dfrac{r_e - r_i}{r_0 - r_e}(r - r_e), \theta, \phi \right), & r_e \leq r \leq r_0, \\ \Phi_c(r,\theta,\phi) = \left(\dfrac{r_i}{r_0} r, \theta, \phi \right), & r \leq r_0. \end{cases} \tag{2.2}$$

Then the folding map can be written (with an abuse of notation) as

$$\Phi^{-1}(\mathbf{x}) = \begin{cases} \mathbf{x}, & |\mathbf{x}| > r_e, \\ -a\mathbf{x} + b\widehat{\mathbf{x}}, & r_i < |\mathbf{x}| < r_e, \\ \dfrac{r_0}{r_i}\mathbf{x}, & |\mathbf{x}| < r_i, \end{cases} \tag{2.3}$$

where a and b are constants defined in (1.9).

Let $\boldsymbol{\kappa}(\mathbf{x})$ be the push-forward by the unfolding map Φ, namely,

$$\boldsymbol{\kappa}(\mathbf{x}) = \begin{cases} \kappa_m |\det \nabla\Phi_m(\mathbf{y})|^{-1} \nabla\Phi_m(\mathbf{y}) \nabla\Phi_m(\mathbf{y})^t, & |\mathbf{x}| > r_e, \\ -\kappa_s |\det \nabla\Phi_s(\mathbf{y})|^{-1} \nabla\Phi_s(\mathbf{y}) \nabla\Phi_s(\mathbf{y})^t, & r_i < |\mathbf{x}| < r_e, \\ \kappa_c |\det \nabla\Phi_c(\mathbf{y})|^{-1} \nabla\Phi_c(\mathbf{y}) \nabla\Phi_c(\mathbf{y})^t, & |\mathbf{x}| < r_i, \end{cases} \tag{2.4}$$

where $\mathbf{x} = \Phi(\mathbf{y})$. By straight-forward computations one can see that $\boldsymbol{\kappa} = \boldsymbol{\epsilon}$ given in (1.8) if we set

$$\kappa_m = 1, \quad \kappa_s = -(\epsilon_s + i\delta), \quad \kappa_c = \epsilon_c. \tag{2.5}$$

Moreover, the solution V_δ to (1.10) is given by

$$V_\delta(\mathbf{x}) = \begin{cases} u_m \circ \Phi^{-1}(\mathbf{x}) & \text{if } |\mathbf{x}| > r_e, \\ u_s \circ \Phi^{-1}(\mathbf{x}) & \text{if } r_i < |\mathbf{x}| < r_e, \\ u_c \circ \Phi^{-1}(\mathbf{x}) & \text{if } |\mathbf{x}| < r_i, \end{cases} \tag{2.6}$$

and by the change of variables $\mathbf{x} = \Phi_s(\mathbf{y})$ and (2.4), we have

$$E_\delta = \Im \int_{\mathbb{R}^3} \boldsymbol{\epsilon}(\mathbf{x}) \nabla V_\delta(\mathbf{x}) \cdot \nabla \overline{V_\delta(\mathbf{x})} = \delta \int_{r_e < |\mathbf{y}| < r_0} |\nabla u_s(\mathbf{y})|^2 . \tag{2.7}$$

Suppose that the source f is supported in $|\mathbf{x}| > R$ for some $R > r_e$. Then the solution u to (2.1) can be expressed in $|\mathbf{x}| < R$ as follows:

$$\begin{cases} u_c(\mathbf{x}) = \displaystyle\sum_{n=0}^{\infty} \sum_{k=-n}^{n} a_n^k |\mathbf{x}|^n Y_n^k(\widehat{\mathbf{x}}), & \text{if } |\mathbf{x}| < r_0, \\ u_s(\mathbf{x}) = \displaystyle\sum_{n=0}^{\infty} \sum_{k=-n}^{n} (b_n^k |\mathbf{x}|^n + c_n^k |\mathbf{x}|^{-n-1}) Y_n^k(\widehat{\mathbf{x}}), & \text{if } r_e < |\mathbf{x}| < r_0, \\ u_m(\mathbf{x}) = \displaystyle\sum_{n=0}^{\infty} \sum_{k=-n}^{n} (e_n^k |\mathbf{x}|^n + d_n^k |\mathbf{x}|^{-n-1}) Y_n^k(\widehat{\mathbf{x}}), & \text{if } r_e < |\mathbf{x}| < R, \end{cases} \tag{2.8}$$

where the coefficients satisfy the following relations resulting from the interface conditions:

$$\begin{aligned} a_n^k r_0^n &= b_n^k r_0^n + c_n^k r_0^{-n-1}, \\ e_n^k r_e^n + d_n^k r_e^{-n-1} &= b_n^k r_e^n + c_n^k r_e^{-n-1}, \\ \kappa_c a_n^k n r_0^n &= \kappa_s (b_n^k n r_0^n - c_n^k (n+1) r_0^{-n-1}), \\ \kappa_s (b_n^k n r_e^n - c_n^k (n+1) r_e^{-n-1}) &= \kappa_m (e_n^k n r_e^n - d_n^k (n+1) r_e^{-n-1}). \end{aligned}$$

By solving this system of linear equations one can see that

$$a_n^k = a_n e_n^k, \quad b_n^k = b_n e_n^k, \quad c_n^k = c_n e_n^k, \quad d_n^k = d_n e_n^k,$$

where

$$a_n = \frac{-\rho^{2n+1}(2n+1)^2 \kappa_m \kappa_s}{(n^2+n)(\kappa_s - \kappa_c)(\kappa_s - \kappa_m) - \rho^{2n+1}((n+1)\kappa_s + n\kappa_c)((n+1)\kappa_m + n\kappa_s)}, \tag{2.9}$$

$$b_n = \frac{-\rho^{2n+1}\kappa_m (2n+1)((n+1)\kappa_s + n\kappa_c)}{(n^2+n)(\kappa_s - \kappa_c)(\kappa_s - \kappa_m) - \rho^{2n+1}((n+1)\kappa_s + n\kappa_c)((n+1)\kappa_m + n\kappa_s)}, \tag{2.10}$$

$$c_n = \frac{-r_e^{2n+1}\kappa_m n (2n+1)(\kappa_s - \kappa_c)}{(n^2+n)(\kappa_s - \kappa_c)(\kappa_s - \kappa_m) - \rho^{2n+1}((n+1)\kappa_s + n\kappa_c)((n+1)\kappa_m + n\kappa_s)}, \tag{2.11}$$

$$d_n = -\frac{n r_e^{2n+1}[\rho^{2n+1}(\kappa_m - \kappa_s)((n+1)\kappa_s + n\kappa_c) + (\kappa_s - \kappa_c)(n\kappa_m + (n+1)\kappa_s)]}{(n^2+n)(\kappa_s - \kappa_c)(\kappa_s - \kappa_m) - \rho^{2n+1}((n+1)\kappa_s + n\kappa_c)((n+1)\kappa_m + n\kappa_s)}. \tag{2.12}$$

Here ρ is defined to be r_e/r_0

Let F be the Newtonian potential of f, as before. Since $u - F$ is harmonic in $|\mathbf{x}| > r_e$ and tends to 0 as $|\mathbf{x}| \to \infty$, we have

$$e_n^k = f_n^k. \tag{2.13}$$

So u_m (the solution in the matrix) is given by

$$u_m(\mathbf{x}) = F(\mathbf{x}) + \sum_{n=0}^{\infty} \sum_{k=-n}^{n} f_n^k d_n |\mathbf{x}|^{-n-1} Y_n^k(\widehat{\mathbf{x}}). \tag{2.14}$$

Since $|d_n| \le C r_0^{2n}$, we have

$$|u_m(\mathbf{x}) - F(\mathbf{x})| \le C \sum_{n=0}^{\infty} \sum_{k=-n}^{n} |f_n^k| r_0^{2n} |\mathbf{x}|^{-n-1} < \infty \tag{2.15}$$

if $|\mathbf{x}| = r > r_0^2 r_e{}^{-1}$. This proves (1.15).

The solution in the shell u_s is given by

$$u_s(\mathbf{y}) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} f_n^k (b_n |\mathbf{y}|^n + c_n |\mathbf{y}|^{-n-1}) Y_n^k(\hat{\mathbf{y}}). \tag{2.16}$$

Using Green's identity and the orthogonality of Y_n^k, we obtain that

$$\begin{aligned}
\int_{r_e<|\mathbf{y}|<r_0} |\nabla u_s(\mathbf{y})|^2 &= \int_{|\mathbf{y}|=r_0} u_s \overline{\frac{\partial u_s}{\partial r}} - \int_{|\mathbf{y}|=r_e} u_s \overline{\frac{\partial u_s}{\partial r}} \\
&= \sum_{n=0}^{\infty} \sum_{k=-n}^{n} |f_n^k|^2 \left[(b_n r_0^n + c_n r_0^{-n-1})(n \overline{b_n} r_0^n - (n+1) \overline{c_n} r_0^{-n-1}) r_0 \right] \\
&\quad - \sum_{n=0}^{\infty} \sum_{k=-n}^{n} |f_n^k|^2 \left[(b_n r_e^n + c_n r_e^{-n-1})(n \overline{b_n} r_e^n - (n+1) \overline{c_n} r_e^{-n-1}) r_e \right] \\
&= \sum_{n=0}^{\infty} \sum_{k=-n}^{n} |f_n^k|^2 \left[n |b_n|^2 (r_0^{2n+1} - r_e^{2n+1}) - (n+1) |c_n|^2 (r_0^{-2n-1} - r_e^{-2n-1}) \right] \\
&\approx \sum_{n=0}^{\infty} \sum_{k=-n}^{n} n |f_n^k|^2 \left(|b_n|^2 r_0^{2n+1} + |c_n|^2 r_e^{-2n-1} \right).
\end{aligned}$$

The estimate (2.7) yields

$$E_\delta \approx \delta \sum_{n=0}^{\infty} \sum_{k=-n}^{n} n |f_n^k|^2 \left(|b_n|^2 r_0^{2n+1} + |c_n|^2 r_e^{-2n-1} \right). \tag{2.17}$$

(i) Suppose that $\epsilon_c = -\epsilon_s = 1$. With the notation in (2.5), we have

$$|(n^2+n)(\kappa_s - \kappa_c)(\kappa_s - \kappa_m) - \rho^{2n+1}((n+1)\kappa_s + n\kappa_c)((n+1)\kappa_m + n\kappa_s)| \approx n^2(\delta^2 + \rho^{2n+1}),$$

and hence

$$|b_n| \approx \frac{\rho^{2n}}{\delta^2 + \rho^{2n}}, \quad |c_n| \approx \frac{\delta r_e^{2n}}{\delta^2 + \rho^{2n}}. \tag{2.18}$$

It then follows from (2.17) that

$$E_\delta \approx \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{2n}}. \tag{2.19}$$

Let

$$N_\delta = \frac{\log \delta}{\log \rho}. \tag{2.20}$$

If $n \le N_\delta$, then we have that $\delta \le \rho^{|n|}$, and hence

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{2n}} &\ge \sum_{n \le N_\delta} \sum_{k=-n}^{n} \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{2n}} \\
&\ge \delta m r_0^{2m} \sum_{k=-m}^{m} |f_m^k|^2 \ge \frac{\delta m}{2m+1} r_0^{2m} \left(\sum_{k=-m}^{m} |f_m^k| \right)^2
\end{aligned}$$

for any $m \le N_\delta$. By taking δ to be ρ^n, $n = 1, 2, \ldots$, we see that if the following holds

$$\limsup_{n\to\infty} (r_e r_0)^{n/2} \sum_{k=-n}^{n} |f_n^k| = \infty, \tag{2.21}$$

then there is a sequence $\{n_k\}$ such that

$$\lim_{k\to\infty} E_{\rho^{|n_k|}} = \infty, \tag{2.22}$$

i.e., weak CALR occurs.

Suppose that the source function f is supported inside the critical radius $r_* = \sqrt{r_e r_0}$ (and outside r_e) and its Newtonian potential F cannot be extended harmonically in $|x| < r_*$. Then

$$\limsup_{n\to\infty} \left(\sum_{k=-n}^{n} |f_n^k| \right)^{1/n} > 1/\sqrt{r_e r_0}. \tag{2.23}$$

and consequently, (2.21) holds. This proves that if the source function f is supported inside the sphere of critical radius r_*, then weak CALR occurs.

If the source function f is supported outside the sphere of critical radius $r_* = \sqrt{r_e r_0}$, then its Newtonian potential F can be extended harmonically in $|x| < r_* + 2\eta$ for $\eta > 0$ and

$$\sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{2n}} \le \sum_{n=0}^{\infty} \sum_{k=-n}^{n} n r_*^{2n} |f_n^k|^2 \le C \|F\|^2_{L^2(\partial B_{r_*+\eta})} < \infty. \tag{2.24}$$

So E_δ is bounded regardless of δ and CALR does not occur.

Suppose that f is supported inside r_* and [GC1] holds. Let $\{n_j\}$ be the sequence satisfying

$$\lim_{j\to\infty} \rho^{n_{j+1}-n_j} \sum_{k=-n_j}^{n_j} n_j r_*^{2n_j} |f_{n_j}^k|^2 = \infty.$$

If $\delta = \rho^\alpha$ for some α, let $j(\alpha)$ be the number in the sequence such that

$$n_{j(\alpha)} \le \alpha < n_{j(\alpha)+1}.$$

Then, we have

$$\begin{aligned} E_\delta \approx \sum_{n\le N_\delta} \sum_{k=-n}^{n} \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{2n}} &\ge \rho^\alpha \sum_{n\le N_\delta} \sum_{k=-n}^{n} \frac{n r_e^{2n} |f_n^k|^2}{\rho^{2n}} \\ &\ge \rho^{|n_{j(\alpha)+1}|-|n_{j(\alpha)}|} \sum_{k=-n_{j(\alpha)}}^{n_{j(\alpha)}} n_{j(\alpha)} r_*^{2n_{j(\alpha)}} |f_{n_{j(\alpha)}}^k|^2 \to \infty \end{aligned}$$

as $\alpha \to \infty$. So CALR takes place. This completes the proof of (i).

To prove (ii) assume that $\epsilon_c \ne -\epsilon_s = 1$. In this case, we have

$$|b_n| \approx \frac{\rho^{2n}}{\delta + \rho^{2n}}, \quad |c_n| \approx \frac{r_e^{2n}}{\delta + \rho^{2n}},$$

and

$$E_\delta \approx \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{4n}}.$$

The rest of proof of (ii) is the same as that for (i).

Suppose now that $-\epsilon_s \neq 1$. If $\epsilon_c = 1$, then we have

$$|b_n| \approx \frac{\rho^{2n}}{\delta + \rho^{2n}}, \quad |c_n| \approx \frac{\delta r_e^{2n}}{\delta + \rho^{2n}},$$

and

$$E_\delta \approx \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\delta(\delta^2 + \rho^{2n}) n r_e^{2n} |f_n^k|^2}{(\delta + \rho^{2n})^2} \leq \sum_{n=0}^{\infty} \sum_{k=-n}^{n} n r_e^{2n} |f_n^k|^2 \leq \left\| \frac{\partial F}{\partial r} \right\|_{L^2(\partial B_e)}^2 .$$

Since the source function f is supported outside the radius r_e, we have

$$\left\| \frac{\partial F}{\partial r} \right\|_{L^2(\partial B_e)} \leq C \|f\|_{L^2(\mathbb{R}^3)},$$

and E_δ is bounded independently of δ. The case when $\epsilon_c \neq 1$ can be treated similarly.

A Gap property of dipoles

In this appendix we show that the Newtonian potentials of dipole source functions satisfy the gap conditions [GC1] and [GC2]. We only prove [GC1] since the other can be proved similarly.

Let f be a dipole in $B_{r_*} \setminus \overline{B}_e$, *i.e.*, $f(\mathbf{x}) = \mathbf{a} \cdot \nabla \delta_{\mathbf{y}}(\mathbf{x})$ for a vector $\mathbf{a}$ and $\mathbf{y} \in B_{r_*} \setminus \overline{B}_e$. Then its Newtonian potential is given by $F(\mathbf{x}) = -\mathbf{a} \cdot \nabla_{\mathbf{y}} G(\mathbf{x} - \mathbf{y})$. It is well-known (see, for example, [13]) that the fundamental solution $G(\mathbf{x} - \mathbf{y})$ admits the following expansion if $|\mathbf{y}| > |\mathbf{x}|$:

$$G(\mathbf{x} - \mathbf{y}) = -\sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{1}{2n+1} Y_n^k(\widehat{\mathbf{x}}) Y_n^k(\widehat{\mathbf{y}}) \frac{|\mathbf{x}|^n}{|\mathbf{y}|^{n+1}}.$$

So we have

$$F(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{1}{2n+1} |\mathbf{x}|^n Y_n^k(\widehat{\mathbf{x}})\, \mathbf{a} \cdot \nabla \left(\frac{1}{|\mathbf{y}|^{n+1}} Y_n^k(\widehat{\mathbf{y}}) \right),$$

and hence

$$f_n^k = \frac{1}{2n+1} \mathbf{a} \cdot \nabla \left(\frac{1}{|\mathbf{y}|^{n+1}} Y_n^k(\widehat{\mathbf{y}}) \right). \tag{A.1}$$

We show that

$$\sum_{k=-n}^{n} n r_*^{2n} |f_n^k|^2 \to \infty \quad \text{as } n \to \infty, \tag{A.2}$$

and hence [GC1] holds. The following lemma is needed.

Lemma A.1 *For any* $\mathbf{a}$ *and* $\widehat{\mathbf{y}}$ *on* S^2 *and for any positive integer* n *there is a homogeneous harmonic polynomial* h *of degree* n *such that*

$$\mathbf{a} \cdot \nabla h(\widehat{\mathbf{y}}) = 1 \tag{A.3}$$

and

$$\max_{|\widehat{\mathbf{x}}|=1} |h(\widehat{\mathbf{x}})| \leq \frac{\sqrt{3}}{n}. \tag{A.4}$$

Proof. After rotation if necessary, we may assume that $\widehat{\mathbf{y}} = (1,0,0)$. We introduce three homogeneous harmonic polynomials of degree n:

$$h_1(\mathbf{x}) := \frac{1}{2n}\left[(x_1+ix_2)^n + (x_1-ix_2)^n\right],$$
$$h_2(\mathbf{x}) := \frac{1}{2ni}\left[(x_1+ix_2)^n - (x_1-ix_2)^n\right],$$
$$h_3(\mathbf{x}) := \frac{1}{2ni}\left[(x_1+ix_3)^n - (x_1-ix_3)^n\right].$$

Then one can easily see that

$$\nabla h_1(\widehat{\mathbf{y}}) = (1,0,0), \quad \nabla h_2(\widehat{\mathbf{y}}) = (0,1,0), \quad \nabla h_3(\widehat{\mathbf{y}}) = (0,0,1).$$

So if we define

$$h = a_1h_1 + a_2h_2 + a_3h_3,$$

then (A.3) holds.

Since

$$\max_{|\widehat{\mathbf{x}}|=1} |h_j(\widehat{\mathbf{x}})| \le \frac{1}{n} \quad \text{for } j = 1,2,3,$$

we obtain (A.4) using the Cauchy-Schwartz inequality. This completes the proof. □

Let $\mathbf{a}$ and $\widehat{\mathbf{y}}$ be two unit vectors, and let h be a homogeneous harmonic polynomial of degree n satisfying (A.3) and (A.4). Then h can be expressed as

$$h(\mathbf{x}) = \sum_{k=-n}^{n} \alpha_k |\mathbf{x}|^n Y_n^k(\widehat{\mathbf{x}}),$$

where

$$\alpha_k = \frac{1}{4\pi}\int_{S^2} h(\widehat{\mathbf{x}}) Y_n^k(\widehat{\mathbf{x}}) dS. \tag{A.5}$$

Because of (A.3), we have

$$1 = \mathbf{a}\cdot\nabla h(\widehat{\mathbf{y}}) \le \sum_{k=-n}^{n} |\alpha_k| \left|\mathbf{a}\cdot\nabla\left(|\mathbf{x}|^n Y_n^k(\widehat{\mathbf{x}})\right)\right|.$$

So there is k, say k_n, between $-n$ and n such that

$$|\alpha_{k_n}| \left|\mathbf{a}\cdot\nabla\left(|\mathbf{x}|^n Y_n^{k_n}(\widehat{\mathbf{x}})\right)\right| \ge \frac{1}{2n+1}. \tag{A.6}$$

On the other hand, from (A.4) and (A.5), it follows by using Jensen's inequality that

$$|\alpha_{k_n}|^2 \le \frac{1}{4\pi}\int_{S^2} |h(\widehat{\mathbf{x}})|^2 |Y_n^{k_n}(\widehat{\mathbf{x}})|^2 dS \le \frac{3}{n^2}.$$

Thus we have

$$\left|\mathbf{a}\cdot\nabla\left(|\mathbf{x}|^n Y_n^{k_n}(\widehat{\mathbf{x}})\right)\right| \ge \frac{n}{\sqrt{3}(2n+1)} \ge C \tag{A.7}$$

for some C independent of n.

Now one can see from (A.1) that

$$|f_n^{k_n}| \ge \frac{C}{n|\mathbf{y}|^{n+1}} \tag{A.8}$$

for some C independent of n. Since $|\mathbf{y}| < r_*$, we obtain that

$$\sum_{k=-n}^{n} n r_*^{2n} |f_n^k|^2 \ge n r_*^{2n} |f_n^{k_n}|^2 \ge \frac{C}{n}\left(\frac{r_*}{|\mathbf{y}|}\right)^{2n} \to \infty \quad \text{as } n \to \infty,$$

as desired. It is worth mentioning that the constants C in the above may be different at each occurrence, but are independent of n.

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