Another proof of the global F-regularity of Schubert varieties

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Abstract

Recently, Lauritzen, Raben-Pedersen and Thomsen proved that Schubert varieties are globally F-regular. We give another proof simpler than the original one.

1. Introduction

Let p be a prime number, k an algebraically closed field of characteristic p, and G a simply connected, semisimple affine algebraic group over k. Let T be a maximal torus of G. We choose a basis Δ of the root system of G. Let B be the negative Borel subgroup of G, and P a parabolic subgroup of G containing B. Then the closure of a B-orbit on G/P is called a Schubert variety.

Recently, Lauritzen, Raben-Pedersen and Thomsen [12] proved that Schubert varieties are globally F-regular, utilizing Bott-Samelson resolution. The objective of this paper is to give another proof of this. Our proof depends on a simple inductive argument utilizing the familiar technique of fibering the Schubert variety as a $\mathbb{P}^1$-bundle over a smaller Schubert variety.

Global F-regularity was first defined by Smith [19]. A projective variety over k is said to be globally F-regular if it admits a strongly F-regular homogeneous coordinate ring. As a corollary, all local rings of a Schubert variety are F-regular, in particular, are F-rational, Cohen-Macaulay and normal.

A globally F-regular variety is Frobenius split. It has long been known that Schubert varieties are Frobenius split [14]. Given an ample line bundle over G/P, the associated projective embedding of a Schubert variety of G/P is projectively normal [16] and arithmetically Cohen-Macaulay [17]. We can prove that the coordinate ring is strongly F-regular indeed.

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Over globally F-regular varieties, there are nice vanishing theorems, one of which yields a short proof of Demazure's vanishing theorem.

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2. Preliminaries

Let p be a prime number, and k an algebraically closed field of characteristic p. For a ring A of characteristic p, the Frobenius map $A \to A$ ($a \mapsto a^p$) is denoted by F or F_A. So F_A^e maps a to a^{p^e} for $a \in A$ and $e \geq 0$.

Let A be a k-algebra. For $r \in \mathbb{Z}$, we denote by $A^{(r)}$ the ring A with the k-algebra structure given by

$$k \xrightarrow{F_k^{-r}} k \to A.$$

Note that $F_A^e \colon A^{(r+e)} \to A^{(r)}$ is a k-algebra map for $e \geq 0$ and $r \in \mathbb{Z}$. For $a \in A$ and $r \in \mathbb{Z}$, the element a viewed as an element in $A^{(r)}$ is occasionally denoted by $a^{(r)}$. So $F_A^e(a^{(r+e)}) = (a^{(r)})^{p^e}$ for $a \in A$, $r \in \mathbb{Z}$ and $e \geq 0$.

Similarly, for a k-scheme X and $r \in \mathbb{Z}$, the k-scheme $X^{(r)}$ is defined. The Frobenius morphism $F_X^e \colon X^{(r)} \to X^{(r+e)}$ is a k-morphism.

A k-algebra A is said to be F-finite if the Frobenius map $F_A \colon A^{(1)} \to A$ is finite. A k-scheme X is said to be F-finite if the Frobenius morphism $F_X \colon X \to X^{(1)}$ is finite. Let A be an F-finite Noetherian k-algebra. We say that A is strongly F-regular if for any non-zerodivisor $c \in A$, there exists some $e \geq 0$ such that $cF_A^e \colon A^{(e)} \to A$ ($a^{(e)} \mapsto ca^{p^e}$) is a split monomorphism as an $A^{(e)}$-linear map [6]. A strongly F-regular F-finite ring is F-rational in the sense of Fedder-Watanabe [3], and is Cohen-Macaulay normal.

Let X be a quasi-projective k-variety. We say that X is globally F-regular if for any invertible sheaf $\mathcal{L}$ over X and any $a \in \Gamma(X, \mathcal{L}) \setminus 0$, the composite

$$\mathcal{O}_{X^{(e)}} \to F_*^e \mathcal{O}_X \xrightarrow{F_*^e a} F_*^e \mathcal{L}$$

has an $\mathcal{O}_{X^{(e)}}$-linear splitting [19], [5]. X is said to be F-regular if $\mathcal{O}_{X,x}$ is strongly F-regular for any closed point x of X.

Smith [19, (3.10)] proved the following fundamental theorem on global F-regularity. See also [20, (3.4)] and [5, (2.6)].

Theorem 1. *Let X be a projective variety over k. Then the following are equivalent*:

1. *There exists some ample Cartier divisor D on X such that the section ring $\bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}(nD))$ is strongly F-regular.*

2. *The section ring of X with respect to each ample Cartier divisor is strongly F-regular.*

3. *There exists some ample effective Cartier divisor D on X such that there exists an $\mathcal{O}_{X^{(e)}}$-linear splitting of $\mathcal{O}_{X^{(e)}} \to F_*^e\mathcal{O}_X \to F_*^e\mathcal{O}(D)$ for some $e \geq 0$ and that the open set $X - D$ is F-regular.*

4. *X is globally F-regular.*

A globally F-regular variety is F-regular. In particular, it is Cohen-Macaulay and normal.

For an affine k-variety $\operatorname{Spec} A$, the following three conditions are equivalent: $\operatorname{Spec} A$ is globally F-regular; A is strongly F-regular; and $\operatorname{Spec} A$ is F-regular.

A globally F-regular variety is Frobenius split in the sense of Mehta-Ramanathan [14]. As the theorem above shows, if X is a globally F-regular projective variety, then the section ring of X with respect to every ample divisor is Cohen-Macaulay normal.

A globally F-regular projective variety X enjoys a nice vanishing theorem. If $\mathcal{L}$ is a numerically effective invertible sheaf, then $H^i(X, \mathcal{L}) = 0$ for $i > 0$. In particular, $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ [19, (4.3)]. It follows that a globally F-regular projective curve is $\mathbb{P}^1$. We also have the following vanishing theorem [19, (4.4)]. Let X be a globally F-regular projective variety and $\mathcal{L}$ a nef big invertible sheaf on X. Then $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < \dim X$.

A projective toric variety over a field of positive characteristic is globally F-regular [19, (6.4)]. Fano varieties with rational singularities in characteristic zero are of globally F-regular type, that is, almost all modulo p reductions of them are globally F-regular [19, (6.3)].

The following lemma is of use later.

Lemma 2 ([4, Proposition 1.2]). *Let $f\colon X \to Y$ be a k-morphism between projective k-varieties. If X is globally F-regular and the associated homomorphism of sheaves of rings $f^\#$ of f, $\mathcal{O}_Y \to f_*\mathcal{O}_X$, is an isomorphism, then Y is globally F-regular.*

Let G be a simply connected, semisimple algebraic group over k, and T a maximal torus of G. We fix a basis Δ of the set of roots of G. Let B be the negative Borel subgroup and P a parabolic subgroup of G containing B. Then B acts on G/P from the left. The closure of a B-orbit of G/P is called a Schubert variety. Any B-invariant closed subvariety of G/P is a Schubert variety. The set of Schubert varieties in G/B is in one-to-one correspondence with the Weyl group $W(G)$ of G. For a Schubert variety X in G/B, there is a unique $w \in W(G)$ such that $X = \overline{BwB/B}$, where the overline denotes the closure operation. For basic notions on algebraic groups, see [2].

We need the following theorem later.

Theorem 3. *A Schubert variety in G/P is a normal variety.*

For a proof, see [16, Theorem 3], [1], [18], and [15].

Let X be a Schubert variety in G/P. Then $\tilde{X} = \pi^{-1}(X)$ is a B-invariant reduced subscheme of G/B, where $\pi\colon G/B \to G/P$ is the canonical projection. It has a dense B-orbit, and actually $\tilde{X}$ is a Schubert variety in G/B.

Let $Y = \rho^{-1}(X)$, where $\rho\colon G \to G/P$ is the canonical projection. Let $\Phi\colon Y \times P/B \to Y \times_X \tilde{X}$ be the Y-morphism given by $\Phi(y, pB) = (y, ypB)$. Since $(y, \tilde{x}B) \mapsto (y, y^{-1}\tilde{x}B)$ gives the inverse, Φ is an isomorphism. Note that $(p_1)_*\mathcal{O}_{Y\times P/B} \cong \mathcal{O}_Y$, where $p_1\colon Y \times P/B \to Y$ is the first projection, since P/B is a k-complete variety and $H^0(P/B, \mathcal{O}_{P/B}) = k$. As Φ is a Y-isomorphism, we see that $(\pi_1)_*\mathcal{O}_{Y\times_X\tilde{X}} \cong \mathcal{O}_Y$, where $\pi_1\colon Y \times_X \tilde{X} \to Y$ is the first projection. As π_1 is a base change of $\pi\colon \tilde{X} \to X$ by the faithfully flat morphism $Y \to X$, we have

Lemma 4. $\pi_*\mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X$. *In particular, if $\tilde{X}$ is globally F-regular, then so is X.*

Let $w \in W(G)$, and $X = X_w$ be the corresponding Schubert variety $\overline{BwB/B}$ in G/B. Assume that w is nontrivial. Then there exists some simple root α such that $l(ws_\alpha) = l(w) - 1$, where s_α is the reflection corresponding to α, and l denotes the length. Set $X' = X_{w'}$ be the Schubert variety $\overline{Bw'B/B}$, where $w' = ws_\alpha$. Let P_α be the parabolic subgroup $Bs_\alpha B \cup B$. Let Y be the Schubert variety $\overline{BwP_\alpha/P_\alpha}$.

The following is due to Kempf [10, Lemma 1].

Lemma 5. *Let $\pi_\alpha\colon G/B \to G/P_\alpha$ be the canonical projection. Then X' is birationally mapped onto Y. In particular, $(\pi_\alpha)_*\mathcal{O}_{X'} = \mathcal{O}_Y$ (by Theorem 3). We have $(\pi_\alpha)^{-1}(Y) = X$, and $\pi|_X\colon X \to Y$ is a $\mathbb{P}^1$-fibration, hence is smooth.*

Let X be a Schubert variety in G/B. Let ρ be the half-sum of positive roots, and set $\mathcal{L} = \mathcal{L}((p-1)\rho)|_X$, where $\mathcal{L}((p-1)\rho)$ is the invertible sheaf on G/B corresponding to the weight $(p-1)\rho$. Note that $\langle \rho, \alpha^\vee \rangle = 1$ for $\alpha \in \Delta$ by [7, Corollary 10.2] (see for the notation, which is relevant here, [8, (II.1.3)]. Under the notation of [7], $(\delta, \alpha^\vee) = 1$.). It follows that $\mathcal{L}$ is ample by [8, Proposition II.4.4]. The following was proved by Ramanan-Ramanathan [16]. See also Kaneda [9].

Theorem 6. *There is a section $s \in H^0(X, \mathcal{L}) \setminus 0$ such that the composite*

$$\mathcal{O}_{X^{(1)}} \to F_*\mathcal{O}_X \xrightarrow{F_*s} F_*\mathcal{L}$$

splits.

Since $\mathcal{L}$ is ample, we immediately have the following.

Corollary 7. *X is globally F-regular if and only if X is F-regular.*

Proof. The 'only if' part is obvious. The 'if' part follows from Theorem 6 and Theorem 1, 3⇒4. ☐

3. Main theorem

Let k be an algebraically closed field, G a simply connected, semisimple algebraic group over k, T a maximal torus of G. We fix a basis of the set of roots of G, and let B be the negative Borel subgroup of G.

In this section we prove the following theorem.

Theorem 8. *Let P be a parabolic subgroup of G containing B, and let X be a Schubert variety in G/P. Then X is globally F-regular.*

Proof. Let $\pi\colon G/B \to G/P$ be the canonical projection, and set $\tilde{X} = \pi^{-1}(X)$. Then $\tilde{X}$ is a Schubert variety in G/B. By Lemma 4, it suffices to show that $\tilde{X}$ is globally F-regular. So in the proof, we may assume that $P = B$.

So, let $X = \overline{BwB/B}$. We proceed by induction on the dimension of X, in other words, $l(w)$. If $l(w) = 0$, then X is a point and X is globally F-regular. Let $l(w) > 0$. Then there exists some simple root α such that $l(ws_\alpha) = l(w) - 1$. Set $w' = ws_\alpha$, $X' = \overline{Bw'B/B}$, $P_\alpha = Bs_\alpha B \cup B$, and $Y = \overline{BwP_\alpha/P_\alpha}$.

By induction assumption, X' is globally F-regular. By Lemma 5 and Lemma 2, Y is also globally F-regular. In particular, Y is F-regular. By Lemma 5, $X \to Y$ is smooth. By [13, (4.1)], X is F-regular. By Corollary 7, X is globally F-regular. □

Corollary 9 (Demazure's vanishing [16], [9]). *Let X be a Schubert variety in G/B, λ a dominant weight, and $\mathcal{L} := \mathcal{L}(\lambda)|_X$. Then $H^i(X, \mathcal{L}) = 0$ for $i > 0$.*

Proof. For any $n \geq 0$ and $\alpha \in \Delta$, $\langle n\lambda + \rho, \alpha^\vee \rangle = n\langle \lambda, \alpha^\vee \rangle + 1 > 0$, since λ is dominant. By [8, Proposition II.4.4], $\mathcal{L}(n\lambda + \rho) = \mathcal{L}(\lambda)^{\otimes n} \otimes \mathcal{L}(\rho)$ is ample. It follows that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}(\rho)|_X$ is ample for any $n \geq 0$. This implies that $\mathcal{L}$ is nef. The assertion follows from Theorem 8 and [19, (4.3)]. □

Let P be a parabolic subgroup of G containing B. Let X be a Schubert variety in G/P. Let $\mathcal{M}_1, \ldots, \mathcal{M}_r$ be effective line bundles on G/P, and set $\mathcal{L}_i := \mathcal{M}_i|_X$. In [11], Kempf and Ramanathan proved that the k-algebra $C := \bigoplus_{\mu \in \mathbb{N}^r} \Gamma(X, \mathcal{L}_\mu)$ has rational singularities, where $\mathcal{L}_\mu = \mathcal{L}_1^{\otimes \mu_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes \mu_r}$ for $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r$. We can prove a very similar result.

Corollary 10. *Let C be as above. Then the k-algebra C is strongly F-regular.*

By [5, Theorem 2.6], $\tilde{C} = \bigoplus_{\mu \in \mathbb{Z}^r} \Gamma(X, \mathcal{L}_\mu)$ is a quasi-F-regular domain. By [5, Lemma 2.4], C is also quasi-F-regular. By [16, Theorem 2], C is finitely generated over k, and is strongly F-regular. □

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