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ANOTHER PROOF OF THE DEFECT RELATION FOR MOVING TARGETS

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1. Introduction. The second main theorem and the defect relation of slow moving targets were discussed in [7], where Stoll gave the bound $n(n+1)$ for the sums of defects. The author generalized this result in [5] and gave in [6] examples of holomorphic mappings and moving targets which have the bound $n+1$. Ru and Stoll [3] then gave the bound $n+1$ in the general case. Since their proof is complicated, however, we give a simpler proof of Ru-Stoll's theorem in this paper.

2. Statement of the result. Let f be a holomorphic mapping of $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$. Let $\tilde{f}=(f_0, \cdots, f_n)$ be its reduced representation, i.e., $\tilde{f}$ is a holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{C}^{n+1}-\{0\}$. Fix $r_0>0$. We define the characteristic function $T(f; r)$ of f by

$$T(f; r)=\frac{1}{2\pi}\int_0^{2\pi}\log\|\tilde{f}(re^{i\theta})\|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log\|\tilde{f}(r_0e^{i\theta})\|d\theta$$

for $r>r_0$. In particular, the characteristic function of a meromorphic function is defined as that of the corresponding holomorphic mapping of $\boldsymbol{C}$ into $P^1(\boldsymbol{C})$.

For $q\geq n$, let g_j be $q+1$ holomorphic mappings of $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ with reduced representations $\tilde{g}_j=(g_{j0}, \cdots, g_{jn})$ $(0\leq j\leq q)$. Assume that the following conditions are satisfied:

(1) $T(g_j; r)=o(T(f; r))$ as $r\to\infty$ $(0\leq j\leq q)$;

(2) g_j $(0\leq j\leq q)$ are in general position, i.e., for any $j_0, \cdots, j_n$ with $0\leq j_0< \cdots < j_n\leq q$,

$$\det(g_{j_k l})_{0\leq k,l\leq n}\not\equiv 0 .$$

By (2), we may assume that $g_{j0}\not\equiv 0$ $(0\leq j\leq q)$ by changing the homogeneous coordinate system of $P^n(\boldsymbol{C})$ if necessary. Then put $\zeta_{jk}=g_{jk}/g_{j0}$ with $\zeta_{j0}\equiv 1$. Let $\mathfrak{K}$ be the smallest subfield containing $\{\zeta_{jk}\,|\,0\leq j\leq q, 0\leq k\leq n\}\cup\boldsymbol{C}$ of the meromorphic function field on $\boldsymbol{C}$. It is easy to check that $T(h; r)=o(T(f; r))$ as $r\to\infty$ for all $h\in\mathfrak{K}$. Furthermore, we assume

(3) f is non-degenerate over $\mathfrak{K}$, i.e., $f_0, \cdots, f_n$ are linearly independent over $\mathfrak{K}$. Put $h_j=g_{j0}f_0+\cdots+g_{jn}f_n$. Then the counting function of g_j for f is defined by

$$N(f, g_j; r)=\frac{1}{2\pi}\int_0^{2\pi}\log|h_j(re^{i\theta})|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log|h_j(r_0e^{i\theta})|d\theta$$

for $r > r_0$. The defect of g_j for f is defined by

$$\delta(f, g_j) = \liminf_{r \to \infty}\left(1 - \frac{N(f, g_j; r)}{T(f; r)}\right).$$

In this situation, Ru and Stoll proved:

THEOREM (Defect relation).

$$\sum_{j=0}^{q} \delta(f, g_j) \leq n+1 .$$

3. Proof of Theorem. Let p be a positive integer. Let $\mathfrak{L}(p)$ be the vector space generated over $\boldsymbol{C}$ by $\{\prod_{\substack{0 \leq j \leq q \\ 0 \leq k \leq n}} \zeta_{jk}^{p_{jk}} \mid p_{jk}$ non-negative integers with $\sum_{\substack{0 \leq j \leq q \\ 0 \leq k \leq n}} p_{jk} = p\}$. Since $\zeta_{j0} = 1$, we have $\mathfrak{L}(p) \subset \mathfrak{L}(p+1)$. Thus we can take a basis $\{b_1, \cdots, b_t\}$ of $\mathfrak{L}(p+1)$ such that $\{b_1, \cdots, b_s\}$ is a basis of $\mathfrak{L}(p)$, where $t = \dim \mathfrak{L}(p+1)$ and $s = \dim \mathfrak{L}(p)$. By (3), we can deduce that $b_j f_k$ $(1 \leq j \leq t, 0 \leq k \leq n)$ are linearly independent over $\boldsymbol{C}$. Put $F_k = h_k / g_{k0}$ for $0 \leq k \leq n$.

First, we prove that $b_j F_k$ $(1 \leq j \leq s, 0 \leq k \leq n)$ are linearly independent over $\boldsymbol{C}$. Assume that $\sum_{\substack{1 \leq j \leq s \\ 0 \leq k \leq n}} c_{jk} b_j F_k \equiv 0$ with $c_{jk} \in \boldsymbol{C}$. Then

$$\sum_{l=0}^{n}\left(\sum_{\substack{1 \leq j \leq s \\ 0 \leq k \leq n}} c_{jk} b_j \zeta_{kl}\right) f_l \equiv 0 .$$

Since f is non-degenerate over $\mathfrak{R}$, we have

$$\sum_{\substack{1 \leq j \leq s \\ 0 \leq k \leq n}} c_{jk} b_j \zeta_{kl} \equiv 0 \qquad (0 \leq l \leq n) .$$

These are expressed in terms of matrices as

$$\left(\sum_{1 \leq j \leq s} c_{j0} b_j, \cdots, \sum_{1 \leq j \leq s} c_{jn} b_j\right)(\zeta_{jk})_{0 \leq j, k \leq n} \equiv (0, \cdots, 0) .$$

By the condition (2), $\det(\zeta_{jk})_{0 \leq j, k \leq n} \not\equiv 0$, hence we have

$$\sum_{1 \leq j \leq s} c_{jk} b_j \equiv 0 \qquad (0 \leq k \leq n) .$$

Since $b_1, \cdots b_s$ are linearly independent over $\boldsymbol{C}$, we obtain $c_{jk} = 0$ $(1 \leq j \leq s,\ 0 \leq k \leq n)$. Hence we conclude that $b_j F_k$ $(1 \leq j \leq s, 0 \leq k \leq n)$ are linearly independent over $\boldsymbol{C}$.

Since $b_j F_k$ $(1 \leq j \leq s, 0 \leq k \leq n)$ are linear combinations of $b_j f_k$ $(1 \leq j \leq t, 0 \leq k \leq n)$ over $\boldsymbol{C}$, we can choose $\beta_{mj}^{kl} \in \boldsymbol{C}$ so that there exists $C \in GL((n+1)t; \boldsymbol{C})$ such that

$$(b_j F_k\ (1 \leq j \leq s, 0 \leq k \leq n),\ h_{mj}\ (s+1 \leq j \leq t, 0 \leq m \leq n)) = (b_j f_k\ (1 \leq j \leq t, 0 \leq k \leq n))C ,$$

where $h_{mj}=\sum_{1\le k\le t, 0\le l\le n}\beta_{mj}^{kl}b_k f_l$ $(s+1\le j\le t, 0\le m\le n)$. Then we have an equality of Wronskian determinants

$$\begin{aligned}&W(b_jF_k\,(1\le j\le s, 0\le k\le n), h_{mj}\,(s+1\le j\le t, 0\le m\le n))\\&\quad=W(b_jf_k\,(1\le j\le t, 1\le k\le n))\cdot\det C\,.\end{aligned}$$

Take a multi-index $\alpha=(\alpha_0,\cdots,\alpha_n)$ with distinct $\alpha_0,\cdots,\alpha_n\in\{0,\cdots,q\}$. We apply the above argument to $F_{\alpha_0},\cdots,F_{\alpha_n}$ instead of $F_0,\cdots,F_n$. Then we denote h_{mj}^{α} for h_{mj} and C_α $(\in \boldsymbol{C}-\{0\})$ for det C. Put

$$W_\alpha=W(b_jF_{\alpha_k}\,(1\le j\le s, 0\le k\le n), h_{mj}^{\alpha}\,(s+1\le j\le t, 0\le m\le n))$$

and

$$W=W(b_jf_k\,(1\le j\le t, 0\le k\le n))\,.$$

Since b_jf_k $(1\le j\le t, 0\le k\le n)$ are linearly independent over $\boldsymbol{C}$, we have $W\not\equiv 0$. Then we have

$$W_\alpha=C_\alpha W\,. \tag{4}$$

For any fixed $z\in\boldsymbol{C}$, we take distinct indices $\alpha_0,\cdots,\alpha_n=\beta_0,\cdots,\beta_{q-n}$ such that

$$|F_{\alpha_0}(z)|\le\cdots\le|F_{\alpha_n}(z)|\le|F_{\beta_1}(z)|\le\cdots\le|F_{\beta_{q-n}}(z)|\le\infty\,. \tag{5}$$

Then we have

$$\log\|\tilde{f}(z)\|\le\log|F_{\beta_j}(z)|+\log^+A(z)\,. \tag{6}$$

for $j=0,\cdots,q-n$, where

$$\int_0^{2\pi}\log^+A(re^{i\theta})d\theta=o(T(f;r)) \tag{7}$$

and $\log^+x=\max(0,\log x)$ for $x\ge 0$. Indeed, let $\gamma_0,\cdots,\gamma_n$ be distinct integers with $0\le\gamma_0,\cdots,\gamma_n\le q$. Then the equalities

$$F_{\gamma_j}=\zeta_{\gamma_j0}f_0+\cdots+\zeta_{\gamma_jn}f_n\quad\text{for}\quad j=0,\cdots,n$$

and (2) admit the representations

$$f_k=\sum_{j=0}^{n}A_{kj}^{\gamma}F_{\gamma_j}\quad\text{for}\quad k=0,\cdots,n\,,$$

where $A_{kj}^{\gamma}\in\mathfrak{R}$ and γ is the multi-index $(\gamma_0,\cdots,\gamma_n)$. Therefore we have

$$|f_k(z)|\le\sum_{j=0}^{n}|A_{kj}^{\alpha}(z)||F_{\beta_l}(z)|\quad\text{for}\quad k=0,\cdots,n\text{ and }l=0,\cdots,q-n$$

by (5), where $\alpha=(\alpha_0,\cdots,\alpha_n)$ and hence

$$\|\tilde{f}(z)\| \leq \sum_{0 \leq k, j \leq n} |A^{\alpha}_{kj}(z)| \, |F_{\beta_l}(z)| \quad \text{for} \quad l=0, \cdots, q-n\,.$$

Here if we put $A=\sum_{\gamma}\sum_{0\leq k,j\leq n}|A^{\gamma}_{kj}|$, where γ ranges over the set $\{\gamma=(\gamma_0, \cdots, \gamma_n) \mid \gamma_0, \cdots, \gamma_n$ distinct and $0\leq\gamma_0, \cdots, \gamma_n\leq q\}$, then we have (7) because of $A^{\gamma}_{kj}\in\mathfrak{R}$ and the concavity of $\log^+$. Now (6) is clear.

By (4), we obtain

$$\text{(8)} \qquad \log\frac{|F_0 \cdots F_q|^s}{|W|} = \log|F_{\beta_1} \cdots F_{\beta_{q-n}}|^s - \log\frac{|W_\alpha|}{|F_{\alpha_0} \cdots F_{\alpha_n}|^s} + c_1$$

$$= \log|F_{\beta_1} \cdots F_{\beta_{q-n}}|^s - \log\frac{|W_\alpha|}{|F_{\alpha_0} \cdots F_{\alpha_n}|^s \|\tilde{f}\|^{(n+1)(t-s)}}$$

$$-(n+1)(t-s)\log\|\tilde{f}\| + c_1$$

for some constant c_1. We put

$$D_\alpha = \frac{|W_\alpha|}{|F_{\alpha_0} \cdots F_{\alpha_n}|^s \|\tilde{f}\|^{(n+1)(t-s)}}\,.$$

Then we obtain

$$\text{(9)} \qquad \int_0^{2\pi} \log^+ D_\alpha(re^{i\theta})d\theta = S(f; r)$$

by the lemma of logarithmic derivatives and the concavity of $\log^+$, where $S(f; r)$ is a quantity which satisfies

$$\text{(10)} \qquad \lim_{r\to\infty,\, r\notin E} S(f; r)/T(f; r) = 0$$

for some subset E of (r_0, ∞) of finite Lebesgue measure. By (8) we have

$$\text{(11)} \qquad \log|F_{\beta_1} \cdots F_{\beta_{q-n}}|^s \leq \log\frac{|F_0 \cdots F_q|^s}{|W|} + \log^+ D_\alpha + (n+1)(t-s)\log\|\tilde{f}\| + c_1\,.$$

By (6) and (11) we get an inequality

$$\text{(12)} \qquad s(q-n)\log\|\tilde{f}\| \leq \log\frac{|F_0 \cdots F_q|^s}{|W|} + \sum_\alpha \log^+ D_\alpha + (n+1)(t-s)\log\|\tilde{f}\|$$

$$+ c_2\log^+ A + c_3$$

on $\boldsymbol{C}$ for some constants c_2 and c_3. By integrating this inequality over the circle $\{z\in\boldsymbol{C} \mid |z|=r\}$ $(r>r_0)$, we obtain

$$s(q-n)T(f;r) \leq s\sum_{j=0}^{q} N(f, g_j; r) + S(f; r) + (n+1)(t-s)T(f; r) + o(T(f; r)) .$$

Therefore we have

$$\sum_{j=0}^{q}\left(1-\frac{N(f, g_j; r)}{T(f; r)}\right) \leq n+1+(n+1)\left(\frac{t}{s}-1\right)+\frac{S(f; r)}{T(f; r)}$$

and hence

$$\sum_{j=0}^{q} \delta(f, g_j) \leq n+1+(n+1)\left(\frac{t}{s}-1\right).$$

By Steinmetz' lemma (cf. [7, Lemma 3.12]), we have

$$\liminf_{p\to\infty} \frac{t}{s} = 1 .$$

Thus we have the defect relation

$$\sum_{j=0}^{q} \delta(f, g_j) \leq n+1 .$$

REMARK. In the situation of § 3, we put

$$N_p(r) = \frac{1}{2\pi}\int_0^{2\pi} \log |W(re^{i\theta})| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log |W(r_0 e^{i\theta})| d\theta ,$$

$\Theta_p = \liminf_{r\to\infty} N_p(r)/T(f; r)$ and $\Theta = \liminf_{p\to\infty} \Theta_p/s$. Then we have

$$\sum_{j=0}^{q} \delta(f, g_j) + \Theta \leq n+1$$

by the inequality (12). It is easy to see that $0 \leq \Theta \leq n+1$. If all ζ_{jk} are constants, then W is the Wronskian determinant of $f_0, \cdots, f_n$ for all p, and Θ can take various values.

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