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Another generalization of Euler's arithmetic function and Menon's identity

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Abstract
We define the k-dimensional generalized Euler function $\varphi_k(n)$ as the number of ordered k-tuples $(a_1, \ldots, a_k) \in \mathbb{N}^k$ such that $1 \le a_1, \ldots, a_k \le n$ and both the product $a_1 \cdots a_k$ and the sum $a_1 + \cdots + a_k$ are prime to n. We investigate some of the properties of the function $\varphi_k(n)$, and obtain a corresponding Menon-type identity.

Keywords Euler's arithmetic function · Menon's identity · Asymptotic formula

Mathematics Subject Classification 11A07 · 11A25 · 11N37

1 Motivation

Jordan's arithmetic function $J_k(n)$ is defined as the number of ordered k-tuples $(a_1, \ldots, a_k) \in \mathbb{N}^k$ such that $1 \le a_1, \ldots, a_k \le n$ and the gcd $(a_1, \ldots, a_k, n) = 1$. It is well known that $J_k(n)$ is multiplicative in n and $J_k(n) = n^k \prod_{p \mid n}(1 - 1/p^k)$. If $k = 1$, then $J_1(n) = \varphi(n)$ is Euler's arithmetic function.

A Menon-type identity concerning the function $J_k(n)$, obtained by Nageswara Rao [7], is given by

$$\sum_{\substack{a_1,\ldots,a_k=1 \\ (a_1,\ldots,a_k,n)=1}}^{n} (a_1 - 1, \ldots, a_k - 1, n)^k = J_k(n)\tau(n) \quad (n \in \mathbb{N}), \tag{1.1}$$

where $\tau(n)$ is the number of divisors of n. If $k = 1$, then (1.1) reduces to Menon's original identity [6].

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Euler's arithmetic function and Menon's identity have been generalized in various directions by several authors. See, e.g., the books [5,8], the papers [4,7,9,11,12], and their references.

The function $X(n) = \#\{(a,b) \in \mathbb{N}^2 : 1 \le a, b \le n, (ab, n) = (a+b, n) = 1\}$ is an analog of Euler's φ-function, and was introduced by Arai and Gakuen [2]. It was shown by Carlitz [3] that the function $X(n)$ is multiplicative and

$$X(n) = n^2 \prod_{p \mid n} \left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right) \quad (n \in \mathbb{N}).$$

Note that if n is even, then $X(n) = 0$. The function $X(n)$ can also be given as

$$X(n) = \varphi(n)^2 \sum_{d \mid n} \frac{\mu(d)}{\varphi(d)} \quad (n \in \mathbb{N}),$$

μ denoting the Möbius function. The corresponding Menon-type identity

$$\sum_{\substack{a,b=1 \\ (ab,n)=(a+b,n)=1}}^{n} (a+b-1, n) = X(n)\tau(n) \quad (n \in \mathbb{N}) \tag{1.2}$$

was deduced by Sita Ramaiah [9, Cor. 10.4]. In fact, (1.2) is a corollary of a more general identity involving Narkiwicz-type regular systems of divisors and k-reduced residue systems.

Recently, identity (1.2) was generalized by Ji and Wang [4,12] to residually finite Dedekind domains, by using Narkiwicz-type regular systems of divisors, and to the ring of algebraic integers, concerning Dirichlet characters modulo n, respectively. Note that in paper [12] identity (1.2) is called the "Arai–Carlitz identity." However, Arai and Carlitz only considered the function $X(n)$ and did not deduce such an identity. We refer to (1.2) as the Sita Ramaiah identity.

It is natural to introduce and to study the following k-dimensional generalization of the function $X(n)$, and to ask if the corresponding generalization of the Sita Ramaiah identity is true for it. These were not investigated in the literature, as far as we know. For $k \in \mathbb{N}$ we define the function $\varphi_k(n)$ as

$$\varphi_k(n) := \sum_{\substack{a_1,\ldots,a_k=1 \\ (a_1\cdots a_k,n)=1 \\ (a_1+\cdots+a_k,n)=1}}^{n} 1. \tag{1.3}$$

Note that $\varphi_1(n) = \varphi(n)$ is Euler's function and $\varphi_2(n) = X(n)$ of above. We investigate some of the properties of the function $\varphi_k(n)$, and obtain a corresponding Menon-type identity. Our main results are included in Sect. 2, and their proofs are presented in Sects. 3 and 4.

We will use the following notations: $\mathrm{id}_k(n) = n^k$, $\mathbf{1}(n) = 1$ $(n \in \mathbb{N})$, $\omega(n)$ will denote the number of distinct prime factors of n, and "$*$" the Dirichlet convolution of arithmetic functions.

2 Main results

In this paper we prove the following results.

Theorem 2.1 *For every $k, n \in \mathbb{N}$,*

$$\varphi_k(n) = \varphi(n)^k \prod_{p|n} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}}\right) \quad (2.1)$$

$$= n^k \prod_{p|n} \left(1 - \frac{1}{p}\right) \left(\left(1 - \frac{1}{p}\right)^k - \frac{(-1)^k}{p^k}\right). \quad (2.2)$$

It is a consequence of Theorem 2.1 that the function $\varphi_k(n)$ is multiplicative. Also, $\varphi_k(n) = 0$ if and only if k and n are both even. Further properties of $\varphi_k(n)$ can be deduced. Its average order is given by the next result.

Theorem 2.2 *Let $k \geq 2$ be fixed. Then*

$$\sum_{n \leq x} \varphi_k(n) = \frac{C_k}{k+1} x^{k+1} + O\left(x^k (\log x)^{k+1}\right),$$

where

$$C_k = \prod_p \left(1 + \frac{1}{p^{k+1}} \left(\left(1 - \frac{1}{p}\right)\left((p-1)^k - (-1)^k\right) - p^k\right)\right).$$

Corollary 2.3 $(k = 2)$ *We have*

$$\sum_{n \leq x} X(n) = \frac{C_2}{3} x^3 + O\left(x^2 (\log x)^3\right),$$

where

$$C_2 = \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3}\right) \approx 0.286747.$$

We have the following generalization of Menon's identity.

Theorem 2.4 *Let f be an arbitrary arithmetic function. Then for every $k, n \in \mathbb{N}$,*

$$\sum_{\substack{a_1,\ldots,a_k=1 \\ (a_1\cdots a_k, n)=1 \\ (a_1+\cdots+a_k, n)=1}}^{n} f((a_1 + \cdots + a_k - 1, n)) = \varphi_k(n) \sum_{d|n} \frac{(\mu * f)(d)}{\varphi(d)}. \quad (2.3)$$

Corollary 2.5 ($f(n)=n$) *For every* $k,n\in\mathbb{N}$,

$$\sum_{\substack{a_1,\ldots,a_k=1\\(a_1\cdots a_k,n)=1\\(a_1+\cdots+a_k,n)=1}}^{n}(a_1+\cdots+a_k-1,n)=\varphi_k(n)\tau(n). \tag{2.4}$$

If $k=1$, then (2.4) reduces to Menon's identity and if $k=2$, then it gives the Sita Ramaiah identity (1.2).

3 Proofs of Theorems 2.1 and 2.2

We need the following lemmas.

Lemma 3.1 *Let* $n,d\in\mathbb{N}$, $d\mid n$, *and let* $r\in\mathbb{Z}$. *Then*

$$\sum_{\substack{a=1\\(a,n)=1\\a\equiv r \pmod d}}^{n}1=\begin{cases}\dfrac{\varphi(n)}{\varphi(d)} & \text{if } (r,d)=1,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 3.1 is known in the literature, usually proved by the inclusion–exclusion principle. See, e.g., [1, Th. 5.32]. The following generalization and a different approach of proof are given in our paper [11].

Lemma 3.2 [11, Lemma 2.1] *Let* $n,d,e\in\mathbb{N}$, $d\mid n$, $e\mid n$ *and let* $r,s\in\mathbb{Z}$. *Then*

$$\sum_{\substack{a=1\\(a,n)=1\\a\equiv r \pmod d\\a\equiv s \pmod e}}^{n}1=\begin{cases}\dfrac{\varphi(n)}{\varphi(de)}(d,e) & \text{if } (r,d)=(s,e)=1 \text{ and } (d,e)\mid r-s,\\ 0 & \text{otherwise.}\end{cases}$$

In the case $e=1$, Lemma 3.2 reduces to Lemma 3.1.

We need to define the following slightly more general function than $\varphi_k(n)$:

$$\varphi_k(n,m):=\sum_{\substack{a_1,\ldots,a_k=1\\(a_1\cdots a_k,n)=1\\(a_1+\cdots+a_k,m)=1}}^{n}1. \tag{3.1}$$

If $m=n$, then $\varphi_k(n,n)=\varphi_k(n)$, given by (1.3).

Lemma 3.3 (recursion formula for $\varphi_k(n,m)$) *Let* $k\geq 2$ *and* $m\mid n$. *Then*

$$\varphi_k(n,m)=\varphi(n)\sum_{d\mid m}\frac{\mu(d)}{\varphi(d)}\varphi_{k-1}(n,d).$$

Proof of Lemma 3.3 We have

$$\begin{aligned}
\varphi_k(n,m) &= \sum_{\substack{a_1,\ldots,a_k=1\\(a_1\cdots a_k,n)=1}}^{n} \sum_{d\mid(a_1+\cdots+a_k,m)} \mu(d)\\
&= \sum_{d\mid m}\mu(d) \sum_{\substack{a_1,\ldots,a_k=1\\(a_1\cdots a_k,n)=1\\a_1+\cdots+a_k\equiv 0 \pmod d}}^{n} 1\\
&= \sum_{d\mid m}\mu(d) \sum_{\substack{a_1,\ldots,a_{k-1}=1\\(a_1\cdots a_{k-1},n)=1}}^{n} \sum_{\substack{a_k=1\\(a_k,n)=1\\a_k\equiv -a_1-\cdots-a_{k-1} \pmod d}}^{n} 1.
\end{aligned}$$

By using Lemma 3.1 we deduce that

$$\begin{aligned}
\varphi_k(n,m) &= \sum_{d\mid m}\mu(d) \sum_{\substack{a_1,\ldots,a_{k-1}=1\\(a_1\cdots a_{k-1},n)=1\\(a_1+\cdots+a_{k-1},d)=1}}^{n} \frac{\varphi(n)}{\varphi(d)}\\
&= \varphi(n)\sum_{d\mid m}\frac{\mu(d)}{\varphi(d)} \sum_{\substack{a_1,\ldots,a_{k-1}=1\\(a_1\cdots a_{k-1},n)=1\\(a_1+\cdots+a_{k-1},d)=1}}^{n} 1\\
&= \varphi(n)\sum_{d\mid m}\frac{\mu(d)}{\varphi(d)}\varphi_{k-1}(n,d),
\end{aligned}$$

where $d \mid m$ and $m \mid n$ imply that $d \mid n$. □

Proof of Theorem 2.1 Let $k, n, m \in \mathbb{N}$ such that $m \mid n$. We show that

$$\varphi_k(n,m) = \varphi(n)^k \prod_{p\mid m}\left(1-\frac{1}{p-1}+\frac{1}{(p-1)^2}-\cdots+(-1)^{k-1}\frac{1}{(p-1)^{k-1}}\right). \tag{3.2}$$

By induction on k. If $k = 1$, then $\varphi_1(n,m) = \varphi(n)$, by its definition (3.1). Let $k \geq 2$. Assume that (3.2) holds for $k-1$ and prove it for k. We have, by using Lemma 3.3,

$$\begin{aligned}
\varphi_k(n,m) &= \varphi(n)\sum_{d\mid m}\frac{\mu(d)}{\varphi(d)}\varphi_{k-1}(n,d)\\
&= \varphi(n)\sum_{d\mid m}\frac{\mu(d)}{\varphi(d)}\varphi(n)^{k-1}\prod_{p\mid d}\Bigg(1-\frac{1}{p-1}+\frac{1}{(p-1)^2}-\cdots\\
&\qquad\qquad +(-1)^{k-2}\frac{1}{(p-1)^{k-2}}\Bigg)
\end{aligned}$$

$$= \varphi(n)^k \prod_{p|m} \left(1 + \frac{\mu(p)}{\varphi(p)} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots \right.\right.$$
$$\left.\left. +(-1)^{k-2} \frac{1}{(p-1)^{k-2}}\right)\right)$$
$$= \varphi(n)^k \prod_{p|m} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}}\right),$$

which proves formula (3.2). Now choosing $m = n$, (3.2) gives identity (2.1), which can be rewritten as (2.2). □

Proof of Theorem 2.2 Let $\varphi_k = \mathrm{id}_k * g_k$, that is, $g_k = \varphi_k * \mu\, \mathrm{id}_k$. Here the function $g_k(n)$ is multiplicative and for any prime power p^ν ($\nu \ge 1$),

$$g_k(p^\nu) = \varphi_k(p^\nu) - p^k \varphi_k(p^{\nu-1}).$$

We obtain from (2.2) that for $\nu \ge 2$,

$$g_k(p^\nu) = p^{(\nu-1)k} \left(1 - \frac{1}{p}\right) \left((p-1)^k - (-1)^k\right)$$
$$-p^k p^{(\nu-2)k} \left(1 - \frac{1}{p}\right) \left((p-1)^k - (-1)^k\right) = 0, \tag{3.3}$$

and for $\nu = 1$,

$$g_k(p) = \varphi_k(p) - p^k = \left(1 - \frac{1}{p}\right) \left((p-1)^k - (-1)^k\right) - p^k$$
$$= -(k+1)p^{k-1} + \cdots + (-1)^k k, \tag{3.4}$$

a polynomial in p of degree $k-1$, with leading coefficient $-(k+1)$. Actually, we have

$$-(k+1)p^{k-1} < g_k(p) < 0 \tag{3.5}$$

for every integer $k \ge 2$ and every prime $p \ge 2$. To see this, note that by Lagrange's mean value theorem,

$$k(p-1)^{k-1} < p^k - (p-1)^k < kp^{k-1},$$

and from (3.4) we deduce that

$$g_k(p) > \left(1 - \frac{1}{p}\right) \left(p^k - kp^{k-1} - (-1)^k\right) - p^k$$
$$= -(k+1)p^{k-1} + kp^{k-2} - (-1)^k \left(1 - \frac{1}{p}\right) > -(k+1)p^{k-1}.$$

On the other hand, $p^k - (p-1)^k > 1$, $(p-1)^k - (-1)^k < p^k$ imply that

$$g_k(p) < \left(1 - \frac{1}{p}\right) p^k - p^k < 0.$$

According to (3.5), $|g_k(p)| < (k+1)p^{k-1}$ holds true for every $k \geq 2$ and every $p \geq 2$, and by (3.3) we deduce that

$$|g_k(n)| \leq (k+1)^{\omega(n)} n^{k-1} \quad (n \in \mathbb{N}).$$

To obtain the desired asymptotic formula we apply elementary arguments. We have

$$\begin{aligned}\sum_{n \leq x} \varphi_k(n) &= \sum_{d \leq x} g_k(d) \sum_{\delta \leq x/d} \delta^k \\ &= \sum_{d \leq x} g_k(d) \left(\frac{1}{k+1} \left(\frac{x}{d}\right)^{k+1} + O\left(\left(\frac{x}{d}\right)^k\right)\right) \\ &= \frac{x^{k+1}}{k+1} \sum_{d=1}^{\infty} \frac{g_k(d)}{d^{k+1}} + O\left(x^{k+1} \sum_{d > x} \frac{|g_k(d)|}{d^{k+1}} \right) + O\left(x^k \sum_{d \leq x} \frac{|g_k(d)|}{d^k} \right).\end{aligned}$$

Here the main term is $\frac{C_k}{k+1} x^{k+1}$ by using the Euler product formula. To evaluate the error terms consider the Piltz divisor function $\tau_{k+1}(n)$, representing the number of ordered $(k+1)$-tuples $(a_1, \ldots, a_{k+1}) \in \mathbb{N}^{k+1}$ such that $a_1 \cdots a_{k+1} = n$. We have $\tau_{k+1}(p^\nu) \geq \tau_{k+1}(p) = k+1$ for every prime power p^ν $(\nu \geq 1)$, and $\tau_{k+1}(n) \geq (k+1)^{\omega(n)}$ for every $n \in \mathbb{N}$.

We obtain

$$\sum_{d > x} \frac{|g_k(d)|}{d^{k+1}} \leq \sum_{d > x} \frac{(k+1)^{\omega(d)}}{d^2} \leq \sum_{d > x} \frac{\tau_{k+1}(d)}{d^2} \ll \frac{(\log x)^k}{x},$$

and

$$\sum_{d \leq x} \frac{|g_k(d)|}{d^k} \leq \sum_{d \leq x} \frac{(k+1)^{\omega(d)}}{d} \leq \sum_{d \leq x} \frac{\tau_{k+1}(d)}{d} \ll (\log x)^{k+1},$$

by using known elementary estimates on the Piltz divisor function. See, e.g., [10, Lemma 3]. This completes the proof. □

4 Proof of Theorem 2.4

Let $M_k(n)$ denote the sum on the left-hand side of (2.3). We have by the convolutional identity $f = (\mu * f) * \mathbf{1}$,

$$M_k(n) = \sum_{\substack{a_1,\ldots,a_k=1 \\ (a_1\cdots a_k,n)=1 \\ (a_1+\cdots+a_k,n)=1}}^{n} \sum_{d\mid(a_1+\cdots+a_k-1,n)} (\mu * f)(d) = \sum_{d\mid n}(\mu * f)(d) \sum_{\substack{a_1,\ldots,a_k=1 \\ (a_1\cdots a_k,n)=1 \\ (a_1+\cdots+a_k,n)=1 \\ a_1+\cdots+a_k\equiv 1 \ (\mathrm{mod}\ d)}}^{n} 1$$

$$= \sum_{d\mid n}(\mu * f)(d) \sum_{\substack{a_1,\ldots,a_k=1 \\ (a_1\cdots a_k,n)=1 \\ a_1+\cdots+a_k\equiv 1 \ (\mathrm{mod}\ d)}}^{n} \sum_{\delta\mid(a_1+\cdots+a_k,n)} \mu(\delta),$$

that is

$$M_k(n) = \sum_{d\mid n}(\mu * f)(d) \sum_{\delta\mid n} \mu(\delta) N_k(n,d,\delta), \tag{4.1}$$

where

$$N_k(n,d,\delta) := \sum_{\substack{a_1,\ldots,a_k=1 \\ (a_1\cdots a_k,n)=1 \\ a_1+\cdots+a_k\equiv 1 \ (\mathrm{mod}\ d) \\ a_1+\cdots+a_k\equiv 0 \ (\mathrm{mod}\ \delta)}}^{n} 1.$$

Next we evaluate the sum $N_k(n,d,\delta)$, where $d \mid n$, $\delta \mid n$ are fixed. If $(d,\delta) > 1$, then $N_k(n,d,\delta) = 0$, the empty sum. So, assume that $(d,\delta) = 1$. If $k = 1$, then by using Lemma 3.2 we deduce

$$N_1(n,d,\delta) := \sum_{\substack{a_1=1 \\ (a_1,n)=1 \\ a_1\equiv 1 \ (\mathrm{mod}\ d) \\ a_1\equiv 0 \ (\mathrm{mod}\ \delta)}}^{n} 1 = \begin{cases} \frac{\varphi(n)}{\varphi(d)} & \text{if } \delta = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

since for each term of the sum $\delta \mid a_1$ and $\delta \mid n$, which gives $\delta \mid (a_1, n) = 1$, so $\delta = 1$.

Lemma 4.1 (Recursion formula for $N_k(n,d,\delta)$) *Let $k \geq 2$, $d \mid n$, $\delta \mid n$, $(d,\delta) = 1$. Then*

$$N_k(n,d,\delta) = \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j\mid d} \mu(j) \sum_{t\mid \delta} \mu(t) N_{k-1}(n,j,t). \tag{4.3}$$

Proof of Lemma 4.1 We have

$$N_k(n,d,\delta) = \sum_{\substack{a_1,\ldots,a_{k-1}=1 \\ (a_1\cdots a_{k-1},n)=1}}^{n} \sum_{\substack{a_k=1 \\ (a_k,n)=1 \\ a_k\equiv 1-a_1-\cdots-a_{k-1} \ (\mathrm{mod}\ d) \\ a_k\equiv -a_1-\cdots-a_{k-1} \ (\mathrm{mod}\ \delta)}}^{n} 1.$$

Using that $(d, \delta) = 1$ and applying Lemma 3.2 we deduce that

$$\begin{aligned}
N_k(n, d, \delta) &= \sum_{\substack{a_1,\dots,a_{k-1}=1 \\ (a_1\cdots a_{k-1},n)=1 \\ (a_1+\cdots+a_{k-1}-1,d)=1 \\ (a_1+\cdots+a_{k-1},\delta)=1}}^{n} \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \\
&= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{\substack{a_1,\dots,a_{k-1}=1 \\ (a_1\cdots a_{k-1},n)=1}}^{n} \sum_{j\mid(a_1+\cdots+a_{k-1}-1,d)} \mu(j) \sum_{t\mid(a_1+\cdots+a_{k-1},\delta)} \mu(t) \\
&= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j\mid d} \mu(j) \sum_{t\mid\delta} \mu(t) \sum_{\substack{a_1,\dots,a_{k-1}=1 \\ (a_1\cdots a_{k-1},n)=1 \\ a_1+\cdots+a_{k-1}\equiv 1 \pmod{j} \\ a_1+\cdots+a_{k-1}\equiv 0 \pmod{t}}}^{n} 1 \\
&= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j\mid d} \mu(j) \sum_{t\mid\delta} \mu(t) N_{k-1}(n, j, t).
\end{aligned}$$

□

Lemma 4.2 *Let $k \geq 2$, $d \mid n$, $\delta \mid n$, $(d, \delta) = 1$. Then*

$$\begin{aligned}
N_k(n, d, \delta) = \frac{\varphi(n)^k}{\varphi(d)\varphi(\delta)} &\prod_{p\mid d} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}}\right) \\
\times &\prod_{p\mid\delta} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}}\right). \qquad (4.4)
\end{aligned}$$

Proof of Lemma 4.2 By induction on k. If $k = 2$, then by the recursion (4.3) and (4.2),

$$\begin{aligned}
N_2(n, d, \delta) &= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j\mid d} \mu(j) \sum_{t\mid\delta} \mu(t) N_1(n, j, t) \\
&= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j\mid d} \mu(j) \sum_{\substack{t\mid\delta \\ t=1}} \mu(t) \frac{\varphi(n)}{\varphi(j)} \qquad (4.5) \\
&= \frac{\varphi(n)^2}{\varphi(d)\varphi(\delta)} \sum_{j\mid d} \frac{\mu(j)}{\varphi(j)} = \frac{\varphi(n)^2}{\varphi(d)\varphi(\delta)} \prod_{p\mid d} \left(1 - \frac{1}{p-1}\right). \qquad (4.6)
\end{aligned}$$

Hence, the formula is true for $k = 2$. Assume it holds for $k - 1$, where $k \geq 3$. Then we have, by the recursion (4.3),

$$N_k(n,d,\delta) = \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j\mid d} \mu(j) \sum_{\substack{t\mid\delta \\ (t,j)=1}} \mu(t) \frac{\varphi(n)^{k-1}}{\varphi(j)\varphi(t)}$$
$$\times \prod_{p\mid j} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}}\right)$$
$$\times \prod_{p\mid t} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-3} \frac{1}{(p-1)^{k-3}}\right),$$

where the condition $(t,j)=1$ can be omitted, since $j \mid d$, $t \mid \delta$ and $(d,\delta)=1$. We deduce that

$$N_k(n,d,\delta) = \frac{\varphi(n)^k}{\varphi(d)\varphi(\delta)} \sum_{j\mid d} \frac{\mu(j)}{\varphi(j)} \prod_{p\mid j} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}}\right)$$
$$\times \sum_{t\mid\delta} \frac{\mu(t)}{\varphi(t)} \prod_{p\mid t} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-3} \frac{1}{(p-1)^{k-3}}\right)$$
$$= \frac{\varphi(n)^k}{\varphi(d)\varphi(\delta)} \prod_{p\mid d} \left(1 - \frac{1}{p-1}\left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}}\right)\right)$$
$$\times \prod_{p\mid\delta} \left(1 - \frac{1}{p-1}\left(\frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-3} \frac{1}{(p-1)^{k-3}}\right)\right),$$

giving (4.4), which completes the proof of Lemma 4.2. □

Now we continue the evaluation of $M_k(n)$. According to (4.1) and Lemma 4.2, we have

$$M_k(n) = \sum_{d\mid n} (\mu * f)(d) \sum_{\substack{\delta\mid n \\ (\delta,d)=1}} \mu(\delta) \frac{\varphi(n)^k}{\varphi(d)\varphi(\delta)} \prod_{p\mid d} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}}\right)$$
$$\times \prod_{p\mid\delta} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}}\right)$$

$$= \varphi(n)^k \sum_{d \mid n} \frac{(\mu * f)(d)}{\varphi(d)} \prod_{p \mid d} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}}\right)$$
$$\times \sum_{\substack{\delta \mid n \\ (\delta,d)=1}} \frac{\mu(\delta)}{\varphi(\delta)} \prod_{p \mid \delta} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}}\right),$$

where the inner sum is

$$\prod_{\substack{p \mid n \\ p \nmid d}} \left(1 + \frac{\mu(p)}{\varphi(p)} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}}\right)\right)$$
$$= \prod_{p \mid n} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}}\right)$$
$$\times \prod_{p \mid d} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}}\right)^{-1}.$$

This leads to

$$M_k(n) = \varphi(n)^k \prod_{p \mid n} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}}\right)$$
$$\times \sum_{d \mid n} \frac{(\mu * f)(d)}{\varphi(d)}$$
$$= \varphi_k(n) \sum_{d \mid n} \frac{(\mu * f)(d)}{\varphi(d)},$$

by using (2.1), finishing the proof of Theorem 2.4.

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References

1. Apostol, T.M.: Introduction to Analytic Number Theory. Springer, Berlin (1976)
2. Arai, M., Gakuen, J.: Problem E 1460. Am. Math. Monthly **68**, 294–295 (1961)
3. Carlitz, L.: Solution to Problem E 1460. Am. Math. Monthly **68**, 932–933 (1961)
4. Ji, Ch., Wang, Y.: Another regular Menon-type identity in residually finite Dedekind domains. Acta Math. Hungar. https://doi.org/10.1007/s10474-020-01038-1
5. McCarthy, P.J.: Introduction to Arithmetical Functions. Universitext, Springer (1986)
6. Menon, P.K.: On the sum $\sum (a-1, n)[(a, n) = 1]$. J. Indian Math. Soc. (N.S.) **29**, 155–163 (1965)
7. Nageswara Rao, K.: On Certain Arithmetical Sums, Lecture Notes in Mathematics, vol. 251, pp. 181–192. Springer, Berlin (1972)
8. Sándor, J., Crstici, B.: Handbook of Number Theory, vol. II. Kluwer Academic Publishers, Dordrecht (2004)
9. Sita Ramaiah, V.: Arithmetical sums in regular convolutions. J. Reine Angew. Math. **303/304**, 265–283 (1978)
10. Tóth, L.: The probability that k positive integers are pairwise relatively prime. Fibonacci Q. **40**, 13–18 (2002)
11. Tóth, L.: Short proof and generalization of a Menon-type identity by Li, Hu and Kim, *Taiwanese J. Math.* **23**, 557–561 (2019). https://projecteuclid.org/euclid.twjm/1537927426
12. Wang, Y., Ji, Ch.: A generalization of Arai-Carlitz's identity. Ramanujan J. https://doi.org/10.1007/s11139-019-00236-y