Progress of Theoretical Physics, Vol. 65, No. 6, June 1981

Another Equilibrium Sequence of Self-Gravitating and Rotating Incompressible Fluid

Yoshiharu ERIGUCHI and Daiichiro SUGIMOTO

Department of Earth Science and Astronomy
University of Tokyo, Komaba, Tokyo 153

(Received February 17, 1981)

A new sequence of gravitational equilibria was obtained for uniformly rotating axisymmetric incompressible fluid. This sequence parts from the Maclaurin sequence at the neutral point against the $P_4(\eta)$ displacement at the surface. It continues into a concave hamburger-like shape, and finally into a toroid. Therefore, this neutral point on the Maclaurin sequence is one of the points of bifurcation.

§ 1. Introduction

It has been said that there are only two axisymmetric equilibrium sequences in the case of self-gravitating, uniformly rotating *incompressible* fluids—Maclaurin spheroids[1] and Dyson-Wong toroids.[2] In the study of post-Newtonian effects on the structure of the Maclaurin spheroids, Chandrasekhar[3] and Bardeen[4] have shown that there is a neutral point on the Maclaurin sequence against the perturbation of $P_4(\eta)$ displacement at the surface where η is one of the spheroidal coordinates. The neutral point corresponds to the eccentricity of the spheroidal configuration $e=e_{cr}=0.98523$. Bardeen[4] has also proved that non-spheroidal configurations can be in gravitational equilibrium so far as the first-order deformation from the Maclaurin spheroid is considered. However, he has not shown whether an equilibrium configuration exists or not in the case of finite deformation from the spheroid.

Recently, Fukushima et al.[5] computed the structure of uniformly rotating polytropes with *small but finite* values of polytropic index. In the case of high angular momentum there appeared a concave hamburger-like shape of equilibrium, and the sequence of such shapes seemed to continue into a toroid.

These facts suggest that there should be another equilibrium sequence of uniformly rotating *incompressible* fluid which bifurcates from the Maclaurin sequence at the point $e=e_{cr}$. It is the aim of the present paper to show that it is the case: We have computed numerically such an intermediate sequence which branches off the spheroids and extends to toroids.

§ 2. Axisymmetric equilibrium

We will treat uniformly rotating axisymmetric incompressible fluid under self-gravitation. The surface of the fluid is written as

$$f(r, \mu)=0\,, \tag{1}$$

where the spherical polar coordinate (r, θ, ϕ) is used and $\mu=\cos\theta$. We will assume that the configuration is plane-symmetric with respect to the plane of $\mu=0$. If we further assume that this equation can be solved uniquely for μ, that is, if μ is a single-valued function of r for $0\leq\mu\leq 1$, the surface can be rewritten as

$$\mu=\mu(r). \tag{2}$$

When matter exists inside the surface of Eq. (2), the gravitational potential $\Phi(r, \mu)$ is written as

$$\begin{aligned}\Phi(r, \mu)&=-G\int d^3\boldsymbol{r}'\frac{\rho}{|\boldsymbol{r}-\boldsymbol{r}'|}\\&=-4\pi G\sum_{n=0}^{\infty}P_{2n}(\mu)\int_0^1 d\mu' P_{2n}(\mu')\left(\int_0^r dr'\frac{r'^{2n+2}}{r^{2n+1}}+\int_r^\infty dr'\frac{r^{2n}}{r'^{2n-1}}\right)\rho\\&=-4\pi G\rho\sum_{n=0}^{\infty}P_{2n}(\mu)\left\{\int_0^{r_1}dr'\frac{r'^{2n+2}}{r^{2n+1}}\tilde{P}_{2n}[\mu(r_1)]\right.\\&\quad\left.+\int_{r_1}^r dr'\frac{r'^{2n+2}}{r^{2n+1}}\tilde{P}_{2n}[\mu(r')]+\int_r^{r_2}dr'\frac{r^{2n}}{r'^{2n-1}}\tilde{P}_{2n}[\mu(r')]\right\}.\end{aligned} \tag{3}$$

Here, ρ is the density, $P_{2n}(\mu)$ is the Legendre polynomial, r_1 and r_2 are the smallest and the largest distance from the center $(r=0)$ to the surface, respectively, and

$$\tilde{P}_{2n}(\mu)=\int_0^\mu d\mu' P_{2n}(\mu'). \tag{4}$$

For a uniformly rotating incompressible fluid, the equilibrium is sustained if and only if the equation

$$\Phi[r, \mu(r)]-\Omega^2 r^2[1-\mu(r)^2]/2=\text{const}\,, \tag{5}$$

is satisfied at all points of the surface (2), where Ω is the angular velocity. If Ω^2 and ρ are given, Eq. (5) is the equation for $\mu(r)$ and r_1/r_2. Alternatively, we solve Eq. (5) for $\mu(r)$ and Ω^2 by specifying the values of r_1/r_2 and ρ.

§ 3. Results and discussion

It was assumed that approximate values of $\mu_0(r)$ and Ω_0 were known for specified values of r_1/r_2 and ρ. Then, Eq. (5) was solved by the Newton-Raphson iteration scheme: We substituted $\mu(r)=\mu_0(r)+\delta\mu(r)$ into Eq. (5), linearized it to the first order in $\delta\mu(r)$, solved it for $\delta\mu(r)$, replaced the assumed $\mu_0(r)$ with newly computed $\mu(r)[=\mu_0(r)+\delta\mu(r)]$, and then whole procedure was repeated until $\delta\mu(r)$ became negligibly small. In practice, we obtained a solution after 2 ~4 iterations.

We started from the point $e=e_{cr}$ on the Maclaurin sequence, and by changing the ratio r_1/r_2 we obtained a sequence of solutions. Physical quantities of the solutions along the sequence are summarized in Table I. The mass M is held

Table I. Properties of the sequence.

$\Omega^2/4\pi G\rho$	$I/M^{5/3}\rho^{-2/3}$	j^2	$T/\|W\|$	$(T+W)/E_0$	
8.506(−2)	5.135(−1)	2.243(−2)	3.648(−1)	−8.531(−4)	(a)*
8.324(−2)	5.223(−1)	2.270(−2)	3.654(−1)	−8.571(−4)	
8.206(−2)	5.259(−1)	2.269(−2)	3.637(−1)	−8.565(−4)	(b)*
8.139(−2)	5.259(−1)	2.251(−2)	3.608(−1)	−8.532(−4)	
8.113(−2)	5.236(−1)	2.224(−2)	3.576(−1)	−8.485(−4)	
8.119(−2)	5.203(−1)	2.198(−2)	3.550(−1)	−8.439(−4)	
8.139(−2)	5.177(−1)	2.181(−2)	3.534(−1)	−8.407(−4)	
8.150(−2)	5.168(−1)	2.177(−2)	3.531(−1)	−8.398(−4)	(c)*
8.183(−2)	5.141(−1)	2.163(−2)	3.519(−1)	−8.377(−4)	
8.236(−2)	5.054(−1)	2.104(−2)	3.461(−1)	−8.272(−4)	
8.174(−2)	4.992(−1)	2.037(−2)	3.378(−1)	−8.145(−4)	
7.944(−2)	5.007(−1)	1.992(−2)	3.279(−1)	−8.056(−4)	
7.556(−2)	5.119(−1)	1.980(−2)	3.228(−1)	−8.031(−4)	(d)*
7.041(−2)	5.334(−1)	2.003(−2)	3.176(−1)	−8.081(−4)	
6.473(−2)	5.662(−1)	2.063(−2)	3.140(−1)	−8.211(−4)	
5.775(−2)	6.118(−1)	2.162(−2)	3.119(−1)	−8.423(−4)	
5.088(−2)	6.728(−1)	2.304(−2)	3.113(−1)	−8.724(−4)	
4.399(−2)	7.534(−1)	2.497(−2)	3.122(−1)	−9.119(−4)	(e)*

The number in parentheses following an entry is the power of ten by which that entry must be multiplied.

* The meridional shapes are shown in Fig. 2.

constant throughout the sequence. The unit of length R_0 is defined by

$$4\pi R_0^3\rho/3=M\,. \tag{6}$$

The angular velocity and the moment of inertia I are in units of $4\pi G\rho$ and $M^{5/3}\rho^{-2/3}$, respectively. The squared dimensionless angular momentum is defined as

$$j^2=J^2/(4\pi G\rho^{-1/3}M^{10/3}), \tag{7}$$

where J is the total angular momentum. The kinetic energy T and the gravitational energy W are defined as

$$T = I\Omega^2/2\,, \tag{8}$$

$$W = \int d^3\boldsymbol{r}\rho\Phi(r,\mu)/2\,. \tag{9}$$

The normalization factor of the total energy is

$$E_0 = (4\pi G)^2 M^5/J^2. \tag{10}$$

The sequence is plotted on $j^2 - \Omega^2/4\pi G\rho$ plane in Fig. 1. Near th neutral point it extends to the direction of increasing angular momentum but toward still lower angular velocity than the Maclaurin spheroid. This is consistent with Bardeen's[4] computation. These results imply that the point $e = e_{cr}$ on the Maclaurin sequence is not only a neutral point against the $P_4(\eta)$ perturbation but also a bifurcation point. Departing from the Maclaurin spheroids, the sequence goes through the transition region, and then comes close to the Dyson-Wong toroids. In our computation the summation in Eq. (3) was taken upto the term $P_{20}(\mu)$. We have checked that the summation upto $P_{40}(\mu)$ term makes a very small difference. So the small difference from Wong's[2] result may have come from the difference in numerical methods. In Fig. 2 the change of the shapes along the sequence is shown in the meridional plane.

Wong[2] discussed that there was a critical angular momentum below which toroids could not exist and that the transition from a spheroid to a toroid was not continuously connected. His argument was based on the fact that his numerical computation did not converge to yield any solution when the *angular momentum* was below a critical value. This did not necessarily rule out the existence of equilibria that are extending into higher values of *angular velocities*. As seen in

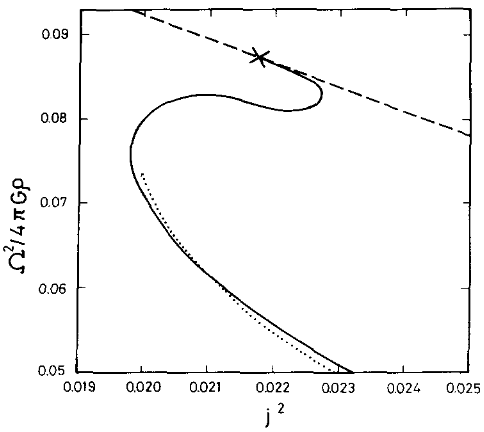

Fig. 1. The squared angular velocity $\Omega^2/4\pi G\rho$ is plotted against $j^2 = J^2/(4\pi G\rho^{-1/3}M^{10/3})$ for our solution (solid curve), for the Maclaurin sequence(dashed curve), and for the Dyson-Wong sequence (dotted curve). The ×-mark denotes the neutral point on the Maclaurin sequence against the $P_4(\eta)$ perturbation. The dotted curve is plotted by using the values which are read from the curve of Fig. 6 of Wong's[2] paper, so it may contain errors to some extent.

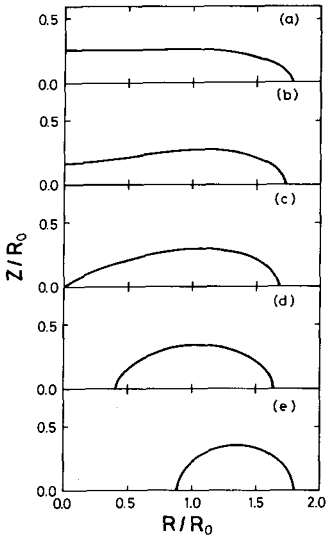

Fig. 2. The change of the meridional shape along the sequence. The cylindrical coordinate $(R,\ Z,\ \phi)$ is used. Physical quantites of each shape (a) through (e) are summarized in corresponding row in Table I.

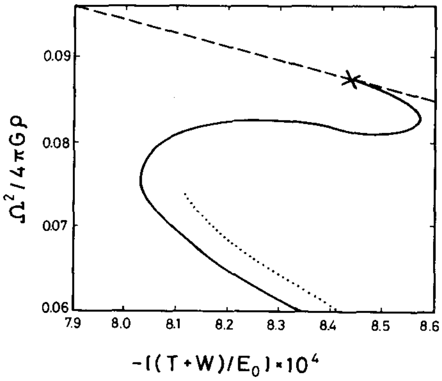

Fig. 3. The squared angular velocity $\Omega^2/4\pi G\rho$ is plotted against the non-dimensional gravitational binding energy $-(T+W)/E_0$. The solid curve is for our sequence, and the dashed curve and the dotted curve are for the Maclaurin sequence and Dyson-Wong sequence, respectively. The ×-mark denotes the neutral point as in Fig. 1.

Fig. 1 such equilibria are actually obtained in the present paper. The critical point which Wong argued corresponds to our model with the local minimum of the gravitational binding energy, i. e., $-(T+W)/E_0=8.031\times10^{-4}$ as seen in Fig. 3. (The difference between Wong's results and ours seems rather large at the first glance, but in reality it is only one percent or so of the binding energy.) This point is also the local minimum of the angular momentum. Above this "critical" value of the binding energy, there are two equilibrium configurations with different values of the angular velocity for the same value of the binding energy. Therefore, the sequence changes from spheroids to toroids continuously.

Acknowledgments

Numerical computations were carried out at the Computer Centre of the University of Tokyo. This research was supported in parts by the Scientific Research Fund of the Ministry of Education, Science and Culture (434010 and 564088).

References

1) See, e. g., S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium* (Yale Univ. Press, New Haven, 1969).
2) C. Y. Wong, Astrophys, J. **190** (1974), 675.

3) S. Chandrasekhar, Astrophys. J. **147** (1967), 334.
4) J. M. Bardeen, Astrophys. J. **167** (1971), 425.
5) T. Fukushima, Y. Eriguchi, D. Sugimoto, G. S. Bisnovatyi-Kogan, Prog. Theor. Phys. **63** (1980), 1957.