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ANOTHER CONVERGENCE THEOREM OF MARTINGALES IN THE LIMIT

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1. Introduction. Let (Ω, F, P) be a probability space and (F_n) an increasing sequence of sub-σ-fields of F. A martingale in the limit is an adapted sequence of integrable random variables such that

$$\lim_{n\to\infty} \sup_{m>n} | X_n - E[X_m | F_n] | = 0 \quad \text{a.s.}$$

Mucci [7], who introduced martingales in the limit, has proved that L^1-bounded martingales in the limit converge a.s., thus extending the Doob martingale convergence theorem. Edgar and Sucheston [6] showed that "several crucial properties possessed by amarts fail for martingales in the limit, namely: the maximal inequality, Riesz decomposition, optional stopping theorem, optional sampling theorem". Recently Bellow and Dvoretzky [1] added another one to this negative list, i.e., the set of L^1-bounded martingales in the limit is not a vector lattice. In the present note we show a convergence theorem of martingales in the limit, which is suggested by Chow's submartingale convergence theorem [2].

2. Convergence theorem. A stopping time is a random variable τ assuming positive integer values and the value $+\infty$, such that $(\tau = n) \in F_n$ for each n. The collection of all stopping times is denoted by T. Chow [2, Th. 3] showed that if $\{X_n, F_n, n \geqq 1\}$ is a submartingale such that $\int_{(\tau<\infty)} X_\tau^+ < \infty$ for every $\tau \in T$, then $\lim_{n\to\infty} X_n$ exists a.s. We establish an analogous result for martingales in the limit.

THEOREM. *Let $\{X_n, F_n, n \geqq 1\}$ be a martingale in the limit such that $\int_{(\tau<\infty)} X_\tau^+ < \infty$ for every $\tau \in T$. Then $\lim_{n\to\infty} X_n$ exists and $> -\infty$ a.s.*

PROOF. Suppose that $P(\lim X_n \text{ exists}) < 1$. Put $V = \{\limsup X_n > a > b > \liminf X_n\}$. Then $P(V) > 0$ for some $a > b$. Without loss of generality we may and do assume that $a=1$ and $b=0$. Let $\varepsilon>0$. By the definition of martingales in the limit there exists an integer r_1 such that

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$$P\Big(\bigcap_{n \geqq r_1} \Big\{\sup_{k>n} |X_n - E[X_k \mid F_n]| < 1/4\Big\}\Big) > 1 - \varepsilon/8 .$$

Because $V \subset \{\limsup X_n > 1\}$, we can find an integer $m_1 > r_1$ such that

$$P\Big(V - \bigcup_{n=r_1}^{m_1} \{X_n > 1\}\Big) < \varepsilon/8 .$$

Put $K_n = \{X_n > 1, \sup_{k>n} |X_n - E[X_k \mid F_n]| < 1/4\}$. Then

$$A_1 = \bigcup_{k=r_1}^{m_1} K_k \in F_{m_1} , \quad P(V - A_1) < \varepsilon/4 .$$

Put $J_n = \{X_n < 0, \sup_{k>n} |X_n - E[X_k \mid F_n]| < 1/4\} A_1$. Then in a similar manner we can find an integer $n_1 > m_1$ such that

$$B_1 = \bigcup_{k=m_1}^{n_1} J_k \ (\subset A_1) \in F_{n_1} , \quad P(VA_1 - B_1) < \varepsilon/4 .$$

Let

$$\sigma_1 = \begin{cases} \min\{k \colon \omega \in K_k\} & \text{on} \quad A_1 , \\ m_1 & \text{on} \quad \Omega - A_1 ; \end{cases}$$

$$\sigma_2 = \begin{cases} m_1 & \text{on} \quad \Omega - A_1 , \\ \min\{k \colon \omega \in J_k\} & \text{on} \quad B_1 , \\ n_1 & \text{on} \quad A_1 - B_1 . \end{cases}$$

Then $\sigma_1, \sigma_2 \in T$, $\sigma_1 \leqq \sigma_2 \leqq n_1$ and $A_1 \in F_{\sigma_1}$, so

$$\int_{A_1} E[X_{n_1} \mid F_{\sigma_2}] = \int_{A_1} E[X_{n_1} \mid F_{\sigma_1}] .$$

Thus we have

$$\begin{aligned} \int_{A_1 - B_1} X_{n_1} &\geqq \int_{A_1 - B_1} X_{n_1} + \int_{B_1} X_{\sigma_2} \\ &\geqq \int_{A_1 - B_1} E[X_{n_1} \mid F_{\sigma_2}] + \int_{B_1} \{E[X_{n_1} \mid F_{\sigma_2}] - 1/4\} \\ &= \int_{A_1} E[X_{n_1} \mid F_{\sigma_1}] - P(B_1)/4 \\ &\geqq \int_{A_1} (X_{\sigma_1} - 1/4) - P(B_1)/4 \geqq P(A_1)/2 , \end{aligned}$$

$$P(VB_1) = P(V) - P(V - VA_1) - P(VA_1 - VB_1) \geqq P(V) - \varepsilon/2 .$$

By induction, for given n_{i-1}, B_{i-1}, there exist m_i, n_i $(n_{i-1} < m_i < n_i)$, A_i, B_i $(B_{i-1} \supset A_i \supset B_i$, $A_i \in F_{m_i}$, $B_i \in F_{n_i})$ such that

$$\int_{A_i - B_i} X_{n_i} \geqq P(A_i)/2 ,$$
$$P(VB_i) \geqq P(VB_{i-1}) - \varepsilon \cdot 2^{-i} \geqq P(V) - \varepsilon \cdot 2^{-1} - \cdots - \varepsilon \cdot 2^{-i} \geqq P(V) - \varepsilon .$$

Put $\tau = n_i$ on $A_i - B_i$ $(i = 1, 2, \cdots)$, and $\tau = \infty$ elsewhere. Then $\tau \in T$. Take ε such that $0 < \varepsilon < P(V)$. Then

$$\int_{(\tau<\infty)} X_\tau^+ = \sum_{i=1}^{\infty} \int_{A_i - B_i} X_{n_i}^+ \geqq (1/2) \cdot \sum P(A_i) \geqq (1/2) \cdot \sum P(VB_i) = \infty .$$

This contradicts the assumption, and we proved that $\lim X_n$ exists a.s.

Now put $V = \{\lim X_n = -\infty\}$ and suppose that $P(V) = \delta > 0$. Note that $E|X_n| < \infty$ for each n. Then, for $\varepsilon > 0$ there exist $M_1 > 1$, m_1, n_1 $(m_1 < n_1)$ such that

$$P(V - A_1) < \varepsilon/4 , \quad P(VA_1 - B_1) < \varepsilon/4 ,$$

where $A_1 = \{X_{m_1} > -M_1 + 1, \sup_{k>m_1} |X_{m_1} - E[X_k \mid F_{m_1}]| < 1\} \in F_{m_1}$ and $B_1 = \{X_{n_1} < -(4/\delta)M_1\}A_1 \in F_{n_1}$. This yields

$$\int_{A_1 - B_1} X_{n_1} = \int_{A_1} X_{n_1} - \int_{B_1} X_{n_1} = \int_{A_1} E[X_{n_1} \mid F_{m_1}] - \int_{B_1} X_{n_1}$$
$$\geqq \int_{A_1} (X_{m_1} - 1) - \int_{B_1} X_{n_1} \geqq -M_1 P(A_1) + (4/\delta) M_1 P(B_1) ,$$
$$P(VB_1) \geqq P(V) - \varepsilon/2 .$$

By induction, for given n_{i-1} and $B_{i-1} \in F_{n_{i-1}}$, there exist $M_i > 1$, $n_i > m_i > n_{i-1}$, A_i, B_i $(B_{i-1} \supset A_i \supset B_i$, $A_i \in F_{m_i}$, $B_i \in F_{n_i})$ such that

$$\int_{A_i - B_i} X_{n_i} \geqq M_i\{-P(A_i) + (4/\delta)P(B_i)\} ,$$
$$P(VB_i) \geqq P(VB_{i-1}) - \varepsilon \cdot 2^{-i} \geqq \delta - \varepsilon .$$

Let $\varepsilon = \delta/2$. Then $P(B_i) \geqq P(VB_i) \geqq \delta/2$, so

$$-P(A_i) + (4/\delta)P(B_i) \geqq -1 + (4/\delta) \cdot (\delta/2) = 1 .$$

Put $\tau = n_i$ on $A_i - B_i$ $(i = 1, 2, \cdots)$, and $\tau = \infty$ elsewhere. Then $\tau \in T$ and

$$\int_{(\tau<\infty)} X_\tau^+ = \sum_{i=1}^{\infty} \int_{A_i - B_i} X_{n_i}^+ \geqq \sum_{i=1}^{\infty} M_i = \infty .$$

This contradicts the assumption. q.e.d.

COROLLARY. *Let $\{X_n, F_n, n \geqq 1\}$ be a martingale in the limit such that $\int_{(\tau<\infty)} |X_\tau| < \infty$ for every $\tau \in T$. Then $\lim_{n\to\infty} X_n$ exists and is finite a.s.*

With a little change in the proof, one can see that Theorem is true also for submartingales. The following example shows that neither $E[(\lim X_n)^-] < \infty$ nor $\lim X_n < \infty$ a.s. is in general true in Theorem even for martingales.

EXAMPLE 1. Let $\Omega = A_0 \supset A_1 \supset A_2 \supset \cdots$, $A_\infty = \bigcap_{n\geqq 0} A_n$, $P(A_n) = a_n$, $X_n = \sum_{k=1}^n y_k I(A_{k-1} - A_k) + x_n I(A_n)$, and $F_n = \sigma(X_1, \cdots, X_n)$. Put $a_n = 2^{-1} + 2^{-n-1}$, $x_n = n$, $y_n = -2^n + n - 2$. Then $\{X_n, F_n, n \geqq 1\}$ is a martingale, and $E[(\lim X_n)^-] = \infty$, $P(\lim X_n = \infty) = P(A_\infty) = 1/2$. Let $\tau \in T$ such that $\int_{(\tau<\infty)} X_\tau^+ > 0$. Then $\int_{(\tau=n)} X_\tau^+ > 0$ for some n. But $(\tau = n) \supset A_n$ in this case, because $(X_n^+ > 0) = A_n$ is an atom of F_n. Thus

$$\int_{(\tau<\infty)} X_\tau^+ \leqq \int_{A_n} X_n^+ + \int_{\Omega - A_n} (X_{\tau\wedge n}^+ + X_{\tau\vee n}^+) = \int_{A_n} X_n^+ + \int_{\Omega - A_n} X_{\tau\wedge n}^+$$
$$\leqq \int \sup_{k\leqq n} X_k^+ < \infty .$$

By Corollary we see that if $\{X_n, F_n, n \geqq 1\}$ is a martingale such that $\int_{(\tau<\infty)} |X_\tau| < \infty$ for every $\tau \in T$, then $\lim X_n$ exists and is finite a.s. Because any L^1-bounded martingale satisfies this condition (see, for example, Neveu [8, p. 76]) and vice versa (Dubins and Freedman [4], Chow [3]), Theorem gives an independent proof of the Doob convergence theorem. For martingales in the limit these conditions are not equivalent, as one can see in the following example.

EXAMPLE 2. In Example 1, put $a_n = 2^{-n}$, $y_n = 0$ and $x_n = 2^n$. Then $\{X_n, F, n \geqq 1\}$ is a martingale in the limit such that $E|X_n| = 1$ for all n. But for $\tau = \sum_{n=1}^\infty n \cdot I(A_n - A_{n+1}) \in T$, we have $\int_{(\tau<\infty)} |X_\tau| = \sum_{n=1}^\infty x_n \cdot 2^{-n-1} = \infty$. Now put $a_n = 2^{-n}$, $y_n = 0$ and $x_n = n \cdot 2^n$. Then $\{X_n, F_n, n \geqq 1\}$ is a martingale in the limit (and a submartingale) such that $\sup_n E|X_n| = \infty$. The same reasoning as in Example 1 shows that $\int_{(\tau<\infty)} |X_\tau| < \infty$ for every $\tau \in T$.

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