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Anomaly-Free Sets of Fermions

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Abstract

We present new techniques for finding anomaly-free sets of fermions. Although the anomaly cancellation conditions typically include cubic equations with integer variables that cannot be solved in general, we prove by construction that any chiral set of fermions can be embedded in a larger set of fermions which is chiral and anomaly-free. Applying these techniques to extensions of the Standard Model, we find anomaly-free models that have arbitrary quark and lepton charges under an additional $U(1)$ gauge group.

1 Introduction

Gauge symmetries successfully describe the electromagnetic, weak, and strong interactions of particle physics. Nevertheless, unitarity and renormalizability of the Standard Model do not necessarily follow from a classical invariance of the Lagrangian under $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge transformations: one-loop $SU(N)$ and $U(1)$ gauge anomalies must also be absent [1, 2, 3, 4].

In order to avoid these local gauge anomalies, the sum over triangle diagrams with gauge bosons as external lines and charged fermions running in the loops must vanish. As a consequence, the sums over loops with more external lines and over higher-order diagrams automatically vanish [5, 2]. Similarly, the sum over triangle diagrams involving two gravitons and one $U(1)$ gauge boson in the external lines must vanish, or else this mixed gravitational-$U(1)$ anomaly will also lead to an explicit breaking of the gauge symmetry by gravitational interactions (for a review see [6]). Finally, the $SU(2)$ gauge symmetry may suffer from a global gauge anomaly unless the number of Weyl fermion doublets is even [7].

A remarkable property of all elementary fermions discovered so far is that they form a chiral set—none of them can have a gauge invariant mass term. The cancellation of anomalies within a chiral set of fermions is highly nontrivial. Given the observed $SU(3)_C \times SU(2)_W$ representations found in the Standard Model, the anomaly cancellation conditions are restrictive enough to uniquely determine the $U(1)_Y$ charges, assuming that not all of them are zero. More strikingly, the *minimal* anomaly-free chiral set of fermions charged under $SU(3)_C \times SU(2)_W \times U(1)_Y$ is exactly given by a Standard Model generation [8]. Therefore, anomaly cancellation provides an explanation for the fermion structure of the Standard Model which is an alternative to the explanation provided by grand unified theories, where an entire Standard Model generation can be embedded in a single, anomaly-free SO(10) representation.

Anomaly cancellation will constrain the charges of the Standard Model fermions under any newly discovered gauge groups, whether these groups follow from grand unification or not. Many models of physics beyond the Standard Model incorporate new gauge groups, and the couplings of the Standard Model fermions to the gauge bosons associated with these groups are completely determined by the spectrum of charges. It is therefore useful to have methods that allow finding, in general, sets of fermions which are anomaly-free under arbitrary gauge groups. Since vector-like pairs of fermions do not contribute to these anomalies, a complete description of anomaly-free sets hinges only on the identification of all *chiral* anomaly-free sets.

Furthermore, if the chiral set of Standard Model fermions is charged under a new gauge group, anomaly cancellation usually dictates the presence of additional fermions. The complete set of fermions under the full gauge group would most likely be chiral. Imagine instead that the full theory were completely vector-like—after the breaking of the extended gauge symmetry to $SU(3)_C \times SU(2)_W \times U(1)_Y$, one would be left with both the observed Standard Model fields and a set of conjugate partners. To avoid mixing with the Standard Model fields, these conjugate partners should have large masses that proceed through electroweak symmetry breaking, which would induce too large corrections to electroweak observables.

In the case of $SU(N)$ gauge groups, comprehensive lists of anomaly-free sets have been identified using numerical methods [9]. By contrast, anomaly-free sets of chiral fermions charged under a $U(1)$ gauge group, or under direct products of gauge groups including at least one $U(1)$ group, have been less thoroughly catalogued, despite the common appearance of extra $U(1)$ groups in connection to flavor symmetry (see, *e.g.,* [10, 11]), supersymmetry breaking (see, *e.g.,* [12]), neutrino masses [13, 14, 15, 16, 17], and many other model building issues (see, *e.g.,* [18]). Many of these constructions depend on the existence of a chiral set of fermions.

If the quarks and leptons have arbitrary charges under a new $U(1)$, then there are a number of gauge anomalies that need to be cancelled. Usually, this is achieved by the inclusion of additional fermions with carefully chosen charges. An alternative is available in the context of string theory, when the $U(1)$ symmetry is spontaneously broken at a scale close to the string scale: the four-dimensional gauge anomalies associated with the $U(1)$ can be cancelled by the Green-Schwartz mechanism [19]. The question of when can anomalies be cancelled by additional fermions has not been given a general answer. It is often stated, though, that there are cases where the anomalies can be cancelled *only* by the Green-Schwartz mechanism [20, 10] (in these cases it is usually said that there is an "anomalous $U(1)$", albeit this is a misleading phrase).

In this paper we prove that any set of fermions with arbitrary charges under a gauge symmetry involving any number of non-Abelian and $U(1)$ groups can be embedded in an anomaly-free chiral set that contains additional fermions—even when any ratio of charges is a rational number. This conclusion is far from obvious: it involves cubic Diophantine equations (*i.e.*, cubic equations with integer solutions), which include for the case of three fermions the (in)famous Fermat's Last Theorem.

We focus on rational charges since such charges seem more natural, but more importantly, they solve a real problem: any $U(1)$ gauge theory eventually hits a Landau pole unless it is embedded in a non-Abelian group, which is possible only if these fermions have commensurate charges (*i.e.*, rational up to a normalization of the gauge coupling) [21]. Even in string theory the gauge charges appear to be commensurate, although we are not aware of a general proof of this statement.

In particular, we find that any "anomalous $U(1)$" that can be made non-anomalous by the Green-Schwartz mechanism can also be made non-anomalous by new fermions. More importantly, our theorem shows that it is possible to add new fermions such that the anomalies cancel for any charges of the Standard Model fermions under a new $U(1)$. This is relevant for the experimental searches for Z' bosons [22, 23], because the Z' couplings to quarks and leptons are fixed, up to an overall normalization, by the $U(1)$-charges.

In Section 2 we discuss $U(1)$ gauge anomalies, and derive our main results. These results are then generalized to any gauge group in Sections 3 and 4. In Section 5 we apply our results to the phenomenologically-interesting case of a $U(1)$ extension of the Standard Model gauge group. Our conclusions are presented in Section 6.

2 $U(1)$ gauge anomalies

Consider a set of n left-handed Weyl fermions with charges z_i, $i = 1, \ldots, n$ under a $U(1)$ gauge theory. The $[U(1)]^3$ and mixed $U(1)$-gravitational anomaly cancellation conditions are given by

$$\sum_{i=1}^{n} z_i^3 = 0 \ , \qquad \sum_{i=1}^{n} z_i = 0 \ . \tag{2.1}$$

We are interested in chiral sets, so the charges of the fermions must satisfy

$$z_i + z_j \neq 0 \tag{2.2}$$

for any i, j, and in particular $z_i \neq 0$. For $n \leq 4$, the first Eq. (2.1) can be easily solved once the constraint given by the second Eq. (2.1) is imposed, and the result is that all the fermions are vector-like, *i.e.*, do not satisfy Eq. (2.2). Hence, at least five chiral fermions are needed to satisfy Eqs. (2.1) [14, 16].

When the charges are arbitrary real numbers, it is evident that there are solutions to Eqs. (2.1) for any $n \geq 5$. In realistic physics theories, however, it is generally expected that the charges are rational numbers up to an overall normalization of the gauge coupling. The reason for that is that the $U(1)$ gauge coupling increases with the energy, and the gauge theory appears to need a cut-off above which some new physics would have to soften the running of the gauge coupling. Typically, that new physics involves the embedding of the $U(1)$ group in a non-Abelian gauge group, which guarantees that the ratio of $U(1)$ charges are rational numbers. We will therefore concentrate on the case where the ratio of any two z_i charges is rational. Furthermore, the overall normalization of the $U(1)$ charges is arbitrary, so we can take all z_i to be integers without loss of generality. For integer charges z_i, Eqs. (2.1) are equivalent to identifying the integer points in the intersection of a cubic hypersurface and a hyperplane in $\mathbb{R}^n$. The first equation (2.1) is a cubic Diophantine equation, and there are no known methods of solving it in general for a fixed but arbitrary value of n.

2.1 Construction of anomaly-free chiral sets

There is often a more straightforward problem that arises in model-building: given a chiral set of fermions which is anomalous, is it possible to include more fermions such that the larger set is chiral and anomaly-free? To address this issue, we make the important observation that any fermion with integer charge z is part of the following anomaly-free set:

$$\left\{ 1 \times (z) \;,\;\; \frac{z}{6}\left(z^2 - 1\right) \times (-2) \;,\;\; \frac{z}{3}\left(z^2 - 4\right) \times (1) \right\} \;. \tag{2.3}$$

where the notation $p \times (x)$ means that there are $|p|$ left-handed fermions with charge $\pm x$, the $+$ and $-$ signs corresponding to $p > 0$ and $p < 0$, respectively. It is not surprising that the two anomaly conditions can be satisfied by two numbers, the numbers of fields of charge 1 and -2. What is nontrivial is that the coefficients p are always integers for any integer z, as a physical number of fields must be. Since fermions with charge $\pm 1, \pm 2$ are central to this construction, we call them basic charges for $U(1)$. If z is one of the basic charges, then the set is vector-like. Otherwise, the set is chiral.

Given a chiral set of charges $S = \{z_i, i = 1, \ldots, n\}$ that may not be anomaly-free, we construct a chiral anomaly free set that consists only of charges in S and, for each one, the appropriate number of fermions with the basic charges, ±1 and ±2, as in Eq. (2.3). If some of the charges in S are themselves basic charges, then we must initially rescale all charges so this is no longer the case. In many cases the resulting anomaly-free set will still contain some vector-like pairs of ±1 and ±2, so the final step is to remove all such pairs. This completes our proof by construction that *for any chiral set of charges S there is a larger set which includes S and is chiral and anomaly-free.*

2.2 Anomaly-free chiral sets with a small number of fermions

The above successful construction of chiral, anomaly-free sets often requires a disturbingly large number of basic charges. We now discuss methods for obtaining smaller sets of fermions which have the advantage of pushing the Landau pole to higher energies.

First, to show that smaller sets are even possible, let us first observe that *for any number $n \geq 5$ of chiral fermions there is a chiral set of $U(1)$ charges that is anomaly free.* To prove this statement it is sufficient to show that there is a chiral set for each $n = 5, \ldots, 9$. An anomaly-free chiral set with arbitrary $n \geq 10$ can always be constructed using linear combinations of chiral sets with $n = 5, \ldots, 9$. In Table 1 we show anomaly-free chiral sets with $n = 5, \ldots, 9$ integer charges that have the maximum charge, chosen to be positive, as small as possible (we include all sets with the two smallest values of the maximum charge).

To reduce the numbers of fermions in a set S we again rely on the construction of Eq. (2.3). Just as the anomaly contribution from a single charge z can be cancelled by the prescribed number of fields with charges $+1$ and -2, the reverse is also true: the anomaly contribution from a number of fields of charges $+1$ and -2 can be cancelled off by a single charge z'. In this way, large numbers of basic charges are exchanged for a single fermion with a large charge.

The numerical techniques that can address the problem of finding small anomaly-free sets are defined on lattices. For our purposes, a lattice is any set of vectors in $\mathbb{R}^n$, that is closed under addition and subtraction (*i.e.*, for any two vectors x, y in a lattice, both $x + y$ and $x - y$ are also in the lattice). Each axis of $\mathbb{R}^n$ represents a possible value of fermion charge, and the coordinates on an axis indicate the *number* of fermions with that charge. Negative coordinates correspond to positive numbers of fermions with the conjugate charge.

The set of chiral, anomaly-free sets forms a lattice, denoted L. For any chiral set

number of fermions	Charges
5	$\{1, 5, -7, -8, 9\}$
	$\{2, 4, -7, -9, 10\}$
6	$\{1, 1, 1, -4, -4, 5\}$
	$\{-1, 2, 3, -5, -5, 6\}$
7	$\{1, 2, 2, -3, -3, -3, 4\}$
	$\{-1, -1, 3, 4, -6, -6, 7\}$
	$\{1, 3, -4, 5, -6, -6, 7\}$
	$\{2, 3, 3, -4, -5, -6, 7\}$
8	$\{1, 1, 2, 3, -4, -4, -5, 6\}$
	$\{2, 2, 2, 2, -5, -5, -5, 7\}$
9	$\{2, 2, 2, -3, -3, 4, -5, -5, 6\}$
	$\{1, 1, 1, 2, -4, 5, -7, -9, 10\}$
	$\{1, -3, 4, 5, 5, -6, -7, -9, 10\}$

Table 1: Anomaly-free chiral sets with $n = 5, \ldots, 9$ integer charges.

X, let $V(X)$ denote the vectorization of X, which is the image of X in the space $\mathbb{R}^n$. For example, if X is the set $\{1, 5, -7, -8, 9\}$, we would define $V(X)$ to be the vector $[1, 0, 0, 0, 1, 0, -1, -1, 1, 0, 0, \ldots]$ corresponding to the fact that there is one fermion of charge 1, zero of charge 2, one of charge -7, etc. Let L denote the set of all such vectorizations $L = \{V(X) \mid X \text{ is chiral and anomaly-free}\}$. We can add elements of L: for any two chiral, anomaly-free sets X and Y, the sum $V(X) + V(Y)$ corresponds to the anomaly-free chiral set that contains all the fermions in both X and Y, followed by the removal of all vector-like pairs. We can similarly subtract any two elements of L to find another element of L; therefore L is a lattice.

For a given z, the vectorization of the construction given in Eq. (2.3) is an element of L, which we call $C(z)$. $C(z)$ contains one fermion with charge z, and the needed number of basic charges to satisfy the anomaly equations. The set $\{C(z_i) \mid z_i \in \pm 3, 4, 5, \ldots\}$ actually spans L: any element of L can be written, by construction, as a unique linear combination of the $C(z_i)$.

It follows that finding the smallest sets of anomaly-free chiral fermions is the same as finding the shortest vectors in L. This problem is called the "Short Vector Problem" and has been studied extensively by mathematicians and computer scientists (for a review, see [24]). Even finding a vector which is at most $\sqrt{2}$ times as long as the shortest vector remains an NP-hard problem, *i.e.*, at least as hard as any nondeterministic polynomial time problem [25]. This means that for very large numbers of fermions, it is impossible to have both accuracy and speed in an algorithm.

To set up the problem concretely, consider searching for an anomaly-free, chiral set with at most N fermions, whose maximum charge is m. A simple iterative approach over all possible numbers of fermions has a time complexity of the order of

$$2^{m-1}\frac{(N+m)!}{N!m!} \ . \tag{2.4}$$

Given a computing power of 10^{10} operations per second, it would take ~ 100 years to find the shortest solution for $N = 30, m = 20$. A better algorithm would be to search over all linear combinations of the basis vectors, $C(z_i)$. This has a time complexity of the order of

$$2^{m-3}\frac{(N+m-2)!}{N!(m-2)!} \ , \tag{2.5}$$

and would take ~ 1 year for $N = 30, m = 20$.

Consider instead the Lenstra-Lenstra-Lovasz (LLL) algorithm [26], readily available in mathematical packages, for attacking the shortest vector problem. The LLL algorithm requires as input the basis vectors $C(z_i)$ for $|z_i| < m$, and outputs a shorter, closer to orthogonal set of basis vectors that span the same lattice. The LLL algorithm has a time complexity of $\mathcal{O}(m^4 \log m)$, and takes $\sim 10^{-5}$ seconds for $m = 20$. Note that this is polynomial in m instead of exponential, and does not involve the number of fermions N (this is possible because the solutions found using the LLL algorithm are by no means guaranteed to be minimal). In fact, they can be up to $2^{\frac{m-1}{2}}$ times larger than the minimal solutions. In practice, however, the solutions found are almost always reasonably short, and the significant decrease in time and ease of implementation make this approach worthwhile.

Since the LLL algorithm actually returns a new basis of short vectors which spans L, the algorithm can easily be adapted to solving another common problem in polynomial time: finding the shortest vector that contains a specific spectrum of fermion charges.

Consider a specific set of charges $\{x_i\}$. To make the LLL algorithm handle this problem, we exchange the basis vectors $C(x_i)$ for the single basis vector $\sum_i C(x_i)$. The output basis set is guaranteed to include at least one short vector that includes the specified charges $\{x_i\}$.

Since we are interested in numbers of fermions which are not particularly large, it may eventually prove useful to adapt even exponential-time solutions to the shortest vector problem in order to identify anomaly-free sets. Although these solutions are exponential in m, a recent algorithm has a time complexity of order $\mathcal{O}(2^{m \log m})$, which take ~ 1 second for $m = 20$ [24].

3 $U(1)_1 \times \cdots \times U(1)_m$

Now consider a set of fermions, ψ_i, $i = 1, \ldots, n$, which are charged under a $U(1)_1 \times \cdots \times U(1)_m$ gauge group. Let us denote the charges of ψ_i under $U(1)_a$, $a = 1, \ldots, m$, by $z_{a,i}$. The construction of anomaly-free sets will proceed as in the case of a single $U(1)$: we will identify the number and structure of basic charges that are needed to cancel off the anomalies for any single fermion.

3.1 $m = 2$

In the case of a $U(1)_1 \times U(1)_2$ gauge group there are six types of anomalies: $[U(1)_1]^3$, mixed $U(1)_1$-gravitational, $[U(1)_2]^3$, mixed $U(1)_2$-gravitational, $[U(1)_1]^2 U(1)_2$ and $U(1)_1 [U(1)_2]^2$. These anomalies cancel if and only if

$$\sum_{i=1}^{n} z_{1,i}^3 = \sum_{i=1}^{n} z_{1,i} = \sum_{i=1}^{n} z_{2,i}^3 = \sum_{i=1}^{n} z_{2,i} = \sum_{i=1}^{n} z_{1,i}^2 z_{2,i} = \sum_{i=1}^{n} z_{1,i} z_{2,i}^2 = 0 \tag{3.1}$$

A set of fermions is chiral with respect to $U(1)_1 \times U(1)_2$ if

$$z_{1,i} + z_{1,j} \neq 0 \quad \text{or} \quad z_{2,i} + z_{2,j} \neq 0 \ , \tag{3.2}$$

for any i and j. Note that a chiral set with respect to $U(1)_1 \times U(1)_2$ may be chiral, partially vector-like, or entirely vector-like with respect to each of the individual $U(1)$'s.

We now show that any set of fermions which is chiral with respect to $U(1)_1 \times U(1)_2$ can be embedded into an anomaly-free set of chiral fermions, as we showed in the previous section for a $U(1)$ gauge theory. This follows from the fact that any fermion with integer charges (z_1, z_2), is part of the following anomaly-free set:

$$\begin{aligned} \Big\{ \ (z_1,z_2) \ , \quad & -\frac{z_1 z_2}{2}(z_1+z_2)\times(1,1) \ , \quad -\frac{z_1 z_2}{2}(z_1-z_2)\times(-1,1) \ , \\ & -\frac{z_1}{6}\left(z_1^2-1\right)\times(2,0) \ , \quad \frac{z_1}{3}\left(z_1^2+3z_2^2-4\right)\times(1,0) \ , \\ & -\frac{z_2}{6}\left(z_2^2-1\right)\times(0,2) \ , \quad \frac{z_2}{3}\left(3z_1^2+z_2^2-4\right)\times(0,1) \ \Big\} \end{aligned} \tag{3.3}$$

where the notation $p\times(x_1,x_2)$ means that there are $|p|$ left-handed fermions with $U(1)_1\times U(1)_2$ charges (x_1,x_2) for $p>0$, or $(-x_1,-x_2)$ for $p<0$. We now have 12 basic pairs of charges $\pm\{(1,1),(1,-1),(2,0),(1,0),(0,2),(0,1)\}$; that are needed to ensure anomaly cancellation. Note that the number of fermions with basic charges prescribed by Eq. (3.3) is automatically an integer. The proof for constructing an anomaly-free chiral set from any chiral set S proceeds exactly as in Section 2.1.

Finding small anomaly-free sets from this construction proceeds through a lattice construction similar to that of Section 2.2. With the larger gauge group $U(1)\times U(1)$, the only change we make is to make each axis of $\mathbb{R}^n$ correspond to a specific (z_1,z_2) charge, instead of a single $U(1)$ charge z. This adaptation works for finding small anomaly-free sets for all of the other gauge groups considered in the remainder of this paper.

3.2 $m \geq 3$

In the case where the number of $U(1)$ gauge groups is $m\geq 3$, there are $m(m^2+3m+8)/6$ equations that must be satisfied to ensure that the theory is anomaly-free:

$$\sum_{i=1}^{n} z_{a,i} = \sum_{i=1}^{n} z_{a,i}z_{b,i}z_{c,i} = 0 \ , \tag{3.4}$$

for any $a,b,c=1,\ldots,m$.

We construct anomaly-free chiral sets by showing that the anomalies of any fermion ψ_i can be cancelled by the anomalies of a set of additional chiral fermions, which is a generalization of the basic charges $\{(1,1),\ (1,-1),\ (2,0),\ (1,0),\ (0,2),\ (0,1)\}$ from Eq. (3.3).

Consider a fermion ψ with charges $(z_1,\ldots,z_m)$. Its $U(1)_aU(1)_bU(1)_c$ anomalies can be cancelled for any unequal $a,b,c=1,\ldots,m$ by a number $(-z_az_bz_c)$ of fermions, labelled by χ_{abc}, $a<b<c$, with charges $(+1,+1,+1)$ under $U(1)_a\times U(1)_b\times U(1)_c$ and charge 0 under all other groups. Then the $[U(1)_a]^2U(1)_b$ anomalies of ψ and χ_{abc} can be cancelled

number of fermions	$U(1)_1$	$U(1)_2$	$U(1)_3$
1	z_1	z_2	z_3
$-z_1z_2z_3$	$+1$	$+1$	$+1$
$-z_1z_2(z_1+z_2-2z_3)/2$	$+1$	$+1$	0
$-z_1z_2(z_1-z_2)/2$	-1	$+1$	0
$-z_2z_3(z_2+z_3-2z_1)/2$	0	$+1$	$+1$
$-z_2z_3(z_2-z_3)/2$	0	-1	$+1$
$-z_3z_1(z_3+z_1-2z_2)/2$	$+1$	0	$+1$
$-z_3z_1(z_3-z_1)/2$	$+1$	0	-1
$-z_1\left(z_1^2-1\right)/6$	$+2$	0	0
$z_1\left(z_1^2-4\right)/3+z_1(z_2^2+z_3^2-z_2z_3)$	$+1$	0	0
$-z_2\left(z_2^2-1\right)/6$	0	$+2$	0
$z_2\left(z_2^2-4\right)/3+z_2(z_3^2+z_1^2-z_3z_1)$	0	$+1$	0
$-z_3\left(z_3^2-1\right)/6$	0	0	$+2$
$z_3\left(z_3^2-4\right)/3+z_3(z_1^2+z_2^2-z_1z_2)$	0	0	$+1$

Table 2: Anomaly-free chiral set of charges under three $U(1)$ groups.

for any unequal $a, b = 1, \ldots, m$ by a set of fermions composed of N^ω_{ab} fermions, labelled by ω_{ab}, $a < b$, with charges $(+1, +1)$ under $U(1)_a \times U(1)_b$ and charge 0 under all other groups, and $N^\omega_{ab}{}'$ fermions, labelled by ω'_{ab}, $a < b$, with charges $(-1, +1)$ under $U(1)_a \times U(1)_b$ and charge 0 under all other groups, where

$$N^\omega_{ab} = z_a z_b \left[\sum_{c=1}^{m} z_c - \frac{3}{2}\left(z_a + z_b\right)\right] \ ,$$
$$N^\omega_{ab}{}' = -\frac{1}{2} z_a z_b \left(z_a - z_b\right) \ . \quad (3.5)$$

The remaining $[U(1)_a]^3$ and mixed $U(1)_a$-gravitational anomalies can be cancelled for any $a = 1, \ldots, m$ by a set of fermions composed of N^ξ_a fermions, labelled by ξ_a, with charges $+2$ under $U(1)_a$ and charge 0 under all other groups, and $N^\xi_a{}'$ fermions, labelled by ξ'_a, with charges +1 under $U(1)_a$ and charge 0 under all other groups, where

$$N_a^{\xi} = -\frac{1}{6} z_a \left(z_a^2 - 1\right) \ ,$$

$$N_a^{\xi\prime} = \frac{1}{3} z_a \left(z_a^2 - 4\right) + z_a \sum_{b \neq a} \left(z_b^2 - z_b \sum_{c\neq a,\, c>b} z_c \right) \ . \tag{3.6}$$

Therefore, we have constructed an anomaly-free chiral set that includes a fermion ψ of arbitrary charges $(z_1, \ldots, z_m)$:

$$\left\{ \psi \ , \ \ -z_a z_b z_c \times (\chi_{abc}) \ , \ \ N_{ab}^{\omega} \times (\omega_{ab}) \ , \ \ N_{ab}^{\omega\prime} \times (\omega'_{ab}) \ , \ \ N_a^{\xi} \times (\xi_a) \ , \ \ N_a^{\xi\prime} \times (\xi'_a) \right\} \tag{3.7}$$

In Table 2 we show the charges in the particular case $m = 3$. Here, the basic charges are the fields χ, ω, ξ.

The proof for constructing an anomaly-free chiral set S' from any set chiral set S proceeds exactly as in Section 2.1.

4 Generalization to any gauge group

We now extend our results to $G \times U(1)$ gauge groups, where G is any non-Abelian group. Consider a set of chiral fermions ψ_i, $i = 1, \ldots, n$, whose charges are (R_i, z_i) under $G \times U(1)$. R_i are some irreducible representations of G. In addition to the $U(1)$ and $U(1)^3$ anomalies, the G^3 and $G^2U(1)$ anomalies also must cancel (all other mixed anomalies are zero). The G^3 anomaly is given by

$$A_{GGG} = \sum_i A(R_i) \ , \tag{4.1}$$

where the anomaly of R_i, $A(R_i)$, is defined by

$$Tr\left(\left\{T^a(R_i), T^b(R_i)\right\} T^c(R_i)\right) = \frac{1}{2} A(R_i) d^{abc} \ . \tag{4.2}$$

The totally symmetric tensor d^{abc} is determined by the anticommutation relation among the group generators $T^a(R_i)$. The $G^2U(1)$ anomaly is given by

$$A_{GG1} = \sum_i C(R_i) z_i \ , \tag{4.3}$$

where the Casimir of R_i, $C(R_i)$, is defined by

$$Tr\left(T^a(R_i) T^b(R_i)\right) = \delta_{ab} C(R_i) \ . \tag{4.4}$$

Finally, the $U(1)$ and $U(1)^3$ anomalies take the following form up to an overall normalization:

$$A_{1gg} = \sum_i d(R_i) z_i \ ,$$
$$A_{111} = \sum_i d(R_i) z_i^3 \ , \tag{4.5}$$

where $d(R_i)$ is the dimension of R_i. The set of fermions ψ_i is anomaly-free if

$$A_{GGG} = A_{GG1} = A_{1gg} = A_{111} = 0 \ . \tag{4.6}$$

If any of these conditions is not satisfied, then we prove by construction that one can add more fermions such that the larger set is both chiral and anomaly-free.

For each fermion with charges (R, z) with $z \neq 0$, we can construct an anomaly-free set

$$\left\{(R,z) \ , \ \frac{z}{6}\left(z^2-1\right) \times (R,-2) \ , \ \frac{z}{3}\left(z^2-4\right) \times (\overline{R},1) \ , \ \frac{1}{6}\left(z+1\right)\left(z+2\right)\left(z-3\right) \times (R,0)\right\} \ , \tag{4.7}$$

where $\overline{R}$ is the conjugate of R: $A(\overline{R}) = -A(R)$, $C(\overline{R}) = C(R)$, $d(\overline{R}) = d(R)$. The notation $p \times (R, x)$ means that if $p \geq 0$ then there are p left-handed fermions with charge (R, x), while if $p < 0$ then there are $-p$ left-handed fermions with charge $(\overline{R}, -x)$. The additional fermions with charge $(R, 0)$ are included to make the entire set vector-like under the G group. We have chosen a basis that is easy to write down explicitly, but in many cases is larger than necessary—one could instead make some of the $(R, 1)$ fermions into $(\overline{R}, 1)$ fermions and remove the appropriate number of $(R, 0)$ fermions leaving at most one fermion with charge 0 under $U(1)$.

To render the entire set $\{\psi_i, \, i = 1, \ldots, n\}$ anomaly-free and chiral, we first rescale the z_i charges to be different than $+1$ and -2, add the fermions for each field ψ_i according to Eq. (4.7), then discard any remaining vector-like pairs. Note that if any of the z_i charges is zero, then one could add other fermions which are neutral with respect to $U(1)$ that belong to nontrivial representations of G such that the entire set is anomaly-free and chiral (as done in [9] for the case where $G = SU(N)$).

Typically, the total number of additional fermions can be further reduced if instead of fermions transforming nontrivially under G we add some fermions which are singlets (belong to the 1 representation of G). For example, consider the case where the set $\{\psi_i$, $i = 1, \ldots, n\}$ is anomaly-free with respect to the non-Abelian group G ($A_{GGG} = 0$). In order to cancel the $G^2U(1)$ anomaly we could add two more fermions with charges (R, z)

and $(\overline{R}, z')$ such that

$$z + z' = -\frac{1}{C(R)} \sum_i C(R_i) z_i \ . \tag{4.8}$$

One may ensure that all the U(1) charges are integers by an appropriate rescaling. The $U(1)$ and $U(1)^3$ anomalies can be finally cancelled by including a number N_1 of fermions with charges $(1, +1)$, and a number N_2 of fermions with charges $(1, -2)$, as prescribed in Section 2.1:

$$\begin{aligned} N_1 &= \frac{1}{3}\left\{\sum_i d(R_i) z_i \left(z_i^2 - 4\right) + d(R)\left[z\left(z^2-4\right) + z'\left(z'^2 - 4\right)\right]\right\} \ , \\ N_2 &= \frac{1}{6}\left\{\sum_i d(R_i) z_i \left(z_i^2 - 1\right) + d(R)\left[z\left(z^2-1\right) + z'\left(z'^2 - 1\right)\right]\right\} \ . \end{aligned} \tag{4.9}$$

The remarkable feature that enables this construction is that N_1 and N_2 are integers for any integer charges z, z', z_i.

This procedure can immediately be extended to groups of the form $G_1 \times \ldots \times G_m \times U(1)$, where G_i are non-Abelian gauge groups. For example, a single fermion ψ with charge $(R_1, \ldots, R_m, z)$ is part of the anomaly free set

$$\begin{aligned} &\Big\{(R_1, \ldots, R_m, z) \ , \quad \frac{z}{6}\left(z^2 - 1\right) \times (R_1, \ldots, R_m, -2) \quad , \\ &\frac{z}{3}\left(z^2-4\right) \times (\overline{R}_1, \ldots, \overline{R}_m, 1) \ , \quad \frac{1}{6}(z+1)(z+2)(z-3) \times (R_1, \ldots, R_m, 0)\Big\} \ . \end{aligned} \tag{4.10}$$

To extend these results to arbitrary $G_1 \times \ldots \times G_m \times U(1)_1 \times \ldots \times U(1)_{m'}$ groups, one may simply use the coefficients and the set of fermions $\chi, \omega, \omega', \xi, \xi'$ described in Section 3.2 in place of the single z charges written above.

5 $U(1)$ extension of the Standard Model gauge group

The results presented in the previous sections have various applications to physics beyond the Standard Model. In this section we study a particularly important application. The elementary fermions discovered in experiments so far, with charges under the $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge group listed in Table 3, may be charged under a new Abelian gauge group, $U(1)_z$, provided this is spontaneously broken. The $U(1)_z$ charges of these Standard Model fermions determine the relative couplings of the Z' boson [the heavy gauge boson associated with $U(1)_z$], and therefore its experimental signatures.

	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$	$U(1)_z$
q_L	3	2	$+1/3$	z_q
u_R	3	1	$4/3$	z_u
d_R	3	1	$-2/3$	z_d
l_L	1	2	-1	z_l
e_R	1	1	-2	z_e

Table 3: Gauge charges of the Standard Model fermions in the presence of a new $U(1)$ group. An index labeling the three generations is implicit.

The discovery of a Z' boson with couplings to the known fermions which are not proportional to hypercharge would imply the existence of certain additional fermions [27] or antisymmetric tensor fields in extra dimensions [19]. Here, as an application of our results, we show that *any* couplings of a Z' boson to the Standard Model fermions are allowed by anomaly cancellation if additional fermions are present. For simplicity, we concentrate on generation-independent couplings. The same method can be easily applied to generation-dependent couplings (in that case, though, there are stronger phenomenological constraints from flavor-changing neutral currents [28, 23]).

The $U(1)_z$ charges of the Standard Model fermions lead in general to six different gauge anomalies: $SU(3)_C^2 U(1)_z$, $SU(2)_W^2 U(1)_z$, $U(1)_Y^2 U(1)_z$, $U(1)_Y U(1)_z^2$, $U(1)_z^3$ and $U(1)_z$. It is imperative to ask whether these anomalies can all be cancelled simultaneously by including additional fermions. According to the prescription outlined in Section 4, one can construct anomaly-free sets for any rational values of the $U(1)_z$ charges.

However, realistic extensions of the Standard Model need additional constraints to be satisfied. One constraint is that there are no new stable particles with fractional electric charges (for a review of experimental limits, see [29]). To avoid fractional electric charges, we choose to introduce only fermions that transform under the Standard Model gauge group in the same representations (or the conjugated ones) as the observed fermions, or are not charged under the Standard Model gauge group. One could relax this restriction, for example by including fermions with larger integer electric charges, but we will not need this freedom here.

Electroweak measurements restrict severely the number of new chiral fermions charged under $SU(2)_W$. In order to satisfy this constraint, we require the new fermions to be

	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$	$U(1)_z$
ψ^l_L ψ^l_R	1	2	-1	z^l_L z^l_R
ψ^e_L ψ^e_R	1	1	-2	z^e_L z^e_R
ψ^d_L ψ^d_R	3	1	$-2/3$	z^d_L z^d_R
ν^j_R , $j = 1, ..., N_1$	1	1	0	-1
ν'^k_R , $k = 1, ..., N_2$	1	1	0	$+2$

Table 4: New fermions which, together with the three Standard Model generations (see Table 3), form an anomaly-free set. The charges under the new $U(1)_z$ gauge group are restricted by Eqs. (5.1) and (5.3), while N_1 and N_2 are given in Eq. (5.4).

vector-like with respect to the $SU(3)_C \times SU(2)_W \times U(1)_Y$, but chiral with respect to $U(1)_z$. Another constraint is that the both the Standard Model fermions and the new ones must have masses, so that some Yukawa-type couplings to Higgs fields need to be gauge invariant. For any specified Higgs sector, this leads to constraints on the $U(1)_z$ charges. However, one can keep the $U(1)_z$ charges of the fermions arbitrary and still give masses to all fermions by including a sufficient number of scalars with $U(1)_z$-breaking VEVs that have higher-dimensional interactions with the fermions.

To allow for completely arbitrary charges for the Standard Model fields under the new $U(1)_z$, the spectrum of fields listed in Table 4 suffices, although other choices are possible as long as there is at least one fermion charged under $SU(3)$, another one charged under $SU(2)_W$ and yet another one charged under $U(1)_Y$. We will eventually show that the anomaly cancellation conditions can be solved for arbitrary rational values of z_q, z_u, z_d, z_e and z_l, as well as rational values of all other charges. Given the freedom in choosing the normalization of the gauge coupling, we can take z_q, z_u, z_d, z_e and z_l to be integers. Our method of approach is to first impose that all of the anomalies cancel except for the $U(1)_z$-gravitational and $U(1)_z^3$ ones. Finding the $U(1)_z$ charges of the ψ fields requires solving three linear equations, corresponding to the $SU(3)^2_C U(1)_z$, $SU(2)^2_W U(1)_z$ and $U(1)^2_Y U(1)_z$ anomalies, and one quadratic equation, corresponding to the $U(1)_Y U(1)^2_z$ anomaly. Note that if we can guarantee that the $U(1)_z$ values of all fields are still rational after having imposed these relations, then the $U(1)_z$ & $U(1)_z^3$ anomaly equations can be cancelled as described in Section 2: by adding the needed number of fields ν_R & ν'_R. The remaining anomaly conditions are not affected by the ν_R & ν'_R fields, so the most difficult part of satisfying the anomaly conditions, the cubic $U(1)_z^3$ equation, is removed from the process.

Notice that we have chosen some of the fermions in Tables 3 and 4 to be right-handed. Their contributions to the anomalies are the same as those of a left-handed fermion in the complex conjugated representation. The three linear equations due to the $SU(3)^2U(1)_z$, $SU(2)^2U(1)_z$, $U(1)_Y^2U(1)_z$ anomalies constrain linear combinations of the charges of the new fields to be

$$\begin{aligned} z_L^d - z_R^d &= -3\left(2z_q - z_u - z_d\right) \ , \\ z_L^l - z_R^l &= -3\left(3z_q + z_l\right) \ , \\ z_L^e - z_R^e &= 3\left(2z_q + z_u + z_e\right) \ . \end{aligned} \tag{5.1}$$

To proceed with the $U(1)_YU(1)_z^2$ anomaly cancellation condition, given by

$$\left(z_L^d\right)^2 - \left(z_R^d\right)^2 + \left(z_L^l\right)^2 - \left(z_R^l\right)^2 + \left(z_L^e\right)^2 - \left(z_R^e\right)^2 = 3\left(z_q^2 - 2z_u^2 + z_d^2 - z_l^2 + z_e^2\right) \ , \tag{5.2}$$

we consider the particular case where the three remaining linear combinations of ψ charges can also be written as linear combinations of z_q, z_u, z_l, z_d and z_e. This reduces the $U(1)_YU(1)_z^2$ anomaly equation to a linear equation in the unknown coefficients which has a three parameter solution for general values of z_q, z_u, z_l, z_d and z_e. We find that the charges of the new fields are given by

$$\begin{aligned} z_L^d &= \left(-2 - \frac{3}{2}a_2 + a_1\right) z_q + \left(1 + \frac{a_1}{2}\right) z_u + 2z_d - \frac{a_2}{2} z_l + \frac{a_1}{2} z_e \ , \\ z_L^l &= \left(-6 + a_2 - a_3\right) z_q - \frac{1}{2}\left(a_2 + a_3\right) z_u - \frac{a_2}{2} z_d - z_l - \frac{a_3}{2} z_e \ , \\ z_L^e &= \left(2 + a_1 - \frac{3}{2}a_3\right) z_q + \left(1 - \frac{a_1}{2}\right) z_u - \frac{a_1}{2} z_d - \frac{a_3}{2} z_l + 2z_e \ , \end{aligned} \tag{5.3}$$

where a_1, a_2, a_3 are arbitrary even integers.

To complete the proof, we add the necessary number of ν_R and ν'_R fields as described in Section 2 to cancel the $U(1)_z$-gravitational and $U(1)_z^3$ anomalies:

$$\begin{aligned}
N_1 &= \frac{1}{3}\sum_f d_f z_f \left(z_f^2 - 4\right) \\
&= \left(z_L^d\right)^3 - \left(z_R^d\right)^3 + \frac{2}{3}\left[\left(z_L^l\right)^3 - \left(z_R^l\right)^3\right] + \frac{1}{3}\left[(z_L^e)^3 - (z_R^e)^3\right] \\
&\quad +6z_q^3 - 3z_u^3 - 3z_d^3 + 2z_l^3 - z_e^3 + 16z_q - 4z_u \ , \\
N_2 &= \frac{1}{6}\sum_f d_f z_f \left(z_f^2 - 1\right) \\
&= \frac{1}{2}\left(N_1 - 12z_q + 3z_u\right) \ . \qquad (5.4)
\end{aligned}$$

where f runs over all fermions, z_f is the $U(1)_z$ charge of the fermion and d_f is the dimensionality of the $SU(3)_C \times SU(2)_W$ representation times ± 1 for left-handed and right-handed fermions, respectively. We emphasize that Eqs. (5.4) yield integer values for N_1 and N_2 for any integers z_q, z_u, z_d, z_e and z_l, and the values of N_1 and N_2 can be reduced using the numerical methods of Section 2.2. Our construction shows that all couplings of a Z' boson to the Standard Model fermions are allowed by anomaly cancellation, so long as additional fermions are present.

For illustration, let us pick some simple $U(1)_z$ charges for the Standard Model fermions, $z_d = z_l = z_e = 0$ and $z_q = z_u = 1$, and compute the number of right-handed neutrinos in Table 4 that need to be included in an anomaly free set. We could use the freedom to choose a_1, a_2 and a_3 in order to minimize N_1 and N_2, but for this simple case we just take $a_1 = a_2 = a_3 = 0$. Therefore, the charges of the ψ fermions follow from Eqs. (5.3) and (5.2): $z_L^d = -1$, $z_R^d = 2$, $z_L^l = z_R^e = -6$, $z_R^l = z_L^e = 3$. Eq. (5.4) then gives $N_1 = -75$ and $N_2 = -42$, which means that there are 75 right-handed neutrinos of $U(1)_z$ charge $+1$ and 42 right-handed neutrinos of $U(1)_z$ charge -2. This large number of right-handed neutrinos can be substantially reduced using Eq. (2.3). For example, the set $\{42 \times (-2)\ ,\ 75 \times (+1)\}$ can be replaced by one of the following sets of five right-handed neutrinos: $\{2 \times (-5)\ ,\ 1 \times (-3)\ ,\ 2 \times (+2)\}$ or $\{1 \times (-6)\ ,\ 2 \times (-3)\ ,\ 1 \times (+2)\ ,\ 1 \times (+1)\}$.

6 Conclusions

The need to embed new $U(1)$ gauge groups in non-Abelian groups forces a focus on integer-valued charges, up to a possible rescaling of the gauge coupling constant. Our results show that anomaly cancellation in a gauge theory, while highly constraining, can occur for *any* set of integer fermion charges through the addition of new integer-charged fields. This is akin to gauging $U(1)_{B-L}$ in the Standard Model: one is forced to add a right-handed neutrino to prevent gauge anomalies from appearing. That such anomaly-free sets exist is obvious when one constructs vector-like sets, but highly non-trivial for chiral integer-valued sets.

The main result is presented in Section 2.1 for fermions charged under a $U(1)$ gauge group, and subsequently extended to any other gauge groups. The key observation is that there always exists a certain *integer* number of basic charges that can cancel off the anomaly from a single fermion. When the sets are large, the numerical techniques discussed in Section 2.2 allow a quick reduction of the set size.

Our solution is a complete description of chiral anomaly-free sets for $U(1)^m$ gauge theories. For gauge groups that have additional non-Abelian factors $G_1 \times \ldots \times G_m \times U(1)^{m'}$ we have concentrated on chiral anomaly-free sets that include vector-like fermions with respect to some of the non-Abelian groups. This is sufficient to prove that any fermion can be included in a larger chiral set of fermions that is anomaly-free. Nevertheless, it would be interesting to extend our results and find a complete description of anomaly-free sets under gauge groups of the form $G_1 \times \ldots \times G_m \times U(1)^{m'}$ which are chiral with respect to each of the G_i groups.

If a gauge extension of the Standard Model is discovered, then we have argued that the full spectrum of the new theory will still be chiral: a completely vector-like theory would leave behind both the observed Standard Model fermions *and* a set of conjugate partners after the extended gauge symmetry breaks to $SU(3)_C \times SU(2)_W \times U(1)_Y$, which is not phenomenologically acceptable. Therefore, our results should have applications to a variety of extensions of the Standard Model. In Section 5 we have presented a particular application: if the Standard Model gauge group is extended to include a new $U(1)$ group, then the Standard Model fermions may have arbitrary rational charges under the new $U(1)$ and still the anomalies would cancel in the presence of certain additional fermions with rational charges.

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