ANOTHER LOOK AT SOBOLEV SPACES

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Dedicated to Alain Bensoussan with esteem and affection

1. Introduction

Our initial concern is to study the limiting behavior of the norms of fractional Sobolev spaces $W^{s,p}, 0 < s < 1$, $1 < p < \infty$, as $s \to 1$. Recall that a commonly used (semi-) norm on $W^{s,p}$, introduced by Gagliardo, is given by

$$\|f\|^p_{W^{s,p}} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+sp}} dxdy$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ (see e.g. Adams [1]). A well-known "defect" of this scale of norms is that $\|f\|_{W^{s,p}}$ does not converge, as $s \nearrow 1$, to $\|f\|_{W^{1,p}}$, given by the (semi-) norm

$$\|f\|^p_{W^{1,p}} = \int_\Omega |\nabla f|^p dx,$$

where $| \ |$ denotes the euclidean norm.

In fact, it is clear that if f is any smooth nonconstant function, then $\|f\|_{W^{s,p}} \to \infty$ as $s \nearrow 1$. The factor$(1-s)^{1/p}$ in front of $\|f\|_{W^{s,p}}$ "rectifies" the situation (see Corollary 2 and Remark 6). This analysis leads us to a new characterization of the Sobolev space $W^{1,p}, 1 < p < \infty$.

First, an easy observation

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Theorem 1. *Assume* $f \in W^{1,p}(\Omega)$, $1 \le p < \infty$ *and let* $\rho \in L^1(\mathbb{R}^N)$, $\rho \ge 0$. *Then*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho(x-y) dx dy \le C \|f\|^p_{W^{1,p}} \|\rho\|_{L^1}$$

where C *depends only on* p *and* Ω.

Next, we take a sequence (ρ_n) of radial mollifiers, i.e.

$$\rho_n(x) = \rho_n(|x|), \quad \rho_n \ge 0, \quad \int \rho_n(x) dx = 1$$

and

$$\lim_{n \to \infty} \int_\delta^\infty \rho_n(r) r^{N-1} dr = 0 \quad \text{for every } \delta > 0.$$

Theorem 2. *Assume* $f \in L^p(\Omega), 1 < p < \infty$. *Then*

$$\lim_{n \to \infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_n(x-y) dx dy = K_{p,N} \|f\|^p_{W^{1,p}},$$

with the convention that $\|f\|_{W^{1,p}} = \infty$ *if* $f \notin W^{1,p}$. *Here* $K_{p,N}$ *depends only on* p *and* N.

When $p = 1$ we have the following variants

Theorem 3. *Assume* $f \in W^{1,1}(\Omega)$. *Then*

$$\lim_{n \to \infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|} \rho_n(x-y) dx dy = K_{1,N} \|f\|_{W^{1,1}},$$

where $K_{1,N}$ *depends only on* N.

Theorem 3'. *Assume* $f \in L^1(\Omega)$. *Then* $f \in BV(\Omega)$ *if and only if*

$$\liminf_{n \to \infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|} \rho_n(x-y) dx dy < \infty,$$

and then

$$C_1\|f\|_{BV} \le \liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy$$
$$\le \limsup_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy \le C_2\|f\|_{BV}. \tag{1}$$

Here C_1 and C_2 depend only on Ω, and

$$\|f\|_{BV} = \int_\Omega |\nabla f| = \sup\left\{\int_\Omega f \text{ div } \varphi \ ; \ \varphi \in C_0^\infty(\Omega;\mathbb{R}^N),\ |\varphi(x)| \le 1 \text{ on } \Omega\right\}.$$

Remark 1. In dimension $N = 1$ we can prove that for every $f \in BV(0,1)$

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy = \int_0^1 |f'|.$$

We do not know whether a similar conclusion holds when $N \ge 2$ (even for a special sequence of mollifiers), i.e., whether

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy = K_{1,N}\int_\Omega |\nabla f|.$$

Here are some simple consequences of the above results (and their proofs), where K denotes various constants depending only on p and N.

Corollary 1. *Assume $f \in W^{1,p}(\Omega)$ with $1 \le p < \infty$. Then*

$$\lim_{n\to\infty} \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)\ dy = K\ \ |\nabla f(x)|^p \ \textit{in}\ L^1(\Omega).$$

Corollary 2. *Assume $f \in L^p(\Omega), 1 < p < \infty$. Then*

$$\lim_{\varepsilon\to 0}\ \varepsilon\ \|f\|^p_{W^{1-\varepsilon,p}} = K\ \ \|f\|^p_{W^{1,p}}.$$

Remark 2. In the special case where $p = 2$ and $\Omega = \mathbb{R}^N$ a similar conclusion follows from the result of Masja and Nagel [5] using the Fourier characterization of H^s.

Corollary 3. *Assume $f \in L^p(\Omega), 1 < p < \infty$. Then*

$$\lim_{\varepsilon \to 0} \ \varepsilon^{-N} \iint\limits_{|x-y|<\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^p} \ dxdy = K\|f\|^p_{W^{1,p}}.$$

Corollary 4. *Assume $f \in L^p(\Omega), 1 < p < \infty$. Then*

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \iint\limits_{|x-y|>\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^{N+p}} dxdy = K\|f\|^p_{W^{1,p}}.$$

Remark 3. P. Mironescu and I. Shafrir [7] have studied related limits, e.g., when $N = 1$ and $f \in BV(0,1)$,

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \iint\limits_{|x-y|>\varepsilon} \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy.$$

The case where $f \in BV(\Omega)$ is not fully satisfactory; we have only partial results, for example

Corollary 5. *Assume $f \in L^1(\Omega)$. Then*

$$\begin{aligned} C_1\|f\|_{BV} &\le \liminf_{\varepsilon \to 0} \varepsilon \int\limits_{\Omega} \int\limits_{\Omega} \frac{|f(x)-f(y)|}{|x-y|^{N+1-\varepsilon}} dxdy \\ &\le \limsup_{\varepsilon \to 0} \varepsilon \int\limits_{\Omega} \int\limits_{\Omega} \frac{|f(x)-f(y)|}{|x-y|^{N+1-\varepsilon}} dxdy \le C_2\|f\|_{BV}. \end{aligned}$$

Remark 4. In particular when $f = \chi_A$ is the characteristic function of a measurable set $A \subset \Omega$ having finite perimeter, then

$$\|\chi_A\|_{BV} \le C \liminf_{\varepsilon \to 0} \varepsilon \int\limits_{\Omega \setminus A} \int\limits_{A} \frac{dxdy}{|x-y|^{N+1-\varepsilon}}.$$

Combining this with the Sobolev inequality yields

$$\big(|A||\Omega \setminus A|\big)^{(N-1)/N} \le C \ \liminf_{\varepsilon \to 0} \varepsilon \int\limits_{\Omega \setminus A} \int\limits_{A} \frac{dxdy}{|x-y|^{N+1-\varepsilon}}.$$

In particular, if A is a measurable subset of $\Omega \subset \mathbb{R}^N, N \geq 1$, such that

$$\int\limits_{\Omega \setminus A} \int\limits_{A} \frac{dxdy}{|x-y|^{N+1}} < \infty,$$

then either $|A| = 0$ or $|\Omega \setminus A| = 0$. This fact was already established in Bourgain, Brezis and Mironescu [2] (Appendix B) with a different proof (see also Bourgain, Brezis and Mironescu [3] and Brezis [5]).

2. Proofs

Proof of Theorem 1. By standard extension we may always assume that $f \in W^{1,p}(\mathbb{R}^N)$ and then there is some constant $C = C(p, \Omega)$ such that

$$\Big(\int\limits_{\mathbb{R}^N} |f(x+h) - f(x)|^p dx \Big)^{1/p} \leq |h| \|f\|_{W^{1,p}(\mathbb{R}^N)} \leq C|h| \|f\|_{W^{1,p}(\Omega)}, \tag{2}$$

for all $f \in W^{1,p}$ and $h \in \mathbb{R}^N$ (see, e.g., Brezis [4], Proposition IX.3). By (2), we obtain

$$\begin{aligned} \int\limits_{\Omega} \int\limits_{\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho(x-y) dxdy &\leq \int\limits_{\mathbb{R}^N} \frac{\rho(h)}{|h|^p} \int\limits_{\mathbb{R}^N} |f(x+h) - f(x)|^p dxdh \\ &\leq C^p \|f\|^p_{W^{1,p}} \int\limits_{\mathbb{R}^N} \rho(h) dh = C^p \|f\|^p_{W^{1,p}} \|\rho\|_{L^1}. \end{aligned}$$

Proof of Theorem 2. For $f \in L^p$, let

$$F_n(x,y) \;=\; \frac{|f(x) - f(y)|}{|x-y|} \rho_n^{1/p}(x-y).$$

Assuming first that $f \in W^{1,p}$, we have to prove that

$$\lim_{n \to \infty} \|F_n\|^p_{L^p} \;=\; K \|f\|^p_{W^{1,p}}, \tag{3}$$

for some $K = K_{p,N}$.

By Theorem 1, we have, for any n and $f, g \in W^{1,p}$,

$$\big| \|F_n\|_{L^p} - \|G_n\|_{L^p} \big| \leq \|F_n - G_n\|_{L^p} \leq C \|f - g\|_{W^{1,p}}, \tag{4}$$

for some constant C independent of n, f and g. Therefore it suffices to establish (3) for f in some dense subset of $W^{1,p}$, e.g., for $f \in C^2(\bar{\Omega})$.

Fix some $f \in C^2(\bar{\Omega})$. Then

$$\frac{|f(x)-f(y)|}{|x-y|} = \left|(\nabla f)(x)\cdot\frac{x-y}{|x-y|}\right| + O(|x-y|).$$

For each fixed $x \in \Omega$, let $R = \text{dist}\,(x, \partial\Omega)$. We have

$$\int_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dy =$$

$$\int_{B(x,R)} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dy + \int_{\Omega\setminus B(x,R)} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dy. \tag{5}$$

Clearly, the last integral in (5) tends to 0 as $n \to \infty$. On the other hand,

$$\begin{aligned}
&\int_{B(x,R)} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dy \\
&= \int_0^R \rho_n(r) \int_{|y-x|=r} \left(\left|(\nabla f)(x)\cdot\frac{x-y}{|x-y|}\right|^p + O(|x-y|^p)\right)d\sigma dr \\
&= \int_0^R \rho_n(r) \int_{|\omega|=r} \left(\left|(\nabla f)(x)\cdot\frac{\omega}{|\omega|}\right|^p + O(r^p)\right)d\sigma dr \\
&= K|\nabla f(x)|^p \int_0^R |S^{N-1}|r^{N-1}\rho_n(r)dr + O(\int_0^R r^{N+p-1}\rho_n(r)dr),
\end{aligned}$$

where $K = K_{p,1} = 1$ for all $p \geq 1$ and for $N \geq 2$, $p \geq 1$,

$$K = K_{p,N} = \frac{1}{|S^{N-1}|}\int_{\omega\in S^{N-1}} |\omega\cdot e|^p d\sigma;$$

here e is any unit vector in $\mathbb{R}^N$.

Therefore,

$$\int_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dy \longrightarrow K|\nabla f(x)|^p, \quad \forall x \in \Omega. \tag{6}$$

If L is such that $|f(x) - f(y)| \leq L|x - y|, \ \ \forall x, y \in \Omega$, then

(7) $$\int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(x - y) dy \leq L^p, \quad \forall x \in \Omega.$$

Hence, for $f \in C^2(\bar{\Omega})$, (3) follows by dominated convergence from (6) and (7).

In order to complete the proof of Theorem 2, it suffices to prove that, if $f \in L^p$ and

(8) $$A_p = \liminf_{n \to \infty} \left[\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(x - y) dx dy \right]^{1/p} < \infty,$$

then $f \in W^{1,p}$. We will use the following

Lemma 1. *Assume $f \in L^1(\mathbb{R}^N)$, $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\rho \in L^1(\mathbb{R}^N)$, ρ radial, $\rho \geq 0$ and let e be any unit vector in $\mathbb{R}^N$. Then*

$$\left| \int_{x \in \mathbb{R}^N} f(x) dx \int_{(y-x)\cdot e \geq 0} \frac{\varphi(y) - \varphi(x)}{|y - x|} \rho(y - x) dy \right| \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|}{|x - y|} |\varphi(y)| \rho(x - y) dx dy.$$

Note that the integral on the l.h.s. makes sense and is finite since $|\varphi(y) - \varphi(x)| \leq L|x - y|$; the integral on the r.h.s. makes sense but is possibly infinite.

Proof. For any $\delta > 0$ set

$$\rho_\delta(t) = \begin{cases} 0 & \text{if } t < \delta \\ \rho(t) & \text{if } t > \delta. \end{cases}$$

It suffices to prove the lemma when ρ is replaced by ρ_δ and then pass to the limit as $\delta \to 0$.

Note that the two functions

$$|f(x)||\varphi(y)| \frac{\rho_\delta(y - x)}{|y - x|} \quad \text{and} \quad |f(x)||\varphi(x)| \frac{\rho_\delta(y - x)}{|y - x|}$$

are integrable on $\mathbb{R}^N \times \mathbb{R}^N$; therefore we have

$$\begin{aligned} I &= \int_{x \in \mathbb{R}^N} f(x) dx \int_{(y-x)\cdot e \geq 0} \frac{\varphi(y) - \varphi(x)}{|y - x|} \rho_\delta(y - x) dy \\ &= \iint_{(y-x)\cdot e \geq 0} f(x)\varphi(y) \frac{\rho_\delta(y - x)}{|y - x|} dx dy - \iint_{(y-x)\cdot e \geq 0} f(x)\varphi(x) \frac{\rho_\delta(y - x)}{|y - x|} dx dy \\ &= I_1 - I_2. \end{aligned}$$

Changing x into y and y into x in I_2 yields

$$I_2 = \iint\limits_{(x-y)\cdot e\geq 0} f(y)\varphi(y)\frac{\rho_\delta(x-y)}{|x-y|}dxdy$$
$$= \iint\limits_{(x-y)\cdot e\leq 0} f(y)\varphi(y)\frac{\rho_\delta(x-y)}{|x-y|}dxdy,$$

where the last equality holds since ρ_δ is radial. Hence we obtain

$$I = \iint\limits_{(y-x)\cdot e\geq 0} \varphi(y)\frac{f(x)-f(y)}{|y-x|}\rho_\delta(y-x)dxdy$$
$$\leq \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|f(x)-f(y)|}{|y-x|}|\varphi(y)|\rho_\delta(x-y)dxdy,$$

which is the desired conclusion.

Proof of Theorem 2 completed. Let $\varphi \in C_0^\infty(\Omega)$ (extended by 0 outside Ω) and let e be a unit vector in $\mathbb{R}^N$. As above, for every $x \in \mathbb{R}^N$,

$$\int\limits_{(y-x)\cdot e\geq 0} \frac{\varphi(y)-\varphi(x)}{|y-x|}\rho_n(y-x)dy \overset{n\to\infty}{\longrightarrow} K\nabla\varphi(x)\cdot e \tag{9}$$

where

$$K = \frac{1}{2|S^{N-1}|}\int\limits_{\omega\in S^{N-1}} |\omega\cdot e|d\sigma = \frac{1}{2}K_{1,N}$$

depends only on N.

Applying Lemma 1 with f replaced by $\bar{f}$,

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x\in\Omega \\ 0, & \text{if } x\notin\Omega, \end{cases}$$

we obtain

$$
\begin{aligned}
J_n &= \Big| \int\limits_{\Omega} f(x)dx \int\limits_{(y-x)\cdot e \geq 0} \frac{\varphi(y)-\varphi(x)}{|y-x|}\rho_n(y-x)dy \Big| \\
&\leq \int\limits_{\mathbb{R}^N} dx \int\limits_{supp\,\varphi} \frac{|f(x)-f(y)|}{|x-y|}|\varphi(y)|\rho_n(x-y)dy \\
&\leq \int\limits_{\Omega} dx \int\limits_{\Omega} \frac{|f(x)-f(y)|}{|x-y|}|\varphi(y)|\rho_n(x-y)dy + \int\limits_{\mathbb{R}^N\setminus\Omega} dx \int\limits_{suppp\,\varphi} |f(y)||\varphi(y)|\frac{\rho_n(x-y)}{|x-y|}dy \\
&= J_{1,n} + J_{2,n}.
\end{aligned}
\tag{10}
$$

By Hölder we have

$$
J_{1,n} \leq \Big(\int\limits_{\Omega}\int\limits_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y) \Big)^{1/p} \|\varphi\|_{L^{p'}}
$$

and

$$
J_{2,n} \leq \frac{1}{d}\|\varphi\|_{L^{p'}}\|f\|_{L^p} \int\limits_{|\xi|>d} \rho_n(\xi)d\xi
$$

where $d = \text{dist }(\mathbb{R}^N \setminus \Omega,\ supp\varphi)$, so that $J_{2,n} \to 0$ as $n \to \infty$.

Passing to the limit in (10) as $n \to \infty$ yields

$$
K\Big| \int\limits_{\Omega} f(x)\big(\nabla\varphi(x)\cdot e\big) \Big| \leq A_p\|\varphi\|_{L^{p'}},
$$

where A_p is defined in (8). Choosing $e = e_i$, $i = 1, 2, ..., N$, we obtain

$$
\Big| \int\limits_{\Omega} f\frac{\partial\varphi}{\partial x_i} \Big| \leq \frac{A_p}{K}\|\varphi\|_{L^{p'}}
$$

and consequently $f \in W^{1,p}$. The proof of Theorem 2 is complete.

Proof of Corollary 1. The conclusion is clear when $f \in C^2(\bar{\Omega})$. For a general $f \in W^{1,p}$, the statement follows by density using (4).

The proof of Theorem 3 is the same as the first part of the proof of Theorem 2, since smooth functions are dense in $W^{1,1}$.

Proof of Theorem 3'. The last inequality in (1) is proved as in Theorem 1. The first inequality in (1) is proved as in the second part of the proof of Theorem 2 (using duality).

In fact, a more precise computation in Lemma 1 yields

$$\Big|\iint\limits_{(y-x)\cdot e\geq 0} \frac{\varphi(y)-\varphi(x)}{|y-x|}\rho(y-x)dxdy\Big| + \Big|\iint\limits_{(y-x)\cdot e\leq 0} \frac{\varphi(y)-\varphi(x)}{|y-x|}\rho(y-x)dxdy\Big|$$
$$\leq \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|f(x)-f(y)|}{|x-y|}|\varphi(y)|\rho(x-y)dxdy.$$

If we proceed as above we then obtain

$$K_{1,N}\Big|\int\limits_{\Omega} f(x)\big(\nabla\varphi(x)\cdot e\big)dx\Big| \leq A_p\|\varphi\|_{L^{p'}}.$$

In particular, when $p=1$, $N=1$ and $\Omega=(0,1)$, we find

$$\Big|\int_0^1 f(x)\varphi'(x)dx\Big| \leq \liminf_{n\to\infty}\int_0^1\int_0^1 \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy, \quad \forall\varphi\in C_0^\infty \text{ with } |\varphi|\leq 1,$$

i.e.,

$$\|f\|_{BV} \leq \liminf_{n\to\infty}\int_0^1\int_0^1 \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy. \tag{11}$$

On the other hand, for every $f\in BV(0,1)$ we have, as in the proof of Theorem 1,

$$\limsup_{n\to\infty}\int_0^1\int_0^1 \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy \leq \|f\|_{BV}. \tag{12}$$

Combining (11) and (12) we see that for every $f\in L^1(0,1)$,

$$\lim_{n\to\infty}\int_0^1\int_0^1 \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy = \|f\|_{BV} \tag{13}$$

which is the content of Remark 1 for $N=1$.

3. The case of a sequence (f_n)

In the previous sections f was a fixed function. Throughout this section we assume that (f_n) is a sequence of functions in $L^p(\Omega)$ satisying the uniform estimate

$$\int_\Omega \int_\Omega \frac{|f_n(x) - f_n(y)|^p}{|x-y|^p} \rho_n(x-y) dx dy \le C_0, \tag{14}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N, 1 \le p < \infty$, and (ρ_n) is a sequence of radial mollifiers. Without loss of generality, we may also assume the normalization condition

$$\int_\Omega f_n(x) dx = 0, \quad \forall n. \tag{15}$$

Theorem 4. *Assume (14), (15) and*

$$\text{for each } n, \text{ the function } t \in (0, \infty) \mapsto \rho_n(t) \text{ is non-increasing .} \tag{16}$$

Then the sequence (f_n) is relatively compact in $L^p(\Omega)$ and (up to a subsequence) we may assume that $f_n \to f$ in $L^p(\Omega)$. Moreover,

a) if $1 < p < \infty$, then $f \in W^{1,p}(\Omega)$ and $\|f\|^p_{W^{1,p}} \le C(p, \Omega) C_0$,

b) if $p = 1$, then $f \in BV(\Omega)$ and $\|f\|_{BV} \le C(\Omega) C_0$.

Remark 5. In view of Theorems 2 and 3, the additional assumption (16) may seem artificial. Actually, it is possible to slightly weaken (16); for example we may assume

$$\rho_n(t) \ge C_1 \rho_n(s), \quad \forall\, n, \quad \forall t \le s, \tag{17}$$

for some C_1 independent of n, t, s.

However, the conclusions of Theorem 4 fail for **general** $\rho_n' s$. We shall give below a counterexample where the sequence (f_n) need not be relatively compact in L^p (Counterexample 2).

Here are two examples of interest

Corollary 6. *For $1 \le p < \infty$, let (f_ε) be a family of functions in $L^p(\Omega)$ such that*

$$\iint_{|x-y|<\varepsilon} \frac{|f_\varepsilon(x) - f_\varepsilon(y)|^p}{|x-y|^p} dxdy \le C_0 \varepsilon^N.$$

Then, up to a subsequence, (f_ε) converges in $L^p(\Omega)$ to some $f \in W^{1,p}(\Omega)$ (for $1 < p < \infty$) or $f \in BV(\Omega)$ (for $p = 1$).

Corollary 7. *For $1 < p < \infty$, let $f_\varepsilon \in W^{1-\varepsilon,p}(\Omega)$. Assume that*

$$\varepsilon \|f_\varepsilon\|^p_{W^{1-\varepsilon,p}} \le C_0.$$

Then, up to a subsequence, (f_ε) converges in $L^p(\Omega)$ (and, in fact, in $W^{1-\delta,p}(\Omega)$, for all $\delta > 0$) to some $f \in W^{1,p}(\Omega)$.

Proof of Theorem 4. The heart of the proof consists of showing that (f_n) is relatively compact in L^p. The rest is done as in the second part of the proof of Theorem 2.

Without loss of generality, we may assume that $\Omega = \mathbb{R}^N$ and that supp $f_n \subset B$, a ball in $\mathbb{R}^N$ of diameter 1. This can be achieved by extending each function f_n by reflection across the boundary in a neighborhood of $\partial\Omega$. Using the monotonicity assumption (16), we see that assumption (14) still holds.

In order to prove compactness in L^p, we rely on the following variant of the Riesz-Fréchet-Kolmogorov theorem(which can be proved combining the arguments in Brezis [4], Théorème IV.25 and Corollaire IV.27) : let, for $\delta > 0, \Phi_\delta$ be the mollifier

$$\Phi_\delta = \frac{1}{|B_\delta(0)|}\chi_{B_\delta(0)}.$$

A sequence (f_n) is relatively compact in $L^p(\Omega)$ if and only if

$$\|f_n\|_{L^p} \le C \tag{18}$$

and

$$\lim_{\delta \to 0} (\limsup_{n\to\infty} \|f_n - f_n * \Phi_\delta\|_{L^p}) = 0. \tag{19}$$

For each n and $t > 0$, let

$$F_n(t) = \int\limits_{\omega\in S^{N-1}} \int\limits_{\mathbb{R}^N} |f_n(x+t\omega) - f_n(x)|^p dx d\sigma$$
$$= \frac{1}{t^{N-1}} \int\limits_{|h|=t} \int\limits_{\mathbb{R}^N} |f_n(x+h) - f_n(x)|^p dx d\sigma.$$

Using the triangle inequality, we obtain

$$F_n(2t) \le 2^p F_n(t). \tag{20}$$

In terms of F_n, assumption (14) can be expressed as

$$\int_0^1 t^{N-1} \frac{F_n(t)}{t^p} \rho_n(t) dt \le C_0. \tag{21}$$

We claim that

$$\int |f_n(x)|^p dx \ \le \ C \int_0^1 t^{N-1} F_n(t) dt \tag{22}$$

and

$$\int |f_n(x) - (f_n * \Phi_\delta)(x)|^p dx \le C\delta^{-N} \int_0^\delta t^{N-1} F_n(t) dt, \tag{23}$$

for some C independent of n and δ.

We prove for example (23):

$$\begin{aligned}\int |f_n(x) - (f_n * \Phi_\delta)(x)|^p dx &= \int \left| f_n(x) - \frac{1}{|B_1|\delta^N} \int\limits_{|y-x|<\delta} f_n(y) dy \right|^p dx \\ &= \frac{1}{(|B_1|\delta^N)^p} \int \left| \int\limits_{|y-x|<\delta} (f_n(x) - f_n(y)) dy \right|^p dx \\ &\le \frac{1}{|B_1|} \delta^{-N} \iint\limits_{|y-x|<\delta} |f_n(x) - f_n(y)|^p dx dy \\ &= \frac{1}{|B_1|} \delta^{-N} \int\limits_{|h|<\delta} \left(\int |f_n(x+h) - f_n(x)|^p dx \right) dh \\ &= C\delta^{-N} \int_0^\delta t^{N-1} F_n(t) dt.\end{aligned}$$

The proof of (22) is similar, since

$$f_n(x) = f_n(x) - \frac{1}{|B|}\int\limits_B f_n(y)dy.$$

We are going to establish below the key inequality

$$\delta^{-N}\int_0^\delta t^{N-1}\frac{F_n(t)}{t^p}dt \le C\Big(\int_0^\delta t^{N-1}\frac{F_n(t)}{t^p}\rho_n(t)dt\Big)\Big/\Big(\int\limits_{|x|<\delta}\rho_n(x)dx\Big). \tag{24}$$

Assuming (24) has been proved, we proceed as follows : since

$$\lim_{n\to\infty}\int\limits_{|x|<\delta}\rho_n(x)dx = 1,$$

by combining (21) with (24) we find

$$\delta^{-N}\int_0^\delta t^{N-1}\frac{F_n(t)}{t^p}dt \le C \quad \text{for } n \ge n_\delta. \tag{25}$$

In particular, we have

$$\delta^{-N}\int_0^\delta t^{N-1}F_n(t)dt \le C\delta^p \quad \text{for } n \ge n_\delta. \tag{26}$$

Inequalities (18), (19) –and thus the conclusion of Theorem 4– follow from (22), (23) and (26).

It remains to establish inequality (24). Note that it is a particular case ($g(t) = \frac{F_n(t)}{t^p}, h(t) = \rho_n(t)$) of the following variant of an inequality due to Chebyshev:

Lemma 2. *Let $g, h : (0,\delta) \to \mathbb{R}_+$. Assume that $g(t) \le g(t/2), t \in (0,\delta)$, and that h is non-increasing.*

Then, for some $C = C(N) > 0$,

$$\int_0^\delta t^{N-1}g(t)h(t)dt \ge C\delta^{-N}\int_0^\delta t^{N-1}g(t)dt\int_0^\delta t^{N-1}h(t)dt.$$

Proof of Lemma 2. It suffices to consider the case $\delta = 1$; the general case follows by scaling. We have

$$\begin{aligned}\int_0^1 t^{N-1} g(t)h(t)dt &= \sum_{j\geq 0} \int_{1/2^{j+1}}^{1/2^j} t^{N-1} g(t)h(t)dt \\ &= \sum_{j\geq 0} \frac{1}{2^{Nj}} \int_{1/2}^1 s^{N-1} g\left(\frac{s}{2^j}\right) h\left(\frac{s}{2^j}\right) ds \\ &= \int_{1/2}^1 s^{N-1} \sum_{j\geq 0} \frac{1}{2^{Nj}} g\left(\frac{s}{2^j}\right) h\left(\frac{s}{2^j}\right) ds,\end{aligned} \tag{27}$$

and a similar equality holds for $\int\limits_0^1 t^{N-1} g(t)dt$. We recall the classical Chebyshev inequality: if $G, H : X \to \mathbb{R}, \mu$ a positive measure on X and

$$(G(x) - G(y))(H(x) - H(y)) \geq 0, \quad \forall x, y \in X,$$

then

$$\int\limits_X GH d\mu \geq \frac{1}{\mu(X)} \int\limits_X G d\mu \int\limits_X H d\mu.$$

In particular, if $\alpha_j \geq 0$ and the sequences $(a_j), (b_j)$ have the same monotonicity, then

$$\sum \alpha_j a_j b_j \geq \frac{1}{\sum \alpha_j} \sum \alpha_j a_j \sum \alpha_j b_j. \tag{28}$$

Since for each $s \in (1/2, 1)$, the sequences $(g(\frac{s}{2^j}))$ and $(h(\frac{s}{2^j}))$ are non-decreasing, (28) with $\alpha_j = \frac{1}{2^{Nj}}$ yields

$$\sum_{j\geq 0} \frac{1}{2^{Nj}} g\left(\frac{s}{2^j}\right) h\left(\frac{s}{2^j}\right) \geq C \sum_{j\geq 0} \frac{1}{2^{Nj}} g\left(\frac{s}{2^j}\right) \sum_{j\geq 0} \frac{1}{2^N j} h\left(\frac{s}{2_j}\right). \tag{29}$$

Now clearly, for each $s \in (1/2, 1)$ and each $j \geq 1$,

$$\frac{1}{2^{Nj}} h\left(\frac{s}{2^j}\right) \geq \frac{1}{2^{Nj}} h\left(\frac{1}{2^j}\right) \geq C \int_{1/2^j}^{1/2^{j-1}} t^{N-1} h(t)dt,$$

for some C depending only on N, so that

$$\sum_{j\geq 0} \frac{1}{2^{Nj}} h\left(\frac{s}{2^j}\right) \geq C \int_0^1 t^{N-1} h(t)dt. \tag{30}$$

It follows from (29) and (30) that

$$\sum_{j\geq 0}\frac{1}{2^{Nj}}g\left(\frac{s}{2^j}\right)h\left(\frac{s}{2^j}\right)\geq C\int_0^1 t^{N-1}h(t)dt\sum_{j\geq 0}\frac{1}{2^{Nj}}g\left(\frac{s}{2^j}\right). \tag{31}$$

Inserting (31) into (27), we find

$$\begin{aligned}\int_0^1 t^{N-1}g(t)h(t)dt &\geq C\int_0^1 t^{N-1}h(t)dt\int_{1/2}^1 s^{N-1}\sum_{j\geq 0}\frac{1}{2^{Nj}}g\left(\frac{s}{2^j}\right)ds\\ &= C\int_0^1 t^{N-1}h(t)dt\ \int_0^1 t^{N-1}g(t)dt.\end{aligned}$$

The proof of Theorem 4 is complete.

Returning to Corollary 7, we still have to prove that, for any fixed $\delta>0$, we have, for small $\varepsilon>0$,

$$\|f_\varepsilon\|_{W^{1-\delta,p}}\leq C.$$

Considering the same functions $F_\varepsilon(t)$ as above (relative to the parameter ε instead of n) we have to prove that

$$\int_0^1\frac{F_\varepsilon(t)}{t^{(1-\delta)p+1}}dt\leq C,\ \text{for small } \varepsilon>0, \tag{32}$$

under the assumption

$$\varepsilon\int_0^1\frac{F_\varepsilon(t)}{t^{(1-\varepsilon)p+1}}dt\ \leq C. \tag{33}$$

The proof of (32) is similar to that of Lemma 2, so we just sketch it. We start by rewriting (32) and (33) as

$$\int_0^1\frac{1}{t^{1-\delta p}}\frac{F_\varepsilon(t)}{t^p}dt\leq C \tag{34}$$

and

$$\int_0^1\frac{1}{t^{1-\delta p}}\frac{F_\varepsilon(t)}{t^p}\frac{\varepsilon}{t^{(\delta-\varepsilon)p}}dt\leq C. \tag{35}$$

We apply Lemma 2 with $\delta = 1$, $N = \delta p$, $g(t) = \frac{F_\varepsilon(t)}{t^p}$, $h(t) = \frac{\varepsilon}{t^{(\delta-\varepsilon)p}}$, and take $0 < \varepsilon < \delta$. We find

$$\int_0^1 \frac{1}{t^{1-\delta p}} \frac{F_\varepsilon(t)}{t^p} \frac{\varepsilon}{t^{(\delta-\varepsilon)p}} dt \geq C \int_0^1 \frac{1}{t^{1-\delta p}} \frac{F_\varepsilon(t)}{t^p} dt \tag{36}$$

for some C depending on δ and p, but not on ε.

Remark 6. If we renorm the $W^{s,p}(\Omega)$ spaces by

$$|f|^p_{W^{s,p}} = \begin{cases} (1-s)\|f\|^p_{W^{s,p}}, & 0 < s < 1 \\ \|f\|^p_{W^{1,p}}, & s = 1, \end{cases}$$

the above computation yields

$$|f|_{W^{\sigma,p}} \leq C|f|_{W^{s,p}}, \quad 0 < \sigma < s \leq 1$$

for some constant C **independent of s and σ**.

Counterexample 1: a sequence (f_n) unbounded in L^p and a sequence of radial mollifiers(ρ_n) such that

$$\int_\Omega \int_\Omega \frac{|f_n(x) - f_n(y)|^p}{|x-y|^p} \rho_n(x-y) dx dy \leq C. \tag{37}$$

We take $\Omega = (0,1)$. Fix some function $f \in L^p_{loc}(\mathbb{R})$, non-constant, periodic of period 1, such that

$$\int_0^1 f(x)dx = 0 \text{ (e.g. } , f(x) = \sin (2\pi x)).$$

Define $g_n(x) = f(nx)$, so that $\|g_n\|^p_{L^p(\Omega)} = \int_0^1 |f(x)|^p dx = C$.

Clearly, $\int_0^1 |g_n(x \pm \frac{1}{n}) - g_n(x)|^p dx = 0$. Since the translations are continuous in L^p, we may find some $0 < \delta_n < \frac{1}{2n}$ such that $\int_0^1 |g_n(x+h) - g_n(x)|^p dx \leq \frac{1}{n^{2p}}$ for $|h \pm \frac{1}{n}| < \delta_n$.

Let $\rho_n = \frac{1}{4\delta_n}(\chi_{(\frac{1}{n}-\delta_n, \frac{1}{n}+\delta_n)} + \chi_{(-\frac{1}{n}-\delta_n, -\frac{1}{n}+\delta_n)})$. Then clearly

$$\int_\Omega \int_\Omega \frac{|g_n(x) - g_n(y)|^p}{|x-y|^p} \rho_n(x-y) dx dy \leq \frac{C}{n^p}.$$

Finally, the functions $f_n = n g_n$ satisfy the desired inequality (37) and $\|f_n\|_{L^p(\Omega)} \sim n$.

Counterexample 2: the sequence (g_n) constructed above is bounded in L^p, is not relatively compact in L^p, and yet it satisfies

$$\int_\Omega \int_\Omega \frac{|g_n(x) - g_n(y)|^p}{|x-y|^p} \rho_n(x-y) dx dy \leq C.$$

References

[1]. R. A. Adams, *Sobolev spaces*, Acad. Press (1975).
[2]. J. Bourgain, H. Brezis and P. Mironescu, *Lifting in Sobolev spaces*, J. d'Analyse **80** (2000), 37–86.
[3]. J. Bourgain, H. Brezis and P. Mironescu, in preparation.
[4]. H. Brezis, *Analyse fonctionnelle*, Masson, Paris (1983).
[5]. H. Brezis (with the assistance of P. Mironescu), *Lectures on the Ginzburg-Landau Vortices*, in preparation.
[6]. W. Masja and J. Nagel, *Über äquivalente Normierung der anisotropen Funktionalräume* $H^{\mu}(\mathbb{R}^n)$, Beiträge zur Analysis **12** (1978), 7-17.
[7]. P. Mironescu and I. Shafrir, in preparation.

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