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Anomalous Transport Processes in Anisotropically Expanding Quark-Gluon Plasmas

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We derive an expression for the anomalous viscosity in an anisotropically expanding quark-gluon-plasma, which arises from interactions of thermal partons with dynamically generated color fields. The anomalous viscosity dominates over the collisional viscosity for large velocity gradients or weak coupling. This effect may provide an explanation for the apparent "nearly perfect" liquidity of the matter produced in nuclear collisions at RHIC without the assumption that it is a strongly coupled state.

§1. Introduction

Measurements of the so-called elliptic flow parameter v_2 of hadrons emitted in noncentral collisions of heavy nuclei at the Relativistic Heavy Ion Collider (RHIC) are in remarkably good agreement with the predictions of ideal relativistic fluid dynamics.[1)] In order to achieve the agreement with the data, the hydrodynamical calculations need to assume a rapid initial equilibration within a time[2)] $\tau_\mathrm{i} < 1$ fm/c and an extremely small ratio of the shear viscosity η of the fluid to its entropy density s.[3)] In fact, the ratio η/s cannot be much larger than the conjectured lower bound $(4\pi)^{-1}$ for this quantity.[4)]

These findings have led to the general conclusion that the matter produced at RHIC is strongly interacting with a large opacity, i.e. small mean free path. However, this does not necessarily have to be the case: In a previous paper[5)] we argued that the unavoidable color field instabilities in a rapidly expanding quark-gluon plasma generate an anomalous viscosity, which is smaller than the collisional viscosity in the weak coupling limit and for early times after the onset of the longitudinal expansion. The expression derived in Ref. 5) for the anomalous viscosity,

$$\frac{\eta_\mathrm{A}}{s} = \frac{1}{g^{3/2}}\left(\frac{(N_c^2-1)T\tau}{10b_0N_c}\right)^{1/2}, \tag{1·1}$$

where N_c denotes the number of colors and b_0 is a presently unknown numerical coefficient, is parametrically (in the coupling constant g) smaller than the collisional viscosity[6)]

$$\frac{\eta_\mathrm{C}}{s} = \frac{d_f}{g^4\ln(\sqrt{4\pi}/g)}, \tag{1·2}$$

where $d_f \approx 5$ for three light quark flavors. The anomalous viscosity is generated by the diffusive transport of quasithermal partons in the turbulent color fields created by Weibel-type instabilities. Such instabilities always arise when the parton momentum distribution is anisotropic,[7)–9)] as is necessarily the case when the quark-gluon plasma expands with a preferred direction. The nonlinear interaction of the unstable modes has been shown to result in a turbulent cascade of field energy into increasingly short wavelength modes,[10),11)] the characteristic signature of a turbulent plasma.

The purpose of this article is to present additional details of the calculation outlined in our previous paper. The article is structured as follows. In §2 we discuss the evidence from lattice QCD calculations for a quasi-particle nature of the quark-gluon plasma, even in the vicinity of the transition temperature T_c. This evidence is derived from the off-diagonal flavor structure of the quark susceptibilities. On the other hand, the RHIC data on jet quenching combined with entropy constraints demand a plasma almost entirely composed of gluons. The anomalous transport properties owing to the presence of intense coherent color fields can alleviate these constraints. In §3 we give a heuristic derivation of the anomalous viscosity $\eta_{\rm A}$. In §4 we derive the transport equation for quarks and gluons in a turbulent plasma. Section 5 contains a brief reminder of the definition of the shear viscosity. In §6 we present details of the formal derivation of $\eta_{\rm A}$ within the framework of the theory of transport phenomena in turbulent plasmas. We summarize our results in §7. Two appendices give an alternative derivation of the transport equation in the presence of plasma turbulence and provide details on the plasma instability structure, respectively.

§2. The quark-gluon plasma at RHIC

There is now overwhelming consensus that matter in (approximate) thermal equilibrium and with an energy density far in excess of 1 GeV/fm^3 is produced in nuclear collisions at RHIC.[1),12)] The evidence leading to this conclusion rests on two key observations. Firstly, all stable hadrons including multistrange baryons are emitted with chemical equilibrium abundances[13),14)] and thermal transverse momentum spectra that are boosted by a collective transverse ("radial") flow.[15),16)] Secondly, the hadron spectra measured in noncentral collisions show an azimuthal quadrupole anisotropy ("elliptic" flow) characteristic of the fluid dynamical expansion of a fireball with oval shape.[17),18)] Plotted as a function of the transverse kinetic energy and scaled by the number of valence quarks of the hadron ($n = 2$ for mesons and $n = 3$ for baryons), the flow parameter v_2 exhibits a universal dependence.[19)] This has been interpreted as evidence for a partonic origin of the observed flow pattern, suggesting that the transverse expansion of the matter is generated during a phase in which it contains independent quasi-particles with the quantum numbers of quarks.[20)–24)]

The observed magnitude of the elliptic flow requires an early onset of the period during which the expansion is governed by fluid dynamics (earlier than 1 fm/c after first impact) and nearly ideal fluid properties with a viscosity-to-entropy density ratio $\eta/s \ll 1$.[25)–28)] This result, interpreted in a quasi-particle picture, in turn requires that the mean free path in the medium must be extremely short, less than the average particle spacing. A short mean free path is also deduced from the strong suppression of the emission of hadrons with a transverse momentum p_T of several GeV/c or more.[29),30)] This phenomenon, commonly referred to as "jet quenching", is attributed to a large rate of energy loss of the energetic parton in the medium before it fragments into the observed hadron.[31)–39)] Since the energy loss is inversely proportional to the mean free path of the parton, this observation again requires a very short mean free path or, equivalently, a large scattering cross section.

There are other, purely theoretical reasons for believing that strongly interacting matter makes a rapid transition in a narrow temperature range around $T_c \approx 170$ MeV from a hadronic resonance gas to a plasma whose thermodynamic properties can be well described by quasiparticle excitations with the quantum numbers of quarks and gluons. Phenomenological attempts to describe the thermodynamic variables in terms of noninteracting quasiparticles with an effective thermal mass work astonishingly well for all temperatures down to T_c.[40)–43)] Rigorous resummation techniques using effective quasiparticle propagators for quarks and gluons are also quite successful above $(1.5 - 2)T_c$.[44),45)]

Even more compelling is the comparison of the temperature dependence of the diagonal and off-diagonal quark flavor susceptibilities calculated on the lattice[46)] with expectations from a quasiparticle picture. An especially sensitive quantity is the ratio between the baryon number-strangeness correlation $\langle \mathcal{B}S \rangle$ and the strangeness fluctuation $\langle S^2 \rangle$,[47)] which rapidly changes from the behavior characteristic of a hadron gas to the temperature independent value of a quasiparticulate quark-gluon plasma at T_c.[48)] This result strongly suggests that the quasiparticles that carry flavor must have the same quantum numbers as quarks almost immediately above T_c (for an alternate view, see Ref. 49)).

However, given the experimental evidence for the low viscosity and strong color opacity of the matter, the interpretation of the data in terms of the quasiparticle picture is problematic: In a quasiparticle picture the conjectured lower bound $\eta/s \geq 1/4\pi$ corresponds to an extremely short mean free path. Using the relation $\eta \approx n\lambda_f \bar{p}/2$ from standard kinetic theory, where n denotes the particle density and λ_f is the mean free path, as well as the equations $\bar{p} = 3T$ and $s \approx 4n$ holding for massless particle, one finds $\lambda_f \geq (3\pi T/2)^{-1}$. This implies that at the lower viscosity bound the mean free path must not be larger than about half the average distance between quasiparticles. This observation is corroborated by simulations of the parton transport within the framework of the Boltzmann equation with binary elastic scattering, which require cross sections up to twenty times larger than expected on the basis of perturbative QCD in order to reproduce the elliptic flow data.[50)] Therefore a collisional origin of such a low viscosity is not easily compatible with the notion that the medium is composed of well-defined quasiparticles. The quasiparticle picture is further cast into doubt by recent calculations of the spectral densities of correlators of the stress-energy tensor in an exactly solvable, strongly coupled quantum field theory ($N = 4$ supersymmetric Yang-Mills theory).[51),52)] These do not reveal peak-like structures that can be attributed to quasiparticle excitations, with the sole exception of the hydrodynamical sound mode.

One possible resolution of the puzzle is to argue[53)] that the quark-gluon plasma near T_c is actually strongly coupled. This argument is bolstered by the fact that results from solvable strongly coupled gauge theories are in qualitative agreement with certain aspects of QCD near T_c, such as reduction of the energy density with respect to the ideal gas limit,[54] and with experimentally obtained values of transport coefficients, such as the viscosity,[4),55),56] energy loss parameter,[57] and quark diffusion constant.[58)–60] It has been argued[49),61),62] that the microscopic structure of such a system is dominated by complex bound states of the elementary constituents. Note, however, that the diagonal and off-diagonal quark flavor susceptibilities calculated on the lattice[46] strongly constrain — and in many cases rule out — the existence of bound states of the elementary constituents above T_C. In addition, a recent calculation of the shear viscosity over entropy ratio in weakly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory yielded a result many times smaller than the corresponding weak-coupling result in QCD.[63] This finding therefore may actually suggest that the ratio η/s of QCD near the transition point is several times larger than the lower viscosity bound of $(4\pi)^{-1}$ for this quantity.[4]

The other possible resolution is that the transport properties of the quark-gluon plasma under the experimentally relevant conditions are not governed by collisional processes involving perturbative interactions among elementary excitations, but by collective phenomena. This situation is not uncommon in plasmas, where coherent fields can be spontaneously generated due to instabilities in the field equations in the presence of the medium. The occurrence of unstable field modes is a familiar phenomenon in electromagnetic plasmas with a charged particle distribution that is locally not fully equilibrated. The most relevant of these for our purposes is the instability discovered by Weibel, which arises when the momentum distribution of charged particles is anisotropic.[64]

It has been known for some time[7),8),65] that similar instabilities exist in quark-gluon plasmas with a parton momentum distribution that is not in thermal equilibrium. As a result of these instabilities long-range color fields can be excited with amplitudes far above the thermal level. The generic nature of such color instabilities has been recognized only in recent years.[9),66] Most work exploring the consequences of these instabilities[67)–70] has been focused on the early stage of the collision, when the momentum distribution is highly anisotropic and far from equilibrium. The fields generated by the instabilities drive the parton distribution rapidly toward local isotropy and thus toward the hydrodynamical regime.[71] However, the expansion of the quark-gluon plasma under its own pressure ensures that the matter never reaches complete equilibrium, and thus the presence of the color instabilities persists even during the period when the matter evolves by viscous hydrodynamical expansion, although the effect of collisions may weaken the color instabilities.[72] Since the size of the deviation from kinetic equilibrium is proportional to the viscosity itself, color instabilities are especially important when the quark-gluon plasma is weakly coupled and the collisional shear viscosity is large. By producing an anomalous contribution to the shear viscosity, extended color fields of a large amplitude present a mechanism that may be responsible for the observed small shear viscosity of the rapidly expanding quark-gluon plasma studied at RHIC.[5]

§3. Anomalous viscosity — Heuristic derivation

Anomalous contributions to transport coefficients caused by the action of turbulent fields are well known in electromagnetic plasmas. As we already mentioned, the condition for the spontaneous formation of extended electromagnetic fields in a plasma is the existence of instabilities in the field equations due to the interaction with the charged particles. This condition is satisfied in electromagnetic plasmas with an anisotropic momentum distribution.[64] The treatment of the consequences for transport processes is based on the formalism of particle propagation in turbulent plasmas originally developed by Dupree.[73),74] The term *plasma turbulence* refers to the spectral distribution of the field excitations, which follows a power law, in analogy to the Kolmogorov spectrum of vortex excitations in a fluid with fully developed turbulence. Such plasmas are characterized by strongly excited random field modes in certain regimes of instability, which coherently deflect the charged particles and thus reduce the effective mean free path. The scattering by turbulent fields in electromagnetic plasmas is known to greatly reduce the heat conductivity[75),76] and shear viscosity[77),78] of the plasma and to increase the energy loss of charged particles propagating through it.[79]

Following Abe and Niu[78] the contribution from turbulent fields to plasma transport coefficients is called *anomalous*. We shall see below that this designation is justified by the fact that the viscous corrections to the hydrodynamic energy-stress tensor due to the turbulent fields are nonlinear in the velocity gradient, but exhibit a sublinear dependence. As a consequence, the viscous effects in a quark-gluon plasma with a large imprinted velocity gradient are much smaller than expected from the usual linear theory. As the formation mechanism of quasi-thermal QCD matter in relativistic heavy ion collisions naturally leads to a large longitudinal velocity gradient in the direction of the beam axis, the relevance of such anomalous contributions to various transport coefficients, including the shear viscosity, is not unexpected, especially during the early stages of the expansion.

Before describing the derivation of the anomalous shear viscosity η_A of an expanding quark-gluon plasma in detail, it is useful to give a heuristic argument for its dependence on the amplitude of turbulent plasma fields. This argument will also elucidate the reason for the dominance of the anomalous viscosity at weak coupling. According to classical transport theory, the shear viscosity is given by an expression of the form[80]

$$\eta \approx \frac{1}{3} n\bar{p}\lambda_f, \tag{3·1}$$

where n denotes the particle density, $\bar{p}$ is the thermal momentum, and λ_f the mean free path. For a weakly coupled quark-gluon plasma, $n \approx 5T^3$ and $\bar{p} \approx 3T$. The mean free path depends on the mechanism under consideration. The collisional shear viscosity η_C is obtained by expressing the mean free path in terms of the transport cross section

$$\lambda_f^{(C)} = (n\,\sigma_{tr})^{-1}. \tag{3·2}$$

Using the perturbative QCD expression[80)]

$$\sigma_{\rm tr} \approx \frac{5g^4}{4\bar{p}^2} \ln \frac{\sqrt{4\pi}}{g} \tag{3·3}$$

for the transport cross section in a quark-gluon plasma yields the result

$$\eta_C \approx \frac{T}{\sigma_{\rm tr}} \approx \frac{18\pi s}{25 g^4 \ln(\sqrt{4\pi}/g)}, \tag{3·4}$$

where we use the relation $s \approx 4n$ valid for ulrarelativistic particles. This agrees parametrically with the result (1·2) for the collisional shear viscosity in leading logarithmic approximation.[6),*)]

The anomalous viscosity is determined by the same relation (3·1) for η, but the mean free path is now obtained by counting the number of color field domains a thermal parton has to traverse in order to "forget" its original direction of motion. If we denote the field strength generically by $\mathcal{B}^a$ (a denotes the color index), a single coherent domain of size $r_{\rm m}$ causes a momentum deflection of the order of $\Delta p \sim gQ^a\mathcal{B}^a r_{\rm m}$, where Q^a is the color charge of the parton. If different field domains are uncorrelated, the mean free path due to the action of the turbulent fields is given by

$$\lambda_{\rm f}^{\rm (A)} = r_{\rm m} \langle (\bar{p}/\Delta p)^2 \rangle \sim \frac{\bar{p}^2}{g^2 Q^2 \langle \mathcal{B}^2 \rangle r_{\rm m}}. \tag{3·5}$$

The anomalous shear viscosity thus takes the form:

$$\eta_{\rm A} \sim \frac{n\,\bar{p}^3}{3g^2 Q^2 \langle \mathcal{B}^2 \rangle r_{\rm m}} \sim \frac{9sT^3}{4g^2 Q^2 \langle \mathcal{B}^2 \rangle r_{\rm m}}, \tag{3·6}$$

which agrees with expression (16) of Ref. 5), if we identify $r_{\rm m}$ with the memory time $\tau_{\rm m}$ for relativistic partons.

The argument now comes down to an estimate of the average field intensity $\langle \mathcal{B}^2 \rangle$ and size $r_{\rm m}$ of a domain. We first note that the size is given by the characteristic wavelength of the unstable field modes. Near thermal equilibrium, the parameter describing the influence of hard thermal partons on the soft color field modes is the color-electric screening mass $m_{\rm D} \sim gT$. Introducing a dimensionless parameter ξ for the magnitude of the momentum space anisotropy,[9)] the wave vector domain of unstable modes is $k^2 \leq \xi m_{\rm D}^2$ (see Appendix B). Thus $r_{\rm m} \sim \xi^{-1/2}(gT)^{-1}$. The exponential growth of the unstable soft field modes is saturated, when the nonlinearities in the Yang-Mills equation become of the same order as the gradient term: $g|A| \sim k$, which implies that the field energy in the unstable mode is of the order of $g^2\langle \mathcal{B}^2 \rangle \sim k^4 \sim \xi^2 m_{\rm D}^4$. The denominator in Eq. (3·6) thus has the characteristic size, at saturation:

$$g^2 Q^2 \langle \mathcal{B}^2 \rangle r_{\rm m} \sim \xi^{3/2} m_{\rm D}^3 \sim \xi^{3/2} (gT)^3. \tag{3·7}$$

*) The logarithm in Eq. (3·3) is the weak coupling limit of the function[80)] $I(\alpha_s) = (2\alpha_s+1)\ln(1+\alpha_s^{-1}) - 2$, which never becomes negative. In fact, $\alpha_s^2 I(\alpha_s) \to 1/6$ for $\alpha_s \gg 1$, which suggests that η/s approaches a constant in the strong coupling limit. Of course, the derivation of Eq. (3·3) becomes invalid in this limit.

Inserting this result into the expression (3·6) gives the following relation for the anomalous viscosity:

$$\eta_{\rm A} \sim \frac{s}{g^3 \xi^{3/2}}. \tag{3·8}$$

We conclude that the anomalous viscosity will be smaller than the collisional viscosity, if the coupling constant g is sufficiently small and the anisotropy parameter ξ is sufficiently large.

§4. Diffusive Vlasov-Boltzmann equation

In order to present a more rigorous derivation of the anomalous viscosity of a turbulent quark-gluon plasma, we first need to derive the appropriate transport equation. The propagation of quasi-thermal partons in the presence of soft, locally coherent color fields and hard collisions among the partons is described by a Vlasov-Boltzmann equation:[81)]

$$v^\mu \frac{\partial}{\partial x^\mu} f(\boldsymbol{r}, \boldsymbol{p}, t) + g\boldsymbol{F}^a \cdot \nabla_p f^a(\boldsymbol{r}, \boldsymbol{p}, t) + C[f] = 0\,. \tag{4·1}$$

Here $f(\boldsymbol{r}, \boldsymbol{p}, t)$ denotes the usual parton distribution in phase space, which sums over all parton colors. $f^a(\boldsymbol{r}, \boldsymbol{p}, t)$ denotes the color octet distribution function, which weights each parton with its color charge Q^a. Both, f and f^a can be defined in the semiclassical formalism[82)] as the moments of the distribution function $\tilde{f}(\boldsymbol{r}, \boldsymbol{p}, Q, t)$ in an extended phase space that includes the color sector:

$$f(\boldsymbol{r}, \boldsymbol{p}, t) = \int dQ\, \tilde{f}(\boldsymbol{r}, \boldsymbol{p}, Q, t)\,, \tag{4·2a}$$

$$f^a(\boldsymbol{r}, \boldsymbol{p}, t) = \int dQ\, Q^a \tilde{f}(\boldsymbol{r}, \boldsymbol{p}, Q, t)\,. \tag{4·2b}$$

In Eq. (4·1) we have used the covariant notation $v^\mu = p^\mu/p^0$ with $p^\mu = (E_p, \boldsymbol{p})$. $\boldsymbol{v} = \boldsymbol{p}/E_p$ denotes the velocity of a parton with momentum $\boldsymbol{p}$ and energy E_p. Furthermore,

$$\boldsymbol{F}^a = \mathcal{E}^a + \boldsymbol{v} \times \mathcal{B}^a \tag{4·3}$$

denotes the color Lorentz force, and $C[f]$ stands for the collision term. (We will specify $C[f]$ later.)

The color octet distribution f^a satisfies a transport equation of its own, which couples it to phase space distributions of even higher color-$SU(3)$ representations. In the vicinity of the equilibrium distribution, however, it makes sense to truncate the hierarchy at the level of the color singlet and octet distributions and to retain only the lowest terms in the gradient expansion. We also note that the color octet distribution function vanishes in equilibrium, $f^a_0 = 0$, implying that f^a is at least of first order in the perturbation.

The transport equation for f^a then reads:[81),82)]

$$v^\mu \frac{\partial f^a}{\partial x^\mu} + g f_{abc} A^b_\mu v^\mu f^c + \frac{gC_2}{N_c^2 - 1} \boldsymbol{F}^a \cdot \nabla_p f + C^a[f, f^a] = 0, \tag{4·4}$$

where C_2 denotes the quadratic Casimir invariant of the color representation of the thermal partons.

Before we linearize the Vlasov-Boltzmann equation (4·1), we must rewrite it in a form applicable to the case of a turbulent quark-gluon plasma. In order to do so, we need to make additional assumptions about the field distribution in the Vlasov force term. Based on the arguments presented in §2, we shall assume that the color field is turbulent, i.e. random with a certain spatial and temporal correlation structure for fields at different space-time points. This assumption will allow us to rewrite the force term involving the color octet distribution function f^a into a dissipative (Langevin) term acting on the color singlet distribution f.

We shall derive the diffusion term in two different ways. In this section, we will use linear response theory to calculate f^a from the color singlet distribution for given color fields. The diffusion term is then obtained after substituting the result into the force term in Eq. (4·1) and choosing appropriate correlation functions for the color fields. In Appendix A, we shall present another derivation based on the extended distribution function $\tilde{f}(\boldsymbol{r}, \boldsymbol{p}, Q, t)$. There we also discuss the difference between our approach and the one due to Dupree[73] for turbulent abelian plasmas.

In order to resolve Eq. (4·4) for the color octet distribution, we Fourier transform the dependence on the space-time coordinate $x^\mu = (t, \boldsymbol{r})$:

$$f^a(\boldsymbol{p}, x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} f^a(\boldsymbol{p}, k). \tag{4·5}$$

We allow for an arbitrary particle distribution in momentum space, but neglect any space-time dependence of the singlet distribution $f(\boldsymbol{p})$. Ignoring the collision term and, for the moment, the gauge connection associated with the space-time derivative, the solution of Eq. (4·4) is given by[83]

$$f^a(\boldsymbol{p}, k) = -ig\frac{C_2}{N_c^2 - 1}(v \cdot k + i\epsilon)^{-1} \boldsymbol{F}^a(k) \cdot \nabla_p f(\boldsymbol{p}), \tag{4·6}$$

where $v \cdot k \equiv v^\mu k_\mu = k^0 - \boldsymbol{v} \cdot \boldsymbol{k}$. The gauge connection has the effect of adding a path-ordered factor

$$U_{ac}(x, x') = P \exp\left(-\int_{x'}^{x} f_{abc} A_\mu^b \, dx^\mu\right), \tag{4·7}$$

which parallel transports the gauge fields from x' to x. Returning to coordinate space, Eq. (4·6) then takes the form

$$f^a(\boldsymbol{p}, x) = -ig\frac{C_2}{N_c^2 - 1}\int \frac{d^4k}{(2\pi)^4}\int d^4x' \, U_{ab}(x, x')\frac{e^{ik\cdot(x'-x)}}{v \cdot k + i\epsilon}\boldsymbol{F}^b(x') \cdot \nabla_p f(\boldsymbol{p}). \tag{4·8}$$

Inserting this solution for f^a into Eq. (4·1) for the singlet distribution function f, the Vlasov force term takes the following form:

$$g\boldsymbol{F}^a(x) \cdot \nabla_p f^a(\boldsymbol{p}, x) = -\frac{ig^2 C_2}{N_c^2 - 1}\boldsymbol{F}^a(x) \cdot \nabla_p \int \frac{d^4k}{(2\pi)^4}\int d^4x' \, U_{ab}(x, x') \times \frac{e^{ik\cdot(x'-x)}}{v \cdot k + i\epsilon}\boldsymbol{F}^b(x') \cdot \nabla_p f(\boldsymbol{p}). \tag{4·9}$$

We now need to invoke the argument that the soft color fields are turbulent and that their action on the quasi-thermal partons in Eq. (4·9) can be described by taking an ensemble average, which can be factorized in the form

$$\langle F_i^a(x)U_{ab}(x,x')F_j^b(x')f(\boldsymbol{p})\rangle = \langle F_i^a(x)U_{ab}(x,x')F_j^b(x')\rangle \bar{f}(\boldsymbol{p}), \qquad (4\cdot10)$$

where $\bar{f} \equiv \langle f\rangle$. We furthermore assume that the correlation functions of fields at different space-time points x and x' depend only on $|x-x'|$ and fall off rapidly with correlation time τ and correlation length σ:

$$\langle \mathcal{E}_i^a(x)U_{ab}(x,x')\mathcal{E}_j^b(x')\rangle = \langle \mathcal{E}_i^a\mathcal{E}_j^a\rangle\, \Phi_\tau^{(\mathrm{el})}\left(|t-t'|\right)\, \tilde{\Phi}_\sigma^{(\mathrm{el})}\left(|\boldsymbol{x}-\boldsymbol{x}'|\right), \qquad (4\cdot11\mathrm{a})$$

$$\langle \mathcal{B}_i^a(x)U_{ab}(x,x')\mathcal{B}_j^b(x')\rangle = \langle \mathcal{B}_i^a\mathcal{B}_j^a\rangle\, \Phi_\tau^{(\mathrm{mag})}\left(|t-t'|\right)\, \tilde{\Phi}_\sigma^{(\mathrm{mag})}\left(|\boldsymbol{x}-\boldsymbol{x}'|\right). \qquad (4\cdot11\mathrm{b})$$

Examples satisfying these assumptions are the Gaussian correlators

$$\Phi_\tau^{(\mathrm{el/mag})}\left(|t-t'|\right) = \exp\left[-(t-t')^2/2\tau_{\mathrm{el/mag}}^2\right], \qquad (4\cdot12\mathrm{a})$$

$$\tilde{\Phi}_\sigma^{(\mathrm{el/mag})}\left(|\boldsymbol{x}-\boldsymbol{x}'|\right) = \exp\left[-(\boldsymbol{x}-\boldsymbol{x}')^2/2\sigma_{\mathrm{el/mag}}^2\right]. \qquad (4\cdot12\mathrm{b})$$

Finally, we assume that the color-electric and -magnetic fields are uncorrelated: $\langle \mathcal{E}_i^a(x)U_{ab}(x,x')\mathcal{B}_j^b(x')\rangle = 0$.

The reality of the correlation functions (4·12) and their symmetry with respect to exchange of the two space-time arguments express the chaotic nature of the plasma. It is here where the reversibility of the mean field dynamics is explicitly broken and the dissipative nature of the turbulent plasma is introduced. The existence of plasma instabilities and the associated exponential growth of the unstable modes, which correspond to the presence of positive Lyapunov exponents in the coupled field-particle system, forms the essential physical basis for our argument.

Since the right-hand side of Eq. (4·9) must be real, under the conditions outlined above only the imaginary part of the propagator

$$\mathrm{Im}\,\frac{1}{v\cdot k + i\epsilon} = -\pi\delta(v\cdot k) \qquad (4\cdot13)$$

can contribute. Performing first the integral over k^0, then the integral over $\boldsymbol{k}$ and finally the integral over $\boldsymbol{x}'$, we obtain

$$\begin{aligned}\langle g\boldsymbol{F}^a\cdot\nabla_p f^a\rangle &= -\frac{g^2C_2}{N_c^2-1}\left[\tau_{\mathrm{m}}^{\mathrm{el}}\langle\mathcal{E}_i^a\mathcal{E}_j^a\rangle\,\frac{\partial^2}{\partial p_i\partial p_j} + \tau_{\mathrm{m}}^{\mathrm{mag}}\langle\mathcal{B}_i^a\mathcal{B}_j^a\rangle\,(\boldsymbol{v}\times\nabla_p)_i(\boldsymbol{v}\times\nabla_p)_j\right]\bar{f}(\boldsymbol{p})\\ &\equiv -\nabla_p\cdot D(\boldsymbol{p})\cdot\nabla_p\,\bar{f}(\boldsymbol{p}). \end{aligned} \qquad (4\cdot14)$$

Here we make use of the identity $(\boldsymbol{v}\times\mathcal{B})\cdot\nabla_p = -\mathcal{B}\cdot(\boldsymbol{v}\times\nabla_p)$ and introduce the memory time

$$\tau_{\mathrm{m}}^{\mathrm{el/mag}} = \frac{1}{2}\int_{-\infty}^{\infty} dt'\,\Phi_\tau^{(\mathrm{el/mag})}\left(|t-t'|\right)\,\tilde{\Phi}_\sigma^{(\mathrm{el/mag})}\left(|\boldsymbol{v}(t-t')|\right). \qquad (4\cdot15)$$

The precise matrix structure of the correlators (4·11) may differ between the various stages of a relativistic heavy ion collision. Initially, color-magnetic field modes exhibit the strongest growth rate,[84] but in the later turbulent steady-state regime all spatial components of the color field modes populated by the instabilities are expected to be of comparable strength.[10),11)] For completeness, we shall discuss both scenarios.

When we consider only the color-magnetic fields initially generated by the plasma instability, which point in a transverse direction, we can expect the ensemble average to be transverse in space:

$$\text{(A)} \qquad \langle \mathcal{B}_i^a \mathcal{B}_j^a \rangle = \frac{1}{2}(\delta_{ij} - \delta_{iz}\delta_{jz})\langle \mathcal{B}^2 \rangle \,; \qquad \langle \mathcal{E}_i^a \mathcal{E}_j^a \rangle \approx 0 \,. \tag{4·16}$$

We shall call this case *Scenario A*. In the late turbulent phase all spatial components of the field correlators are of approximately equal size, and we write

$$\text{(B)} \qquad \langle \mathcal{B}_i^a \mathcal{B}_j^a \rangle = \frac{1}{3}\delta_{ij}\langle \mathcal{B}^2 \rangle \,; \qquad \langle \mathcal{E}_i^a \mathcal{E}_j^a \rangle = \frac{1}{3}\delta_{ij}\langle \mathcal{E}^2 \rangle \,. \tag{4·17}$$

We shall call this case *Scenario B*. Employing the notation $-i\boldsymbol{p} \times \nabla_p = \boldsymbol{L}^{(p)}$ for the generator of rotations in momentum space, we can write the diffusive term in the transverse color-magnetic field dominated Scenario A as

$$\nabla_p \cdot D \cdot \nabla_p = -\frac{g^2 C_2}{2(N_c^2-1)E_p^2}\langle \mathcal{B}^2 \rangle \, \tau_{\rm m}^{\rm mag} \left[(\boldsymbol{L}^{(p)})^2 - (L_z^{(p)})^2 \right] , \tag{4·18}$$

and in Scenario B with isotropically turbulent color-electric and color-magnetic field excitations as

$$\nabla_p \cdot D \cdot \nabla_p = \frac{g^2 C_2}{3(N_c^2-1)} \left[\langle \mathcal{E}^2 \rangle \, \tau_{\rm m}^{\rm el} \, \nabla_p^2 - \langle \mathcal{B}^2 \rangle \, \tau_{\rm m}^{\rm mag} \, \frac{(\boldsymbol{L}^{(p)})^2}{E_p^2} \right] . \tag{4·19}$$

Note that the operator ∇_p^2 associated with random color-electric fields yields a nonvanishing contribution when acting on the equilibrated momentum distribution $f_0(\boldsymbol{p})$. This is different for the operator associated with random color-magnetic fields, $(\boldsymbol{L}^{(p)})^2$, which only yields a nonvanishing contribution when acting on the anisotropic part of the momentum distribution given by $f_1(\boldsymbol{p})$. This is not surprising, because randomly distributed electric fields are well known to lead to an increase in the average energy of the plasma particles,[94] corresponding to a heating of the plasma. In contrast, color-magnetic fields only contribute to the isotropization of the momentum distribution and do not cause plasma heating.

§5. Linear response theory

We now have motivated the replacement of the Vlasov-Boltzmann equation (4·1) by Dupree's ensemble averaged, diffusive Vlasov-Boltzmann equation

$$v^\mu \frac{\partial}{\partial x^\mu}\bar{f} - \nabla_p \cdot D \cdot \nabla_p \bar{f} + \langle C[f] \rangle = 0 \,. \tag{5·1}$$

We next need to expand this equation up to linear terms in the ensemble averaged perturbation $\delta\bar{f}$ of the parton distribution. The drift term gives rise to gradients of the collective variables T and u^μ, which are considered to be of the same size as terms linear in $\delta\bar{f}$; the collision term vanishes at equilibrium and contributes only at first order in $\delta\bar{f}$; and the Vlasov term will require special consideration.

The general linear response (Chapman-Enskog) formalism assumes a small perturbation of the thermal equilibrium distribution

$$f_0(\boldsymbol{p}) = \left(e^{\beta u\cdot p} \mp 1\right)^{-1}, \tag{5·2}$$

where $\beta = 1/T$ denotes the inverse temperature and the upper/lower sign applies to bosons/fermions, respectively. Here u^μ denotes the four-velocity of the equilibrated medium and $u \cdot p \equiv u^\mu p_\mu$. Using the relation $T\,\partial f_0/\partial(u\cdot p) = -f_0(1 \pm f_0)$, one expresses the perturbed distribution function in the form:

$$f(\boldsymbol{p},\boldsymbol{r}) = f_0(\boldsymbol{p}) + \delta f(\boldsymbol{p},\boldsymbol{r}) = f_0(\boldsymbol{p})[1 + f_1(\boldsymbol{p},\boldsymbol{r})(1 \pm f_0(\boldsymbol{p}))], \tag{5·3}$$

where the function $f_1(\boldsymbol{p},\boldsymbol{r})$ can be considered as (minus) the change in the argument of the equilibrium distribution:[85)]

$$f(\boldsymbol{p},\boldsymbol{r}) \approx \left(e^{\beta u\cdot p - f_1(\boldsymbol{p},\boldsymbol{r})} \mp 1\right)^{-1}. \tag{5·4}$$

While the relativistic dissipative hydrodynamics can be cast in a manifestly covariant formalism,[86),87)] this is not necessary for the derivation of the transport coefficients, which are Lorentz invariants. It is thus convenient[88)] to work in the local rest frame of the fluid, i.e. in the frame where $\boldsymbol{u}(x) = 0, u^0 = 1$ and thus $u \cdot p = p^0 = E_p$ at the considered space-time point x. This choice also implies the relation $\partial u^0/\partial x^\mu = 0$.

In the derivation of transport coefficients one assumes that the particle distribution is slowly varying, so that the deviation from the equilibrium distribution is homogeneous in space and proportional to gradients of the equilibrium parameters. For the shear viscosity one uses the local perturbation[*)]

$$f_1(\boldsymbol{p},\boldsymbol{r}) = -\frac{\bar{\Delta}(p)}{E_p T^2} p_i p_j (\nabla u)_{ij}, \tag{5·5}$$

where $\bar{\Delta}(p)$ is a scalar function of the momentum p, which measures the magnitude of the deviation from equilibrium, and $(\nabla u)_{ij}$ denotes the traceless symmetrized velocity gradient:

$$(\nabla u)_{ij} \equiv \frac{1}{2}(\nabla_i u_j + \nabla_j u_i) - \frac{1}{3}\delta_{ij}\nabla\cdot\boldsymbol{u}. \tag{5·6}$$

For Bjorken's[89)] boost invariant flow field $u_z = z/\tau$ we have

$$(\nabla u)_{ij} = \frac{1}{3\tau}\mathrm{diag}(-1,-1,2). \tag{5·7}$$

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*) Our definition makes $\bar{\Delta}(\boldsymbol{p})$ dimensionless. To compare with Ref. 6), identify $\bar{\Delta}(p) = -\chi(p)E_p/\boldsymbol{p}^2$. In order to compare with Ref. 85), identify $f_1(p) = \chi(p)/T$ and $\bar{\Delta}(p) = -g(p)T/E_p$. Also note that $(\nabla u)_{ij}$ projects out the traceless part of $p_i p_j$.

The connection to the shear viscosity is made by comparing the microscopic definition of the stress tensor

$$T_{ik} = \int \frac{d^3p}{(2\pi)^3 E_p}\, p_i p_k f(\boldsymbol{p}, \boldsymbol{r}) \tag{5·8}$$

with the macroscopic definition of the viscous stress:

$$T_{ik} = T_{ik}^{(0)} + \delta T_{ik} = P\delta_{ik} + \varepsilon u_i u_k - 2\eta(\nabla u)_{ik} - \zeta\delta_{ik}\nabla\cdot\boldsymbol{u}, \tag{5·9}$$

where ε and P are the equilibrium energy density and pressure, and where η and ζ denote the shear and bulk viscosity, respectively.

Combining Eqs. (5·3), (5·5) and (5·8) one finds

$$\begin{aligned}\delta T_{ik} &= \int \frac{d^3p}{(2\pi)^3 E_p}\, p_i p_k f_1(\boldsymbol{p}) f_0(\boldsymbol{p})(1 \pm f_0(\boldsymbol{p})) \\ &= \frac{1}{T}(\nabla u)_{mn} \int \frac{d^3p}{(2\pi)^3 E_p^2}\, p_i p_k \bar{\Delta}(p) p_m p_n \frac{\partial f_0}{\partial E_p} \\ &= -\eta(\nabla u)_{mn}[\delta_{im}\delta_{kn} + \delta_{in}\delta_{km} + \delta_{ik}\delta_{mn}],\end{aligned} \tag{5·10}$$

with the shear viscosity coefficient

$$\eta = -\frac{1}{15T}\int \frac{d^3p}{(2\pi)^3}\, \frac{\boldsymbol{p}^4}{E_p^2} \bar{\Delta}(p) \frac{\partial f_0}{\partial E_p}. \tag{5·11}$$

For the special case (5·7) of boost-invariant longitudinal flow, the perturbation of the equilibrium distribution takes the form

$$f_1(\boldsymbol{p}) = -\frac{\bar{\Delta}(p)}{E_p T^2}\, |\nabla u| \left(p_z^2 - \frac{\boldsymbol{p}^2}{3}\right), \tag{5·12}$$

with (note the normalization):

$$|\nabla u| \equiv \left[\frac{3}{2}(\nabla u)_{ij}(\nabla u)_{ji}\right]^{1/2} = \frac{1}{\tau}, \tag{5·13}$$

and the anisotropy of the stress tensor (5·10) is[3),90)]

$$2\delta T_{xx} = 2\delta T_{yy} = -\delta T_{zz} = \frac{4}{3}\,\eta|\nabla u|\,. \tag{5·14}$$

§6. Shear viscosity

6.1. *Drift term*

We begin with the evaluation of the drift term. The dominant contribution comes from the r-dependence of the local equilibrium distribution. Because the form of this term does not depend on ensemble averaging, we shall omit the "bar" symbol in this section. Using a dot to denote the partial time derivative, one finds in the local rest frame (recall that $\boldsymbol{v} = \boldsymbol{p}/E_p$):

$$v^\mu \frac{\partial}{\partial x^\mu} f_0(\boldsymbol{p}) = -f_0(1 \pm f_0) \left[\dot\beta E_p - \beta\dot{\boldsymbol{u}} \cdot \boldsymbol{p} + \boldsymbol{v} \cdot \nabla\beta - \beta \boldsymbol{v} \cdot \nabla(\boldsymbol{u} \cdot \boldsymbol{p})\right]. \tag{6·1}$$

This expression can be simplified with the help of the energy-momentum conservation law in the presence of color fields:

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = F^{a\mu\nu} j^a_\nu . \tag{6·2}$$

With the help of the expression for $T^{\mu\nu}$ in terms of the momentum space distribution function, we find

$$\begin{aligned}\frac{\partial T^{\mu\nu}}{\partial x^\mu} &= \int \frac{d^3p}{(2\pi)^3 E_p} p^\mu p^\nu \frac{\partial f_0}{\partial x^\mu} \\ &= -\int \frac{d^3p}{(2\pi)^3 E_p} p^\mu p^\nu f_0(1 \pm f_0) \left(E_p \frac{\partial\beta}{\partial x^\mu} - \beta \boldsymbol{p} \cdot \frac{\partial \boldsymbol{u}}{\partial x^\mu}\right) \\ &= \frac{\partial\beta}{\partial x^\mu}\frac{\partial T^{\mu\nu}}{\partial\beta} - \frac{\partial \boldsymbol{u}}{\partial x^\mu} \cdot \beta\frac{\partial}{\partial\beta} \int \frac{d^3p}{(2\pi)^3 E_p^2} p^\mu p^\nu \boldsymbol{p} f_0(p).\end{aligned} \tag{6·3}$$

Using the expression (4·8) for the color octet distribution induced by the color field and following the same arguments that led to the relation (4·14), we obtain for the ensemble average of the right-hand side of Eq. (6·2):

$$\begin{aligned}\langle F^{a\mu\nu}(x) j^a_\nu(x)\rangle &= g \int \frac{d^3p}{(2\pi)^3 E_p} \langle F^{a\mu\nu} p_\nu f^a(\boldsymbol{p}, x)\rangle \\ &= -\frac{g^2 C_2}{N_c^2 - 1} \int \frac{d^3p}{(2\pi)^3 E_p} p_\nu \tau_{\rm m} \langle F^{a\mu\nu} \boldsymbol{F}^a\rangle \cdot \nabla_p \bar f(\boldsymbol{p}).\end{aligned} \tag{6·4}$$

The time-like component ($\nu = 0$) of the expressions (6·3) and (6·4) is easily evaluated to yield the relation

$$\dot\beta \frac{\partial\varepsilon}{\partial\beta} - (\nabla \cdot \boldsymbol{u})\beta\frac{\partial P}{\partial\beta} = \langle \mathcal{E}^a \cdot \boldsymbol{j}^a\rangle = \frac{m_{\rm D}^2}{3}\langle\mathcal{E}^2\rangle\tau_{\rm m}^{\rm el} , \tag{6·5}$$

where $m_{\rm D}$ is the Debye screening mass defined as*)

$$m_{\rm D}^2 = -\frac{g^2 C_2}{N_c^2 - 1} \int \frac{d^3p}{(2\pi)^3 E_p} \boldsymbol{p} \cdot \nabla_p \bar f(\boldsymbol{p}) = \left(\frac{N_c}{3} + \frac{N_f}{6}\right) g^2 T^2 , \tag{6·6}$$

and the final expression holds for a noninteracting plasma of massless quarks and gluons in thermal equilibrium. The space-like components yield, after some calculations:

$$(\nabla\beta - \beta\dot{\boldsymbol{u}})\frac{\partial P}{\partial\beta} = -\langle \mathcal{E}^a j^{a0} + \boldsymbol{j}^a \times \mathcal{B}^a\rangle = -g \int \frac{d^3p}{(2\pi)^3} \langle \boldsymbol{F}^a f^a(\boldsymbol{p})\rangle = 0 . \tag{6·7}$$

*) In the notation of Ref. 5) the coefficient of $g^2T^2$is given by the expression $\frac{N_c \nu_2' \zeta(2)}{(N_c^2-1)\pi^2}$.

The right-hand side vanishes by virtue of Eq. (4·8), because the momentum integral reduces to a surface term at infinity.

We can now use the relations (6·5) and (6·7) to eliminate the time derivatives $\dot{\beta}$ and $\dot{\boldsymbol{u}}$ from the drift term (6·1), obtaining

$$v^\mu \frac{\partial}{\partial x^\mu} f_0(\boldsymbol{p}) = f_0(1 \pm f_0) \left[\frac{(\boldsymbol{p}\cdot\nabla)(\boldsymbol{u}\cdot\boldsymbol{p})}{E_p T} - \frac{m_{\rm D}^2 \langle \mathcal{E}^2 \rangle \tau_{\rm m}^{\rm el} E_p}{3T^2(\partial\varepsilon/\partial T)} - \frac{\partial P/\partial T}{T(\partial\varepsilon/\partial T)} E_p (\nabla\cdot\boldsymbol{u}) \right] . \tag{6·8}$$

The first term in the square brackets can be expressed in terms of the traceless velocity gradient (5·6) and a term which can be combined with the last term in the brackets, yielding

$$\left(\frac{\boldsymbol{p}^2}{3E_p^2} - \frac{\partial P/\partial T}{\partial\varepsilon/\partial T} \right) \frac{E_p}{T} (\nabla\cdot\boldsymbol{u}) . \tag{6·9}$$

For a gas of massless noninteracting partons, $\varepsilon = 3P$ and $\boldsymbol{p}^2 = E_p^2$, causing the term proportional to the divergence of the collective velocity to vanish, in accordance with the expectation that the bulk viscosity of a scale invariant system must be zero. Thus our final expression for the drift term is

$$v^\mu \frac{\partial}{\partial x^\mu} f_0(\boldsymbol{p}) = f_0(1 \pm f_0) \left[\frac{p_i p_j}{E_p T} (\nabla u)_{ij} - \frac{m_{\rm D}^2 \langle \mathcal{E}^2 \rangle \tau_{\rm m}^{\rm el} E_p}{3T^2(\partial\varepsilon/\partial T)} \right] . \tag{6·10}$$

Equation (6·5) describes the heating of the plasma by the coherent color field. The coefficient of $\langle \mathcal{E}^2 \rangle$,

$$\sigma_{\rm A} = \frac{1}{3} m_{\rm D}^2 \tau_{\rm m}^{\rm el} , \tag{6·11}$$

is the effective color conductivity of the turbulent plasma (see Appendix B of Ref. 83)). This expression for the conductivity differs from the collisional conductivity obtained for a plasma in equilibrium in the absence of coherent color fields.[91),92)] In analogy to the anomalous shear viscosity $\eta_{\rm A}$ of the turbulent plasma, which is the main object of this article, $\sigma_{\rm A}$ may be called the *anomalous* color conductivity.

6.2. *Force term*

Next we come to the diffusive Vlasov force term in Eq. (5·1). Because they are associated with different operators in momentum space (see Eq. (4·19)), the color-electric and color-magnetic contributions to the diffusion term need to be considered separately. Since f_0 depends solely on $E_p = |\boldsymbol{p}|$ and hence $\boldsymbol{L}^{(p)} f_0 = 0$, the color-magnetic contribution to the diffusion term affects only the deviation of the particle distribution from equilibrium. On the other hand, the color-electric contribution to the diffusion term affects the full momentum space distribution, because $\nabla_p^2 f_0 \neq 0$. The difference is easy to understand: color-magnetic fields only change the direction of the momentum of a parton, leading to a rearrangement of particles within the equilibrium distribution, but not to a modification of the distribution itself. Color-electric fields, on the other hand, accelerate partons and thus lead, on average, to a heating of the thermal distribution. In the macroscopic formulation, this difference is related to the fact that color-electric fields induce a color current in the parton distribution, which interacts dissipatively with the color-electric field. We have already discussed this effect at the end of the previous section.

We first consider the diffusion operator (4·18) for Scenario A. Since the angular dependence of the perturbation $f_1(\boldsymbol{p})$ in Eq. (5·5) has the form of a quadrupole in momentum space, it is an eigenfunction of the magnetic diffusion operator. For the perturbation (5·12) associated with the Bjorken flow, $f_1(\boldsymbol{p}) \sim Y_{20}(\hat{\boldsymbol{p}})$, implying that

$$\left[(L^{(p)})^2 - (L_z^{(p)})^2\right] f_1(\boldsymbol{p}) = \left[(L^{(p)})^2\right] f_1(\boldsymbol{p}) = 6\, f_1(\boldsymbol{p}). \tag{6·12}$$

The color-magnetic part of the diffusion term in Scenario A thus takes the form

$$\nabla_p \cdot D^{\rm mag} \cdot \nabla_p \bar{f}(\boldsymbol{p}) = \frac{3\, C_2\, \bar{\Delta}\, g^2 \langle \mathcal{B}^2 \rangle\, \tau_{\rm m}^{\rm mag}}{(N_c^2-1) E_p^3 T^2}\, f_0(1 \pm f_0) p_i p_j (\nabla u)_{ij}\,. \tag{6·13}$$

The diffusion term for Scenario B yields (with $p = |\boldsymbol{p}|$):

$$\begin{aligned}\nabla_p \cdot D \cdot \nabla_p \bar{f}(\boldsymbol{p}) = {} & \frac{C_2\, g^2 \langle \mathcal{E}^2 \rangle\, \tau_{\rm m}^{\rm el}}{3(N_c^2-1)p} \frac{\partial^2}{\partial p^2}[p f(\boldsymbol{p})] \\ & + \frac{2 C_2 \bar{\Delta} g^2}{(N_c^2-1) E_p T^2} \left(\frac{\langle \mathcal{E}^2 \rangle\, \tau_{\rm m}^{\rm el}}{p^2} + \frac{\langle \mathcal{B}^2 \rangle\, \tau_{\rm m}^{\rm mag}}{E_p^2} \right) \\ & \times f_0(1 \pm f_0) p_i p_j (\nabla u)_{ij}\,. \end{aligned} \tag{6·14}$$

The rotationally symmetric part of the first term describes the heating of the parton distribution by the turbulent electric fields discussed in conjunction with the anomalous color conductivity (6·11). The anisotropic part of the first term, together with the second term describes the angular diffusion of the parton distribution, which leads to an anomalous viscosity. These terms have the same structure as the force term in Scenario A except that color-electric as well as color-magnetic fields contribute.

6.3. *Collision term*

In the evaluation of the collision term

$$\begin{aligned} C[f](\boldsymbol{p}) = {} & \frac{1}{4E_p} \int \frac{d^3k d^3p' d^3k'}{(2\pi)^5 8 E_k E_p' E_k'} \delta^4(p + k - p' - k') \left| \sum_\alpha M_\alpha(p,k,p',k') \right|^2 \\ & \times \big(f(\boldsymbol{p}) f(\boldsymbol{k}) [1 \pm f(\boldsymbol{p}')][1 \pm f(\boldsymbol{k}')] \\ & \quad - f(\boldsymbol{p}') f(\boldsymbol{k}') [1 \pm f(\boldsymbol{p})][1 \pm f(\boldsymbol{k})] \big)\,, \end{aligned} \tag{6·15}$$

we follow Arnold et al.[6] Note that Eq. (6·15) involves an implicit summation over parton flavors and helicities. Since the collision term vanishes at equilibrium owing to detailed balance, the leading contribution is linear in $f_1(\boldsymbol{p})$. In first approximation, the collision term thus gives rise to a linear integral operator of the form

$$I[f_1](\boldsymbol{p}) = \int \frac{d^3k}{(2\pi)^3} d\sigma_{12} v_{\rm rel} f_0(\boldsymbol{p}) f_0(\boldsymbol{k}) \left[f_1(\boldsymbol{p}) + f_1(\boldsymbol{k}) - f_1(\boldsymbol{p}') - f_1(\boldsymbol{k}') \right]\,, \tag{6·16}$$

where

$$d\sigma_{12} = \frac{1}{8\sqrt{p\cdot k}} \frac{d^3p'd^3k'}{16\pi^2 E'_p E'_k} \delta^4(p+k-p'-k') \left| \sum_\alpha M_\alpha \right|^2 [1 \pm f_0(\boldsymbol{p}')][1 \pm f_0(\boldsymbol{k}')] \quad (6\cdot17)$$

denotes the differential cross section for the scattering process $\boldsymbol{p}, \boldsymbol{k} \to \boldsymbol{p}', \boldsymbol{k}'$ and $v_{\rm rel}$ is the relative velocity. We also note that ensemble average of the collision term required in the diffusive Vlasov-Boltzmann equation (5·1) simply translates into the averaged distribution function $\bar{f}_1$ in the linearized collision term (6·16): $\langle I[f_1]\rangle = I[\bar{f}_1]$.

For the leading logarithmic limit of the collisional viscosity, it is sufficient to use the contributions to the scattering matrix element M_α with the highest infrared divergence. Using Mandelstam variables s, t, and u, the squared matrix element for one-gluon exchange processes (averaged over initial-state and summed over final-state quantum numbers) is

$$\left|\bar{M}_{(ab)}\right|^2 = \frac{4g^4}{N_c^2-1} \frac{s^2}{t^2} C_2^{(a)} C_2^{(b)} , \quad (6\cdot18)$$

where a and b denote the quantum numbers of the scattering partons, and $C_2^{(a,b)}$ are the quadratic Casimir operators for their color multiplet. The squared matrix element for the quark annihilation process is

$$\left|\bar{M}\right|^2 = \frac{4g^4}{N_c} \left(\frac{u}{t} + \frac{t}{u} \right) (C_2^{(\rm f)})^2 . \quad (6\cdot19)$$

Finally, the Compton scattering process doubles the contribution from quark annihilation.[6)]

6.4. *Anomalous viscosity*

We are now ready to calculate the coefficient of shear viscosity. We begin by ignoring the collision term and calculate only the contribution of the diffusive Vlasov term, i.e. the anomalous shear viscosity. Equating the first term in (6·10) with the right-hand side of Eq. (6·13) for Scenario A we obtain

$$\bar{\Delta}(p) = \frac{(N_c^2-1)E_p^2 T}{3C_2\, g^2 \langle \mathcal{B}^2 \rangle\, \tau_{\rm m}^{\rm mag}} . \quad (6\cdot20)$$

We note that the diffusive Vlasov equations for quarks and gluons decouple in the absence of collisions, causing the function $\bar{\Delta}(p)$ to take different values for quarks and gluons. Inserting Eq. (6·20) into the relation (5·11) between η and $\bar{\Delta}$ and assuming massless partons, the desired expression for the anomalous shear viscosity due to the action of the coherent color-magnetic fields on massless partons is found to be

$$\eta_{\rm A} = \frac{N_c^2-1}{15\pi^2 C_2 g^2 \langle \mathcal{B}^2 \rangle\, \tau_{\rm m}^{\rm mag}} \int_0^\infty dp\, p^5 f_0(p) , \quad (6\cdot21)$$

where a sum over parton species and helicities is implied. After performing the momentum space integration, one obtains the following anomalous viscosities for gluons and quarks:

$$\eta_{\rm A}^{(g)} = \frac{16\zeta(6)(N_c^2-1)^2}{\pi^2 N_c} \frac{T^6}{g^2\langle \mathcal{B}^2\rangle\, \tau_{\rm m}^{\rm mag}}\,, \tag{6·22a}$$

$$\eta_{\rm A}^{(q)} = \frac{62\zeta(6)N_c^2 N_f}{\pi^2} \frac{T^6}{g^2\langle \mathcal{B}^2\rangle\, \tau_{\rm m}^{\rm mag}}\,. \tag{6·22b}$$

These results differ by a numerical factor from the one obtained previously[5)] by approximating $\bar{\Delta}(\boldsymbol{p})$ as a constant. We emphasize again that, given an ensemble of color fields, gluons and each flavor of quarks generate their separate contribution to the shear viscosity.

The results obtained for Scenario B have a similar, but somewhat more complicated form, because the algebraic equation for $\bar{\Delta}(p)$ is replaced with a second-order differential equation. We will not discuss this case further here and come back to it in the context of the calculation of the complete shear viscosity.

However, we note that the field ensemble is sensitive to the total value of $\eta_{\rm A}$, because the field instabilities are driven by the overall anisotropy of the parton distribution. Possessing a smaller color charge than gluons, quarks develop a larger momentum space anisotropy in an expanding quark-gluon plasma, neglecting the effect of collisions. On the other hand, quarks contribute with a smaller weight than gluons to the color polarization tensor, which drives the instability of soft modes of the color field. Thus, although $\bar{\Delta}(\boldsymbol{p})$ and $\eta_{\rm A}$ are inversely proportional to C_2, the contribution of gluons and quarks to the color field instabilities, which is proportional to $C_2\eta_{\rm A}$, is independent of the magnitude of their color charge.

6.5. *Complete shear viscosity*

The full linearized diffusive Vlasov-Boltzmann equation (5·1) constitutes a linear integral equation for the scalar function $\bar{\Delta}(p)$ characterizing the deviation of the momentum distribution from equilibrium. An exact solution of this equation requires numerical methods. The standard approach[85),93)] makes use of the fact that the kernel of the linearized collision operator is self-adjoint with respect to an appropriately chosen scalar product and has non-negative eigenvalues. As a consequence, the solution of the Boltzmann equation coincides with the minimum of a quadratic functional $W[\bar{f}_1]$ and can thus be obtained from a variational principle. The variational principle can be cast into the form

$$W[\bar{f}_1] \equiv \int \frac{d^3p}{(2\pi)^3} \bar{f}_1(\boldsymbol{p}) \left[v^\mu \frac{\partial f_0(\boldsymbol{p})}{\partial x^\mu} + \frac{1}{2}\left(-\nabla_p \cdot D \cdot \nabla_p \delta \bar{f}(\boldsymbol{p}) + I[\bar{f}_1](\boldsymbol{p})\right)\right] = \text{min.} \tag{6·23}$$

The minimum of Eq. (6·23) defines the optimal solution $f_1(\boldsymbol{p})$ or $\bar{\Delta}(\boldsymbol{p})$ of the transport equation (5·1).

The optimal function $\bar{\Delta}(p)$ can be determined by means of the variational method after an expansion in a complete set of orthogonal functions. A good (up to a few percent) approximation can be obtained by choosing the one-parameter function $\bar{\Delta}(p) = A|\boldsymbol{p}|/T$, where A is a constant. Taking the appropriate moment of the linearized transport equation results in an algebraic equation for A, which can be solved to obtain an approximate analytic expression for the shear viscosity. Because the mean free paths of quarks and gluons are different, we need to introduce different parameters for quarks (A_q) and gluons (A_g).

We now evaluate the momentum integrals in Eq. (6·23) for massless quarks and gluons. For the drift term we obtain

$$\begin{aligned} W_{\rm D}[\bar{f}_1] &= \int \frac{d^3p}{(2\pi)^3} \bar{f}_1(\boldsymbol{p}) v^\mu \frac{\partial}{\partial x^\mu} f_0(p) \\ &= -\frac{1}{T^3} \int \frac{d^3p}{(2\pi)^3 E_p^2} \bar{\Delta}(p) \left[p_i p_j (\nabla u)_{ij}\right]^2 f_0(p)(1 \pm f_0(p)) \boldsymbol{v} \cdot \nabla_r f_0(p) \\ &= -\frac{2|\nabla u|^2}{9\pi^2 T^3} \int_0^\infty dp\, p^4 A f_0(p) \\ &= -\frac{32|\nabla u|^2}{3\pi^2} \zeta(5) T^2 \left[(N_c^2 - 1) A_g + \frac{15}{8} N_c N_f A_q \right]. \end{aligned} \tag{6·24}$$

The diffusive Vlasov term yields (in Scenario A)

$$\begin{aligned} W_{\rm V}^{(\rm A)}[\bar{f}_1] &= -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \bar{f}_1(\boldsymbol{p}) \nabla_p \cdot D \cdot \nabla_p \delta\bar{f}(\boldsymbol{p}) \\ &= \frac{3C_2 g^2 \langle \mathcal{B}^2 \rangle\, \tau_{\rm m}^{\rm mag}}{2T^4(N_c^2-1)} \int \frac{d^3p}{(2\pi)^3 E_p^4} \bar{\Delta}(p)^2 \left[p_i p_j (\nabla u)_{ij}\right]^2 f_0(p)(1 \pm f_0(p)) \\ &= \frac{4|\nabla u|^2}{15 T^5} \frac{C_2 g^2 \langle \mathcal{B}^2 \rangle\, \tau_{\rm m}^{\rm mag}}{N_c^2 - 1} \int_0^\infty dp\, p^3 A^2 f_0(p) \\ &= \frac{16|\nabla u|^2}{5\pi^2 T} \zeta(4) g^2 \langle \mathcal{B}^2 \rangle \tau_{\rm m}^{\rm mag} \left[N_c A_g^2 + \frac{7}{8} N_f A_q^2 \right]. \end{aligned} \tag{6·25}$$

For Scenario B, the diffusive Vlasov term receives contributions from color-electric as well as color-magnetic fields:

$$\begin{aligned} W_{\rm V}^{(\rm B)}[\bar{f}_1] &= -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \bar{f}_1(\boldsymbol{p}) \nabla_p \cdot D \cdot \nabla_p \delta\bar{f}(\boldsymbol{p}) \\ &= \frac{C_2 g^2}{T^4(N_c^2-1)} \int \frac{d^3p}{(2\pi)^3 E_p^2} \left(\frac{\langle \mathcal{E}^2 \rangle\, \tau_{\rm m}^{\rm el}}{p^2} + \frac{\langle \mathcal{B}^2 \rangle\, \tau_{\rm m}^{\rm mag}}{E_p^2} \right) \\ &\qquad \times \bar{\Delta}(p)^2 \left[p_i p_j (\nabla u)_{ij}\right]^2 f_0(p)(1 \pm f_0(p)) \\ &\quad - \frac{C_2 g^2 \langle \mathcal{E}^2 \rangle\, \tau_{\rm m}^{\rm el}}{6T^4(N_c^2-1)} \int \frac{d^3p}{(2\pi)^3 p^3 E_p} \bar{\Delta}(p) \left[p_i p_j (\nabla u)_{ij}\right]^2 \\ &\qquad \times \frac{\partial^2}{\partial p^2} \left[\frac{p^3 \bar{\Delta}(p)}{E_p} f_0(p)(1 \pm f_0(p)) \right] \\ &= \frac{8|\nabla u|^2 C_2 g^2}{45 T^5 (N_c^2-1)} (2\langle \mathcal{E}^2 \rangle\, \tau_{\rm m}^{\rm el} + \langle \mathcal{B}^2 \rangle\, \tau_{\rm m}^{\rm mag}) \int_0^\infty dp\, p^3 A^2 f_0(p) \\ &= \frac{32|\nabla u|^2}{15\pi^2 T} \zeta(4) g^2 (2\langle \mathcal{E}^2 \rangle \tau_{\rm m}^{\rm el} + \langle \mathcal{B}^2 \rangle \tau_{\rm m}^{\rm mag}) \left[N_c A_g^2 + \frac{7}{8} N_f A_q^2 \right]. \end{aligned} \tag{6·26}$$

This result differs from the one obtained for Scenario A only by the substitution

$$\langle\mathcal{B}^2\rangle\tau_{\rm m}^{\rm mag} \longrightarrow \frac{2}{3}(2\langle\mathcal{E}^2\rangle\tau_{\rm m}^{\rm el} + \langle\mathcal{B}^2\rangle\tau_{\rm m}^{\rm mag}) \,. \tag{6·27}$$

Finally, we simply state the result for the collision term in the leading logarithmic approximation and refer to Arnold et al.[6] for details of the calculation:

$$\begin{aligned} W_{\rm C}[\bar{f}_1] &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\bar{f}_1(\boldsymbol{p})I[\bar{f}_1](\boldsymbol{p}) \\ &= \frac{\pi|\nabla u|^2T^2}{45}(N_c^2-1)g^4\ln g^{-1}\left[\frac{1}{9}(2N_c+N_f)\left(N_cA_g^2+\frac{7}{8}N_fA_q^2\right)\right. \\ &\qquad \left. +\frac{\pi^2N_f}{256N_c}(N_c^2-1)(A_g-A_q)^2\right] \,. \end{aligned} \tag{6·28}$$

Note that our expression differs from that of Ref. 6) by an overall factor $4|\nabla u|^2/(45T)$ owing to the different definition of the expectation value (6·23).

In order to calculate the shear viscosity, we have to minimize Eq. (6·23) with respect to A_g and A_q, and then insert the obtained values into the expression

$$\eta = \frac{24\zeta(5)T^3}{3\pi^2}\left[(N_c^2-1)A_g + \frac{15}{8}N_cN_fA_q\right] \tag{6·29}$$

obtained by performing the momentum integral in the formula (5·11) for the shear viscosity. The mininization results in two linear equations for A_g and A_q. We state the expression for Scenario A:

$$\begin{aligned} \frac{32\zeta(5)}{3\pi^2}\begin{pmatrix} N_c^2-1 \\ \frac{15}{8}N_cN_f \end{pmatrix} &= \frac{32\zeta(4)}{5\pi^2}\frac{g^2\langle\mathcal{B}^2\rangle\,\tau_{\rm m}^{\rm mag}}{T^3}\begin{pmatrix} N_cA_g \\ \frac{7}{8}N_fA_q \end{pmatrix} \\ &\quad +\frac{\pi(N_c^2-1)}{45}g^4\ln g^{-1}\left[\frac{2}{9}(2N_c+N_f)\begin{pmatrix} N_cA_g \\ \frac{7}{8}N_fA_q \end{pmatrix}\right. \\ &\qquad \left. +\frac{\pi^2N_f(N_c^2-1)}{128N_c}\begin{pmatrix} A_g-A_q \\ A_q-A_g \end{pmatrix}\right] \,. \end{aligned} \tag{6·30}$$

In the absence of turbulent fields, this array of equations reduces to Eqs. (6.8) and (6.9) of Ref. 6). The result for Scenario B is obtained by means of the substitution (6·27).

It is useful to introduce a concise vector notation for these equations. With the help of the two-vector $A = (A_g, A_q)$, Eq. (6·30) can be written in the symbolic form

$$(a_{\rm A} + a_{\rm C})A = r \,, \tag{6·31}$$

with

$$r = \frac{32\zeta(5)}{3\pi^2}\begin{pmatrix} N_c^2-1 \\ \frac{15}{8}N_cN_f \end{pmatrix} \,, \tag{6·32a}$$

$$a_{\rm A} = \frac{32\zeta(4)}{5\pi^2}\frac{g^2\langle\mathcal{B}^2\rangle\,\tau_{\rm m}^{\rm mag}}{T^3}\begin{pmatrix} N_c & 0 \\ 0 & \frac{7}{8}N_f \end{pmatrix} \,, \tag{6·32b}$$

$$a_{\rm C} = \frac{\pi(N_c^2-1)}{45} g^4 \ln g^{-1} \left[\frac{2}{9}(2N_c+N_f) \begin{pmatrix} N_c & 0 \\ 0 & \frac{7}{8}N_f \end{pmatrix} + \frac{\pi^2 N_f(N_c^2-1)}{128 N_c} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right] . \tag{6·32c}$$

The equation (6·29) for the shear viscosity then takes the form of a scalar product between the vectors r and A:

$$\eta = \frac{3}{4} r \cdot A = \frac{3}{4} r \cdot (a_{\rm A} + a_{\rm C})^{-1} \cdot r \,, \tag{6·33}$$

where we use Eq. (6·31) to express A in terms of r and the matrices $a_{\rm A}$ and $a_{\rm C}$.

The inversely additive property of the contributions of turbulent color fields and parton collisions to the total shear viscosity is apparent in Eq. (6·33). Its origin lies in the additivity of the relaxation rates due to different processes and allows us to write the total shear viscosity as an inverse sum of anomalous and collisional viscosity:

$$\eta^{-1} = \eta_{\rm A}^{-1} + \eta_{\rm C}^{-1} \,, \tag{6·34}$$

with

$$\eta_{\rm A} = \frac{3}{4} r \cdot a_{\rm A}^{-1} \cdot r \,, \qquad \eta_{\rm C} = \frac{3}{4} r \cdot a_{\rm C}^{-1} \cdot r \,. \tag{6·35}$$

Equation (6·34) implies that, whichever contribution to η is smaller, dominates the overall shear viscosity. The anomalous viscosity dominates when $\eta_{\rm A} < \eta_{\rm C}$. Because $\eta_{\rm A}$ grows with a smaller power of g than $\eta_{\rm C}$, the anomalous viscosity dominates for sufficiently weak coupling, and we thus have $\eta \approx \eta_{\rm A}$ for $g \to 0$.

6.6. *Estimate of the anomalous viscosity*

Equation (6·21) shows that $\eta_{\rm A}$ decreases with increasing strength of the turbulent fields. Since the amplitude of these fields grows with the magnitude of the momentum anisotropy, a large anisotropy will result in a small value of $\eta_{\rm A}$. The anomalous mechanism thus exhibits a stable equilibrium in which the shear viscosity regulates itself: The momentum anisotropy grows with $\eta_{\rm A}$, but an increased anisotropy tends to reduce the anomalous viscosity. This leads to a self-consistency condition which determines $\eta_{\rm A}$.

In order to proceed further, we need to explore the dependence of the turbulent fields on the anisotropy of the momentum distribution of the partons. In particular, we must know how large $\langle \mathcal{B}^2 \rangle$ and $\tau_{\rm m}$ are. The coherent color magnetic fields are only generated by the plasma instability when the momentum distribution of partons in the quark-gluon plasma is deformed due to the collective expansion. Analytical studies have shown that the instability always occurs when the momentum distribution is anisotropic. We also know from these studies how the growth rate of the instability depends on the anisotropy of the momentum distribution, but there are no published systematic studies that show how the "saturation" level of the coherent field energy depends on the anisotropy. We will therefore rely on some heuristic arguments for the needed dependences.

The study by Romatschke and Strickland[9)] uses the following parametrization of the anisotropic momentum distribution:

$$f(\boldsymbol{p}) = f_0\left(\sqrt{p^2 + \xi(\boldsymbol{p}\cdot\hat{n})^2}\right) \approx f_0(p) - \frac{\xi(\boldsymbol{p}\cdot\hat{n})^2}{2E_pT} f_0(1 \pm f_0)\,. \tag{6·36}$$

Choosing $\hat{n} = \hat{e}_z$ and subtracting the trace, this corresponds to a perturbation of the equilibrium distribution:

$$f_1(\boldsymbol{p}) = -\frac{\xi}{2E_pT}\left(p_z^2 - \frac{p^2}{3}\right)\,. \tag{6·37}$$

Comparing with Eq. (5·12) this establishes the connection

$$\xi = 2\bar{\Delta}\,\frac{|\nabla u|}{T}. \tag{6·38}$$

The relative anisotropy of the stress tensor is given by

$$2\frac{\delta T_{xx}}{T_{xx}^{(0)}} = 2\frac{\delta T_{yy}}{T_{yy}^{(0)}} = -\frac{\delta T_{zz}}{T_{zz}^{(0)}} = \frac{8}{15}\xi. \tag{6·39}$$

Comparing with Eq. (5·14) we obtain a relation between η and ξ, which takes the form (for a massless parton gas):

$$\xi = \frac{15\eta|\nabla u|}{2T_{00}} = 10\frac{\eta}{s}\,\frac{|\nabla u|}{T}\,. \tag{6·40}$$

The central point of our argument is that the average collective color field energy is a function of the momentum anisotropy. For Scenario A (turbulent color-magnetic fields only) this means: $\langle \mathcal{B}^2\rangle = b(\xi)$. We do not know this function in detail, but we know that $b(0) = 0$, because no instability exists in the absence of a momentum anisotropy. The simplest *ansatz* is a power law, which we will write in the form

$$g^2\langle \mathcal{B}^2\rangle = b_0 g^4 T^4 \xi^n \tag{6·41}$$

with an as yet unknown power n. Following the discussion before Eq. (3·7) we set $n = 2$.[*)] The memory time $\tau_{\rm m}$ can be determined by one of two mechanisms. If the plasma particles move faster than the coherent fields propagate, $\tau_{\rm m}$ will be set by the spatial coherence length of the coherent fields. This coherence length is given by the wavelength of the maximally unstable mode, which is of the order of the Debye length:[**)]

$$\tau_{\rm m} = d_0\xi^{-1/2}(gT)^{-1}. \tag{6·42}$$

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*) This differs from the assumption made in Ref. 5), where we assumed $n = 1$.

**) In the opposite situation, when the color fields evolve rapidly compared with the motion of the plasma particles, the memory time is determined by the decoherence time of the fields interacting with the particles and among themselves via the nonlinearities of the Yang-Mills equations. The case, in which the back reaction of the particle motion on the unstable field modes determines the decoherence time has been treated by Dupree[73)] and by Abe and Niu.[77),78)]

The decoherence time and its dependence on ξ can, in principle, be determined from simulations of the classical Yang-Mills equations. In the absence of such a determination, we shall assume that $\tau_{\rm m}$ is given by Eq. (6·42).

We are now ready to derive the self-consistency condition for the anomalous viscosity in the limit when the collisional viscosity can be neglected. We are interested in the late time, steady state situation where the expansion drives the momentum distribution into a slightly oblate anisotropy along the z-axis. As discussed before, this leads to the growth of unstable mean field modes, which determines the viscosity $\eta_{\rm A}$ which, in turn, controls the size of the momentum space anisotropy. The feedback loop is a stable one, because a large anisotropy leads to large saturation levels of the fields, which reduce the viscosity and thus limit the size of the momentum anisotropy.

One technical complication of the limit of vanishing collisional viscosity is that the momentum distributions of quarks and gluons then attain different anisotropies, because color charges differ and thus their interaction with the turbulent fields is different. This implies that the presently unknown constants b_0 and d_0 receive different contributions from quarks and gluons, and that we should, in principle, distinguish between ξ_g and ξ_q. In order to avoid unnecessary distractions from our argument, and because the numerical constants are not known anyway, we will only consider the gluon contribution to the anomalous viscosity in the following. The generalization to include quarks is straightforward and does not change the functional dependence of the result on g, T, and $|\nabla u|$.

Combining Eqs. (6·41) and (6·42) into a single scaling relation and replacing ξ with $\eta = \eta_{\rm A}$, we obtain

$$g^2 \langle \mathcal{B}^2 \rangle \, \tau_{\rm m} = b_0 d_0 (gT)^3 \xi^{3/2} = b_0 d_0 (gT)^3 \left(\frac{10\eta_{\rm A} |\nabla u|}{sT} \right)^{3/2} . \tag{6·43}$$

We now insert this result into the expressions (6·22a) for the anomalous viscosity and obtain the self-consistency relation:

$$\frac{\eta_{\rm A}}{s} = c_0 \left(\frac{Ts}{g^2 |\nabla u| \eta_{\rm A}} \right)^{3/2} , \tag{6·44}$$

where we have combined all numerical constants into a single factor c_0. We note once again that this relation neglects contributions from the collisional shear viscosity $\eta_{\rm C}$ and does not distinguish between the contributions from quarks and gluons. We also note that the power on the right-hand side differs slightly from that in Ref. 5), because of our different scaling assumptions in Eqs. (6·41) and (6·42). Resolving Eq. (6·44) for $\eta_{\rm A}/s$ finally yields the desired expression for the self-consistent anomalous shear viscosity:

$$\frac{\eta_{\rm A}}{s} = \bar{c}_0 \left(\frac{T}{g^2 |\nabla u|} \right)^{3/5} . \tag{6·45}$$

Several things are noteworthy about this result. First, if the memory time $\tau_{\rm m}$ is longer than our estimate (6·42), the value of $\eta_{\rm A}$ decreases. Second, the dependence

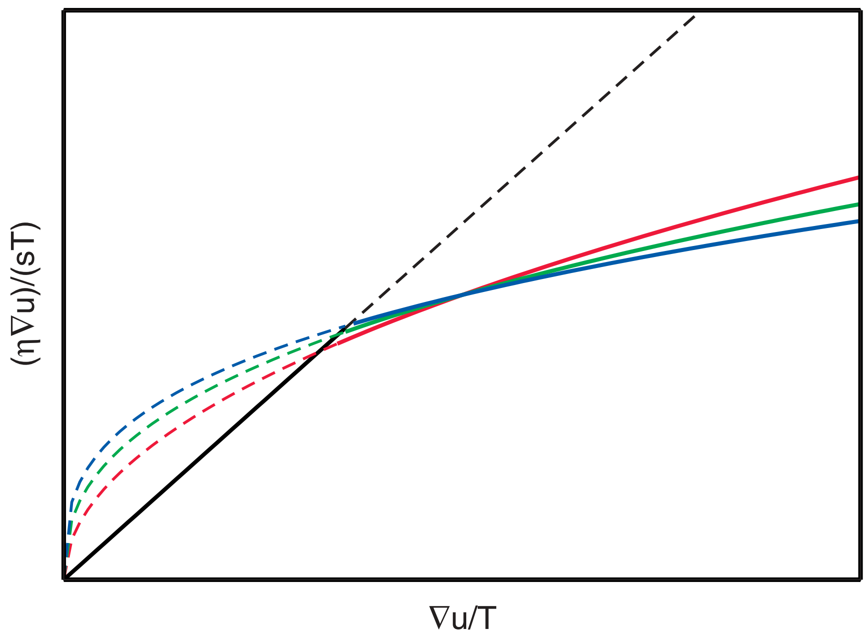

Fig. 1. Schematic representation of the dependence of the collisional and anomalous viscous stress δT_{ik} on the velocity gradient. The collisional viscous stress is shown by the linearly rising black line; the anomalous viscous stress is shown by the (colored) curved lines for three scaling exponents of the turbulent color field energy ($n = 1.5, 2, 2.5$). Solid lines indicate the dominant source of viscous stress in different regions of the scaled velocity gradient $|\nabla u|/T$.

on the gauge coupling ($\sim g^{-6/5}$) is parametrically much weaker than that of the collisional viscosity ($\sim (g^4 \ln(1/g))^{-1}$). Thus, for early times τ and weak coupling $g \ll 1$, the anomalous viscosity will be much smaller than the collisional viscosity and thus dominate the total viscosity according to Eq. (6·34).

It is instructive to consider the role of the anomalous viscosity in the hydrodynamical equations. As Eq. (5·14) shows, the viscous contribution to the stress tensor is proportional to $\eta|\nabla u|$. For the collisional shear viscosity which is, in first approximation, independent of the magnitude of the velocity gradient; this implies that δT_{ik} grows linearly with $|\nabla u|$. The anomalous shear viscosity (6·45), on the other hand, is a decreasing function of the velocity gradient; its contribution to the stress tensor grows like $|\nabla u|^{2/5}$ for our scaling assumptions. The unusual dependence of $\eta_{\rm A}$ on $|\nabla u|$ certainly justifies the term "anomalous viscosity".

The different dependence of the collisional and the anomalous viscous stress tensor on the velocity gradient is shown schematically in Fig. 1. For very small gradients the linear dependence of the collisional viscous stress tensor dominates, but for larger velocity gradients the lower power associated with the anomalous shear viscosity will assert its dominance. The precise location of the crossover between the two domains depends on the value of the numerical constant $\bar{c}_0$, but we can deduce from Eq. (6·45) that the crossover point shifts to lower values of $|\nabla u|$ with decreasing coupling constant g. We also show in the figure the effect of choosing a different power n in the scaling law (6·41) for the energy density of the turbulent color fields. Finally, we note that the decreasing dependence of η_A on $|\nabla u|$ implies that the shear viscosity can, in principle, fall below the Kovtun-Son-Starinets (KSS) bound $\eta/s = (4\pi)^{-1}$.

§7. Conclusions and outlook

In this paper we have presented details of our derivation of the anomalous viscosity in an anisotropically expanding quark-gluon plasma, which arises from interactions of thermal partons with dynamically generated color fields. In the weak coupling limit or for large velocity gradients, the anomalous viscosity is much smaller than the viscosity due to collisions among thermal partons. By reducing the shear viscosity of a weakly coupled, but expanding quark-gluon plasma, this mechanism could possibly explain the observations of the RHIC experiments without the assumption of a strongly coupled plasma state.

Due to the self-consistency condition, the anomalous shear viscosity itself is inversely dependent on the expansion rate of the plasma. This means that the viscous term in the hydrodynamic equation does not depend on the velocity gradient linearly, but sublinearly. This unusual dependence on the velocity gradient justifies the term "anomalous" viscosity. It also implies that the response of the medium to the expansion is nonlinear, in fact less than linear. The usual method to relate the transport coefficient to a correlation function (Kubo formula) that can be measured in full equilibrium is therefore not applicable. Even if the appropriate real-time correlation function could be determined from Euclidean lattice QCD simulations, it would fail to describe the anomalous viscosity. More sophisticated methods, including a treatment of the color instabilities in the expanding quark-gluon plasma, will be necessary.

The presence of anomalous viscosity for a rapidly expanding *anisotropic* quark-gluon plasma has important consequences for the early Universe. The expansion of the quark-gluon plasma in the early Universe is relatively slow and thermalization is therefore always maintained. Thus, the quark-gluon plasma in the early Universe does not generally possess an anomalous shear viscosity. However, when off-equilibrium processes induce a strong local anisotropy, the anomalous viscosity can become important in determining the total viscosity. This could have been the case during the reheating period after inflation and the electroweak phase transition, and may have affected the production of fluctuations in the Universe, baryogenesis, etc.

The approach described here can be applied to other transport properties of an expanding, turbulent quark-gluon plasma. Examples are the coefficient $\hat{q}$ of radiative energy loss of an energetic parton,[95)] the absorption coefficient for bound states of heavy quarks (e.g. J/ψ), and flavor equilibration rates. For the rate of strange quark pair production, we note that a quark pair can be produced in the presence of a mean color field by a single gluon. This process is well known in quantum electrodynamics, where a single photon can produce an electron-positron pair in the presence of a strong electromagnetic field, such as the Coulomb field of a nucleus. In QED the pair production rate grows like Z^2, where Ze is the nuclear charge; in the case of the quark-gluon plasma one would expect the rate to grow like $g^2\langle\mathcal{B}^2\rangle$ and hence be proportional to the expansion rate. A rapid expansion would, therefore, accelerate the rate of light quark pair production. Such an effect may, indeed, contribute to the very high rate of quark pair production seen in numerical solutions of the Dirac equation in longitudinally expanding gluon fields in the color glass condensate picture.[96]

Turbulent color fields can also contribute to the dissociation of heavy quark-antiquark bound states, because the two constituents will be deflected in opposite directions. This effect has been considered for randomly oriented color-electric fields,[97] but not for color-magnetic fields, which should have a similar effect, albeit suppressed by a factor $(v/c)^2$, where v is the velocity of the charmonium. Turbulent color fields will also contribute to the diffusion coefficient for heavy quarks in the plasma. Finally, turbulent color fields can influence the trajectories of the partons contained in a jet created by a hard scattered quark or gluon. Because the fields created by the color instabilities in an expanding medium are polarized transversely to the expansion direction, the color magnetic fields will preferentially deflect the outgoing partons in the longitudinal direction and thus cause a longitudinal broadening of the jet cone. Such an effect has been been observed at RHIC.[98]

In a forthcoming publication we shall further investigate the phenomenological consequences of the anomalous viscosity: in particular we shall focus on jet energy-loss in turbulent color fields and address the question whether the vast array of current data at RHIC can be understood in the framework of a weakly coupled QGP with anomalous transport properties.

Acknowledgements

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Appendix A
—— *Alternative Derivation of the Vlasov Diffusion Term* ——

In this section, we present a different derivation of the Vlasov diffusion term (4·14), based on the evolution of the extended phase-space distribution $\tilde{f}(r,p,Q,t)$, which is given by[81]

$$\left[v^\mu\left(\frac{\partial}{\partial x^\mu}+gf_{abc}Q^aA^b_\mu\frac{\partial}{\partial Q^c}\right)+gQ^a\left(\mathcal{E}^a+\boldsymbol{v}\times\mathcal{B}^a\right)\cdot\nabla_p\right]\tilde{f}(r,p,Q,t)=0. \quad \text{(A·1)}$$

Here we have suppressed any unnecessary vector notation and omitted the collision term. Our alternative derivation of the diffusion term starts from the integral representation for $\tilde{f}$:

$$\tilde{f}(r,p,Q,t) = \int dr_0\, dp_0\, dQ_0\, \delta(r-\bar{r}(t))\, \delta(p-\bar{p}(t))\, \delta(Q-\bar{Q}(t))\, \tilde{f}(r_0,p_0,Q_0,0), \quad (\text{A}\cdot 2)$$

with

$$\frac{d\bar{r}}{dt} = \bar{v} = \frac{\bar{p}}{E_p}\,, \quad (\text{A}\cdot 3\text{a})$$

$$\frac{d\bar{p}}{dt} = gQ^a \left(\mathcal{E}^a(\bar{r}) + \bar{v}\times \mathcal{B}^a(\bar{r})\right)\,, \quad (\text{A}\cdot 3\text{b})$$

$$\frac{d\bar{Q}}{dt} = gf_{abc}Q^b A^c_\mu v^\mu\,. \quad (\text{A}\cdot 3\text{c})$$

and $r_0 = \bar{r}(0)$, $\bar{p}(0) = p_0$, $\bar{Q}(0) = Q_0$. Here $\bar{r}(t)$ denotes the trajectory of a plasma particle, which is found at position r_0 with momentum p_0 and color charge Q_0 at time $t = 0$. The evolving phase space distribution is given the sum over all the trajectories of these (test) particles. When we act on Eq. (A·2) with the Vlasov drift operator $(\partial_t + v\cdot\nabla_r)$ and use the definition (A·3) of $\bar{v}$, we obtain

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_r + gf_{abc}Q^a A^b_\mu v^\mu \frac{\partial}{\partial Q^c}\right)\tilde{f}(r,p,Q,t)$$
$$= -\int dr_0\, dp_0\, \delta(r-\bar{r})\, \frac{d\bar{\boldsymbol{p}}}{dt}\cdot\nabla_p\, \delta(p-\bar{p})\, \tilde{f}(r_0,p_0,Q_0,0), \quad (\text{A}\cdot 4)$$

where we dropped the explicit notation of the time-dependence of $\bar{r}$, $\bar{p}$, and $\bar{Q}$. Since $\bar{p}$ is time independent for vanishing color field, it makes sense to expand the delta function $\delta(p-\bar{p})$ around $(p-p_0)$ for weak fields (or short times). It is important here to remember that (strong) plasma turbulence does not imply strong fields, just a spectral field distribution without phase correlations on all inverse length scales k. The expansion is

$$\delta(p-\bar{p}) = \delta(p-p_0) - \Delta\boldsymbol{p}(t)\cdot\nabla_p\delta(p-p_0) + \cdots\,, \quad (\text{A}\cdot 5)$$

where $\Delta p(t) = \bar{p}(t) - p_0$, implying $d(\Delta p)/dt = d\bar{p}/dt$. The right-hand side of (A·4) then becomes

$$-\int dr_0\, dp_0\, \delta(r-\bar{r})\, \frac{d\Delta\boldsymbol{p}}{dt}\cdot\nabla_p\left[1-\Delta\boldsymbol{p}(t)\cdot\nabla_p\right]\delta(p-p_0)\tilde{f}(r_0,p_0,Q_0,0) + \cdots\,. \quad (\text{A}\cdot 6)$$

The expression now contains a term quadratic in Δp, and it makes sense to take an ensemble average over the color fields. We also integrate by parts with respect to ∇_p and obtain

$$\left\langle\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_r + gf_{abc}Q^a A^b_\mu v^\mu \frac{\partial}{\partial Q^c}\right)\tilde{f}\right\rangle = \left\langle\nabla_p\cdot\Delta\boldsymbol{p}(t)\frac{d\Delta\boldsymbol{p}}{dt}\cdot\nabla_p\,\tilde{f}\right\rangle$$
$$\equiv \nabla_p\cdot D\cdot\nabla_p\langle\tilde{f}\rangle. \quad (\text{A}\cdot 7)$$

In the last step we have introduced the diffusion coefficient D and assumed that we can factorize the ensemble average. Explicitly, the diffusion coefficient is given by $(i, j = x, y, z)$:

$$D_{ij} \equiv \left\langle \Delta p_i(t) \frac{d\Delta p_j}{dt} \right\rangle = g^2 \int_0^t dt' \left\langle Q^a(t') F_i^a(\bar{r}(t'), t') Q^b(t) F_j^b(\bar{r}(t), t)) \right\rangle , \quad \text{(A·8)}$$

where

$$F_i^a(\bar{r}(t), t) = \mathcal{E}_i^a(\bar{r}(t), t) + (\bar{v}(t) \times \mathcal{B}^a(\bar{r}(t), t))_i \, . \quad \text{(A·9)}$$

In the local rest frame of the medium the electric and magnetic components of the color field can be expressed as (using the convention $\epsilon^{0123} = 1$)

$$\mathcal{E}_i^a = F_{i\nu}^a u^\nu \, ; \qquad \mathcal{B}_i^a = \epsilon_{i\lambda\mu\nu} F^{a\lambda\mu} u^\nu \, . \quad \text{(A·10)}$$

If we now argue that the correlation time/length for the color fields is short in comparison with the temporal change of the velocity of a plasma particle, we can take $\bar{v}(t) = \bar{v}(t') = v$ out of the average and are left with the autocorrelation function of the color fields and color charges along a typical particle trajectory.

The time evolution of the color charge $Q^a(t)$ is given by the solution of the last equation (A·3):

$$Q^a(t') = P \, \exp\left(\int_{\bar{r}(t)}^{\bar{r}(t')} f_{abc} A_\mu^b \, dx^\mu \right) Q^c(t) = U_{ac}(\bar{r}(t'), \bar{r}(t)) Q^c(t) \, . \quad \text{(A·11)}$$

Inserting this solution into (A·8), we obtain

$$\begin{aligned} D_{ij} &\equiv \left\langle \Delta p_i(t) \frac{d\Delta p_j}{dt} \right\rangle \\ &= g^2 \int_0^t dt' \left\langle Q^c(t) Q^b(t) F_i^a(\bar{r}(t'), t') U_{ac}(\bar{r}(t'), \bar{r}(t)) F_j^b(\bar{r}(t), t)) \right\rangle . \end{aligned} \quad \text{(A·12)}$$

If we now assume that the distribution of the color charges of partons at a given time t is random and independent of the color fields, $\langle Q^a Q^b \rangle = (N_c^2 - 1)^{-1} C_2 \delta_{ab}$, we recover the expression (4·14) for the Vlasov diffusion term with the field correlators (4·11).

Finally, in order to make contact with the diffusive transport equation (5·1), we need to integrate Eq. (A·7) over the color charges Q to obtain an equation for

$$\bar{f}(r, p, t) = \int dQ \langle \tilde{f}(r, p, Q, t) \rangle \, . \quad \text{(A·13)}$$

Partial integration with respect to Q shows that the third term on the left-hand side of Eq. (A·7) vanishes in view of the relation $\partial Q^a / \partial Q^c = \delta_{ac}$ and the antisymmetry of f_{abc}.

We end this section with a remark concerning abelian plasmas. Our derivation of the diffusive Vlasov term was motivated by the insight that the color instabilities of an anisotropic nonabelian plasma saturate under the action of the nonlinearities of the Yang-Mills equation.[99),100)] This mechanism is, of course, absent in abelian plasmas. The saturation of the instablilities is then caused by the back reaction of the growing soft field modes on the particle distribution.[77)] In order to address this situation, we follow Dupree's argument[73)] that the ensemble average in Eq. (A·8) should be taken over Fourier components of the color field, because these are the slowly varying variables. This is standard practice when dealing with an ensemble of waves. It certainly makes sense for electromagnetic plasmas, where nonrelativistic particles move in fields that propagate at the speed of light. In the case of the quark-gluon plasma, however, the situation is reversed: the thermal partons move with the speed of light, but the soft color fields obey a dispersion relation with a slower propagation speed. Thus it is doubtful whether a similar reasoning would make sense. Nevertheless, for the sake of interest, we outline Dupree's approach (for color-magnetic fields only). One writes

$$\mathcal{B}^a(\bar{r}(t), t) = \sum_k \mathcal{B}^a(k)\, e^{-i\omega_k t + ik\cdot\bar{r}(t)}, \tag{A·14}$$

where $\mathcal{B}(k)$ are the Fourier components of the field. One can then pull the slow variables $\mathcal{B}(k)$ out of the time integral in Eq. (A·8). After factorizing the ensemble average one obtains

$$\nabla_p \cdot D \cdot \nabla_p = -g^2 Q^a Q^b \sum_k \langle \mathcal{B}_i^a(k)\mathcal{B}_j^b(-k)\rangle L_i^{(p)} L_j^{(p)} \left\langle \int_0^t dt' e^{-\omega_k(t-t') + ik\cdot(\bar{r}(t') - \bar{r}(t))} \right\rangle . \tag{A·15}$$

The time integral can be interpreted as an autocorrelation or memory time for the action of the magnetic fields on the particles. In Dupree's approach, the value of the time integral is governed by the effect of the turbulent magnetic fields on the particle trajectory, and can be shown to satisfy a self-consistency condition. We will not pursue this approach further here and refer the interested reader to Abe and Niu's work.[78)]

Appendix B
—— *Color Instabilities near Equilibrium* ——

For convenience, we here state the results for the growth rate of unstable quark-gluon plasma modes in a background distribution of quasi-thermal partons whose momentum distribution is only slightly perturbed away from equilibrium. We follow the notation and derivation of Romatschke and Strickland.[9)]

We assume that the momentum distribution in the rest frame of the medium can be written as

$$f(\boldsymbol{p}) = f_0\left(\sqrt{\boldsymbol{p}^2 + \xi(\boldsymbol{p}\cdot\boldsymbol{n})^2}\right), \tag{B·1}$$

where $\boldsymbol{n}$ is a unit vector defining the orientation of the anisotropy and $\xi \ll 1$. We denote the angle between the wave vector $\boldsymbol{k}$ of the considered field mode and $\boldsymbol{n}$ by θ: $\cos\theta = \hat{\boldsymbol{k}}\cdot\boldsymbol{n}$, where $\hat{\boldsymbol{k}}$ is the unit vector in the direction of $\boldsymbol{k}$. For a given temperature T, the polarization function of the gluon field is then a function of the variables ω,

$k = |\boldsymbol{k}|$, and θ. Because the parton distribution violates spherical symmetry, there are four different components of the polarization tensor, which can be expressed in terms of the functions α, β, γ, and δ. The gluon propagator in medium can be decomposed with the help of the projectors $i, j = 1, 2$, and 3,

$$A_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j \,, \tag{B·2a}$$
$$B_{ij} = \hat{k}_i \hat{k}_j \,, \tag{B·2b}$$
$$C_{ij} = \tilde{n}_i \tilde{n}_j \,, \tag{B·2c}$$
$$D_{ij} = \hat{k}_i \tilde{n}_j + \tilde{n}_i \hat{k}_j \,, \tag{B·2d}$$

where $\tilde{n} = \boldsymbol{n} - \hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{n})$. The resulting expression for the gluon propagator is

$$\Delta_{ij}(k, \omega, \theta) = \Delta_{\mathrm{A}}(A_{ij} - C_{ij}) + \Delta_{\mathrm{G}}[(k^2 - \omega^2 + \alpha + \gamma)B_{ij} + (\beta - \omega^2)C_{ij} - \delta D_{ij}] \tag{B·3}$$

with

$$\Delta_{\mathrm{A}}^{-1}(k, \omega, \theta) = k^2 - \omega^2 + \alpha, \tag{B·4a}$$
$$\Delta_{\mathrm{G}}^{-1}(k, \omega, \theta) = (k^2 - \omega^2 + \alpha + \gamma)(\beta - \omega^2) - \tilde{\boldsymbol{n}}^2 \delta^2. \tag{B·4b}$$

For small values of ξ the functions α, β, γ, and δ are given by

$$\alpha = \Pi_{\mathrm{T}}(z) + \xi \left[\left(\frac{m_D^2}{3} - \Pi_{\mathrm{T}}(z) \right) \frac{z^2}{2} (5\cos^2\theta - 1) - \frac{m_D^2}{3} \cos^2\theta \right. \\ \left. + \frac{1}{2} \Pi_{\mathrm{T}}(z)(3\cos^2\theta - 1) \right], \tag{B·5a}$$

$$\beta = z^2 \Pi_{\mathrm{L}}(z) + \xi z^2 \left[\left(\frac{m_D^2}{3} - z^2 \Pi_{\mathrm{L}}(z) \right) (3\cos^2\theta - 1) \right. \\ \left. + \Pi_{\mathrm{L}}(z)(2\cos^2\theta - 1) \right], \tag{B·5b}$$

$$\gamma = \xi \left(\Pi_{\mathrm{T}}(z) - \frac{m_D^2}{3} \right) (z^2 - 1) \sin^2\theta, \tag{B·5c}$$

$$\delta = \xi \left(4z^2 \frac{m_D^2}{3} - \Pi_{\mathrm{T}}(z)(1 - 4z^2) \right) \cos\theta, \tag{B·5d}$$

where $z = \omega/k$ and

$$\Pi_{\mathrm{T}}(z) = \frac{m_D^2}{2} z^2 \left[1 - \left(\frac{z}{2} - \frac{1}{2z} \right) \ln \frac{z+1}{z-1} \right], \tag{B·6a}$$

$$\Pi_{\mathrm{L}}(z) = m_D^2 \left[\frac{z}{2} \ln \frac{z+1}{z-1} - 1 \right] \tag{B·6b}$$

are the usual expressions for the transverse and longitudinal gluon polarization functions in thermal equilibrium.

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