TOKYO J. MATH.
VOL. 21, NO. 2, 1998

Another Type of Instanton Bundles on $Gr_2(\mathbf{C}^{n+2})$

Yasuyuki NAGATOMO

Sophia University

1. Introduction.

The purpose of the present paper is to show the existence of an anti-self-dual connection on 2-plane complex Grassmannian $Gr_2(\mathbf{C}^{n+2})$, to classify instantons and to describe the moduli space. The reason why we use the terminology "instanton" is that our anti-self-dual connections are nothing but 1-instantons in the case $n=1$ ($\mathbf{C}P^2$). However we also have proved that there exists another generalization of instantons on $\mathbf{C}P^2$ to $Gr_2(\mathbf{C}^{n+2})$ [N-N2]. The structure group reflects the main difference between them. In the present paper, $SU(r)$-bundles are taken into account, while in [N-N2], $Sp(r)$-bundles are considered. By the isomorphism $SU(2)\cong Sp(1)$, our two series of generalizations coincide with instantons on $\mathbf{C}P^2$ in the case $n=1$. On $\mathbf{H}P^n$, which is another typical example of quaternion-Kähler manifolds, there exists a generalization of instantons on 4-dimensional sphere $S^4\cong\mathbf{H}P^1$. This instanton bundle also has $Sp(r)$ as a structure group and so, odd Chern classes of this bundle vanish. Since the cohomology groups $H^{4i}(\mathbf{H}P^n, \mathbf{Z})\cong\mathbf{Z}$ for $i=0, 1, \cdots, n$ and the others vanish, odd Chern classes of an arbitrary bundle on $\mathbf{H}P^n$ necessarily vanish. On the contrary, our examples have the non-vanishing third Chern classes. In higher dimensional case, these are the first examples such that higher degree odd Chern classes do not vanish.

As for the existence of anti-self-dual connections, Mamone-Capria and Salamon first give the above examples of instanton bundles on $\mathbf{H}P^n$ and prove that a well-known Horrocks bundle on $\mathbf{C}P^5$ can be obtained as the pull-back of an anti-self-dual bundle on $\mathbf{H}P^2$ [M-S]. Applying the monad given by Donaldson [D] to higher dimensional case, $Sp(r)$-instanton bundles on $Gr_2(\mathbf{C}^{n+2})$ are exhibited in [N-N2]. In both cases, the typical examples of 1-instantons are homogeneous bundles with canonical connections. The author determines all irreducible homogeneous bundles with anti-self-dual canonical connections over compact quaternion symmetric spaces and give a deformation of canonical connections [Na-3]. Adapting this point of view, we will deform the canonical connection on a direct sum of a line bundle and a homogeneous bundle on $Gr_2(\mathbf{C}^{n+2})$.

To classify anti-self-dual bundles, we make use of the theory of monads on the

Received February 4, 1997

Salamon twistor space.

MAIN THEOREM 1. *Let E be a vector bundle on* $Gr_2(\mathbf{C}^{n+2})$ $(n\geqq 2)$ *which has*

(1) *an anti-self-dual connection with the structure group* $SU(r)$, *where* $r\geqq 3$, *and*

(2) $c_2(E)=xy$, $c_3(E)=xy(x-y)$ *and* $c_4(E)=x^3y-x^2y^2+xy^3$.

We denote by $\tilde{E}$ *the pull back bundle of E on the twistor space F.*

Then, $\tilde{E}$ *is the cohomology bundle of the following monad,*

$$(\mathrm{M}) \qquad \mathcal{O}(-1,0)\to \underline{V}\oplus\mathcal{O}(-1,1)\to\mathcal{O}(0,1)\,,$$

where $\underline{V}$ *is a trivial bundle* $F\times V$ *of rank* $r+1$.

In [M-S], Mamone-Capria and Salamon derive a monad using Beilinson's spectral sequence of $\mathbf{C}P^{2n+1}$. Then they need some vanishing theorems. First a vanishing theorem about an anti-self-dual bundle on higher dimensional positive quaternion-Kähler manifold is obtained by the author [Na-2]. This vanishing theorem and an inductive argument give a complete classification of $Sp(r)$-instantons on $\mathbf{H}P^n$ [K-N]. This is a direct generalization of ADHM-construction [A]. Next it is showed that there exists a spectral sequence for holomorphic bundles on the generalized flag manifold F^{2n+1} which is the twistor space of $Gr_2(\mathbf{C}^{n+2})$ [N-N2] (see §2). (In the case $n=1$, our spectral sequence coincides with Buchdahl's spectral sequence [Bu].) This spectral sequence, combined with extended vanishing theorems [N-N1], implies a classification of $Sp(r)$-instantons on $Gr_2(\mathbf{C}^{n+2})$ [N-N2]. In both cases, since $Sp(r)$ are structure groups, the dual bundles are isomorphic to the original bundles. However, a similar isomorphism can not be carried in our case, because the third Chern class does not vanish. Hence, we need to take a pair of a bundle and its dual into account. (Note that our monad (M) is not self-dual in the sense of [O-S-S, p. 282].) We use a spectral sequence in the case $n=2$ ($Gr_2(\mathbf{C}^4)$) and an induction in higher dimensional cases. In the latter case, a slightly stronger theorem (Main Theorem 1′) will be proved. This argument also makes a proof in [N-N2] be complete. Main Theorem 1 can be regarded as a generalization of a classification by Buchdahl [Bu].

Finally the moduli space will be described.

MAIN THEOREM 2. *The moduli space of anti-self-dual connections on E satisfying the hypothesis in Main Theorem* 1 *is identified with an open cone over* $\mathbf{P}(\mathbf{C}^{n+2})$, *where* $\mathbf{C}^{n+2}$ *is the standard representation space of* $SU(n+2)$.

In [N-N3], following Donaldson [D], we give a description of the moduli of $Sp(n)$-instantons on $Gr_2(\mathbf{C}^{n+2})$ in a coordinate-based fashion by using the embedding of F^{2n+1} into $\mathbf{C}P^{n+1}\times\mathbf{C}P^{n+1*}$. On the other hand, the author shows that this moduli can be described by the representation theory via the Bott-Borel-Weil theorem and this moduli is identified with an open cone over $\mathbf{P}(\bigwedge^2\mathbf{C}^{n+2})$ where $\bigwedge^2\mathbf{C}^{n+2}$ is one of the irreducible representation spaces of $SU(n+2)$ [Na-3]. Now, we also make use of the representation of $SU(n+2)$ in a slightly different manner from [Na-3]. For example, this method makes us enable to observe the degeneration of anti-self-dual connections easily. In this case, the set of singular points is only a quaternion hypersurface $Gr_2(\mathbf{C}^{n+1})$. In the case of $Sp(n)$-instantons on $Gr_2(\mathbf{C}^{n+2})$, one of $\mathbf{H}P^i$'s where $i=0, 1, \cdots, [n/2]$ appears as the singular set [N-N3]. The structure of $SU(n+2)$-orbits in $\mathbf{C}^{n+2}$ and $\bigwedge^2 \mathbf{C}^{n+2}$ causes these phenomena.

2. Preliminaries.

Let M be a connected quaternion-Kähler manifold with non-zero scalar curvature and Z be the Salamon twistor space of M [S].

From the definition, the vector bundle $\bigwedge^2 T^*M$ has the following holonomy invariant decomposition:

$$\bigwedge^2 T^*M = S^2\mathbf{H} \oplus S^2\mathbf{E} \oplus (S^2\mathbf{H} \oplus S^2\mathbf{E})^\perp ,$$

where $\mathbf{H}$ and $\mathbf{E}$ are vector bundles associated with the standard representations of $Sp(1)$ and $Sp(n)$, respectively. For example, $\mathbf{H}$ is a tautological quaternionic line bundle when the base spece is a quaternionic projective space $\mathbf{H}P^n$.

DEFINITION 2.1. An $\omega \in \Omega^2 T^*M$ is called a *self-dual* (resp. *anti-self-dual*) form if $\omega \in \Gamma(S^2\mathbf{H})$ (resp. $\Gamma(S^2\mathbf{E})$).

This definition reduces to the usual one on a 4-dimensional oriented Riemannian manifold in the case $n=1$. We shall investigate metric connections on a complex vector bundle F equipped with a hermitian metric h.

DEFINITION 2.2 ([G-P], [M-S] and [Ni]). A connection ∇ is called (resp. *anti-*)*self-dual* if its curvature 2-form R^∇ is a (resp. anti-)self-dual form.

THEOREM 2.3 ([G-P], [M-S] and [Ni]). *Self-dual and anti-self-dual connections are Yang-Mills connections.*

REMARK. If M is compact, self-dual and anti-self-dual connections actually minimize the Yang-Mills functional ([G-P] and [M-S]). Moreover, it is known that there is an essentially unique non-flat self-dual connection over a simply connected quaternion-Kähler manifold whose dimension is greater than or equal to 8 [Na-1].

Let E be an anti-self-dual bundle on M and $\tilde{E}$ be the pull-back bundle of E on Z. Then, it is known that $\tilde{E}$ is a holomorphic bundle with the induced structure ([M-S] and [Ni]). The author showed a vanishing theorem about $\tilde{E}$ at first and this vanishing theorem was extended as follows.

THEOREM 2.4 ([Na-2] and [N-N]). *Let M be a $4n$-dimensional compact quaternion-Kähler manifold with positive scalar curvature and Z be the twistor space of M. If $\tilde{E}$ is the pull back bundle of E on M which has a unitary structure and an anti-self-dual* *connection, then we have*

$$H^i(Z, \tilde{E}(k))=0 \quad \textit{for} \quad 1\leqq i\leqq n \ \textit{and} \ i+k+1<0\,,$$
$$H^1(Z, \tilde{E}(-2))=0\,,$$
$$H^2(Z, \tilde{E}(-3))=0 \quad \textit{for} \quad n\geqq 2\,,$$

and

$$H^i(Z, \tilde{E}(k))=0 \quad \textit{for} \quad n+1\leqq i\leqq 2n \ \textit{and} \ i+k>0\,,$$
$$H^{2n-1}(Z, \tilde{E}(-2n+1))=0 \quad \textit{for} \quad n\geqq 2\,,$$
$$H^{2n}(Z, \tilde{E}(-2n))=0\,.$$

REMARK. A line bundle $\mathcal{O}(1)$ on the twistor space corresponds to L in [S].

From now on, we focus our attention on a complex Grassmanian manifold of 2-planes:

$$Gr_2(\mathbf{C}^{n+2})=SU(n+2)/S(U(2)\times U(n))\,.$$

The twistor space of this manifold is a generalized flag manifold F^{2n+1}:

$$F^{2n+1}=SU(n+2)/S(U(1)\times U(n)\times U(1))\,.$$

In other words, F^{2n+1} is represented as follows:

$$F^{2n+1}=\{(l, V)\mid 0\in l\subset V\subset \mathbf{C}^{n+2},\ \dim l=1 \ \text{and} \ \dim V=n+1\}\,,$$

where l and V are complex vector subspaces. Then the twistor fibration $\pi: F\rightarrow Gr_2(\mathbf{C}^{n+2})$ is

$$\pi((l, V))=l\oplus V^{\perp}\,.$$

So, note that this is *not* a holomorphic fibration [S]. (When no confusion can arise, the dimension $2n+1$ will be omitted.)

We give a quick review of the geometry of this generalized flag manifold F^{2n+1} and refer to [N-N2] for more details.

Fisrt, we describe the ring structure of the cohomology of F. The twistor space F^{2n+1} of $Gr_2(\mathbf{C}^{n+2})$ has double holomorphic fibrations to $\mathbf{P}^{n+1}$ and $\mathbf{P}^{n+1*}$ such that

$$p_1: (l, V)\rightarrow [l]\,,$$
$$p_2: (l, V)\rightarrow [V]\,.$$

We denote by x and y pull-back elements of $H^2(F, \mathbf{Q})$ of the standard positive generators of $H^2(\mathbf{P}^{n+1}, \mathbf{Z})$ and $H^2(\mathbf{P}^{n+1*}, \mathbf{Z})$ respectively. Then, by Leray-Hirsch theorem, the cohomology ring $H^*(F, \mathbf{Q})$ is generated by x and y. In our proof of Main Theorem 1, we need to know the ring structure of cohomology in the case F^5 $(n=2)$.

Bernstein-Gelfand-Gelfand theorem [B-G-G] gives that there is a relation on F^5 such that

$$(2.1)\qquad \begin{gathered} x^3-x^2y+xy^2-y^3=0\,, \\ x^4=0\,,\quad x^3y-x^2y^2+xy^3=0\,,\quad y^4=0\,, \\ x^3y^2-x^2y^3=0\,. \end{gathered}$$

The fundamental class of F^5 is $x^3y^2=x^2y^3$.

On the other hand, using the twistor fibration, we have $H^*(Gr_2(\mathbf{C}^{n+2}), \mathbf{Q})$ is regarded as a subring of $H^*(F, \mathbf{Q})$ [B-G-G]. Hence, an element of $H^*(Gr_2(\mathbf{C}^{n+2}), \mathbf{Q})$ may be written with x and y.

Next, we introduce a spectral sequence of Beilinson type for a holomorphic vector bundle on the flag manifold. To represent this spectral sequence, we define vector bundles on F.

DEFINITION 2.5. A vector bundle Q_1 denotes the quotient bundle of $P_1^*\Omega_{\mathbf{P}^{n+1}}$ by $\mathcal{O}(-1, -1)$, where $\Omega_{\mathbf{P}^{n+1}}$ is the holomorphic cotangent bundle on $\mathbf{P}^{n+1}$ and in general, $\mathcal{O}(p, q)$ is the line bundle $p_1^*\mathcal{O}(p)\otimes p_2^*\mathcal{O}(q)$. The quotient bundle Q_2 is defined in a similar way:

$$0\to\mathcal{O}(-1, -1)\to p_1^*\Omega_{\mathbf{P}^{n+1}}\to Q_1\to 0\,,$$
$$0\to\mathcal{O}(-1, -1)\to p_2^*\Omega_{\mathbf{P}^{n+1*}}\to Q_2\to 0\,.$$

Using vector bundles Q_1 and Q_2, we can show an analogue of theorem of Beilinson on $\mathbf{P}^m$ and refer to [N-N2] for a proof of the next proposition.

PROPOSITION 2.6. *For an arbitrary holomorphic vector bundle S on F, there exists a spectral sequence $E_r^{p,q}$ converging to*

$$E_\infty^{p,q}=\begin{cases}\sum_{p=0}^{2n+1}E_\infty^{-p,p}=S & \text{if}\ \ p+q=0\\ 0 & \text{otherwise}\,.\end{cases}$$

The E_1-terms satisfy exact sequences

$$\cdots\to\sum_{r=0}^{p-1}H^q(F,\textstyle\bigwedge^r Q_1^*\otimes S(-p, 0))\otimes\bigwedge^{p-1-r}Q_2^*(0, -p)\to E_1^{-p,q}$$
$$\to\sum_{r=0}^{p}H^q(F,\textstyle\bigwedge^r Q_1^*\otimes S(-p, 0))\otimes\bigwedge^{p-r}Q_2^*(0, -p)\to\cdots\,.$$

where, for example, $S(p, q)$ means $S\otimes\mathcal{O}(p, q)$ and $\sum$ denotes the direct sum.

Since the above spectral sequence is used in the case $n=2$, we give vanishing theorems for anti-self-dual bundles on $Gr_2(\mathbf{C}^4)$. These vanishing theorems can be obtained from Theorem 2.4 and an induction argument ([N-N2; Theorem 4.10]).

THEOREM 2.7. *Let E be an anti-self-dual bundle with a hermitian structure on $Gr_2(\mathbf{C}^4)$ and $\tilde{E}$ be the pull back bundle on the twistor space F^5. Then, we have*

$$H^0(F^5, \tilde{E}(p,q))=0 \quad \text{if } p+q\leqq -1, \qquad H^1(F^5, \tilde{E}(p,q))=0 \quad \text{if } p+q\leqq -2,$$
$$H^2(F^5, \tilde{E}(p,q))=0 \quad \text{if } p+q\leqq -4, \qquad H^3(F^5, \tilde{E}(p,q))=0 \quad \text{if } p+q\geqq -2,$$
$$H^4(F^5, \tilde{E}(p,q))=0 \quad \text{if } p+q\geqq -4, \qquad H^5(F^5, \tilde{E}(p,q))=0 \quad \text{if } p+q\geqq -5.$$

The twistor space F^5 is a homogeneous Kähler manifold and line bundles $\mathcal{O}(p,q)$ are homogeneous bundles on F^5. Hence, by the Bott-Borel-Weil theorem, we can know the dimension of the cohomology groups for $\mathcal{O}(p,q)$ (see, for example, [K]).

THEOREM 2.8. *There exist the following formulae:*

$$\dim H^i(F^5, \mathcal{O}(p,q))=(-1)^i\tfrac{1}{12}(p+1)(p+2)(q+1)(q+2)(p+q+3)$$

if

$$\begin{aligned}
&i=0 && \text{for} \quad p\geqq 0 \text{ and } q\geqq 0,\\
&i=2 && \text{for} \quad p\leqq -3 \text{ and } p+q\geqq -2 \quad \text{or} \quad q\leqq -3 \text{ and } p+q\geqq -2,\\
&i=3 && \text{for} \quad p\geqq 0 \text{ and } p+q\leqq -4 \quad \text{or} \quad q\geqq 0 \text{ and } p+q\leqq -4,\\
&i=5 && \text{for} \quad p\leqq -3 \text{ and } q\leqq -3,
\end{aligned}$$

and the other cohomology groups vanish.

Finally, we introduce the Ward correspondence. To do so, we make use of the real structure σ on the twistor space which is induced by the quaternionic structure [S].

WARD CORRESPONDENCE. *There is a one-to-one correspondence between anti-self-dual bundles with unitary structures on a quaternion-Kähler manifold M and holomorphic vector bundles E on the twistor space such that*

(1) *the restricted bundles $E|_{\mathbf{P}^1_x}$ to the fibre $\mathbf{P}^1_x$ are trivial for all $x\in M$, and*
(2) *there is an isomorphism $\tau: E\to\sigma^*\bar{E}^*$ with $(\sigma^*\bar{\tau})^*=\tau$ which induces a positive definite hermitian form on sections of $E|_{\mathbf{P}^1_x}$ for all $x\in M$.*

3. Classification.

In this section, we give a proof of Main Theorem 1. We employ an induction argument about the dimension of the base manifold $Gr_2(\mathbf{C}^{n+2})$. Therefore, we classify anti-self-dual bundles E satisfying the hypothesis of Main Theorem 1 on $Gr_2(\mathbf{C}^4)$ $(n=2)$ at first.

REMARK. Throughout this section, we do not distinguish between an anti-self-dual bundle on $Gr_2(\mathbf{C}^{n+2})$ and its pull-back on F^{2n+1}, and we use the same symbol E for both.

LEMMA 3.1. *Let $\chi(E(p, q))$ be the holomorphic Euler characteristics for $E(p, q)$ on F^5:*

$$\chi(E(p, q))=\sum_{i=0}^{5}(-1)^i \dim H^i(F^5, E(p, q)).$$

Then we have

$$\chi(E(p, q))=r\left\{1+\frac{11}{6}(p+q)+(p+q)^2+\frac{5}{4}pq+\frac{1}{6}(p+q)^3+\frac{7}{6}pq(p+q)\right.$$
$$\left.+\frac{1}{4}pq(p+q)^2+\frac{1}{4}p^2q^2+\frac{1}{12}p^2q^2(p+q)\right\}$$
$$-\left\{3+\frac{5}{2}(p+q)+\frac{3}{2}p+\frac{1}{2}(p+q)^2+2pq+\frac{1}{2}p^2+\frac{1}{2}pq(p+q)\right\}.$$

PROOF. Note that $c_4(E)=0$, because of the relation (2.1). A direct computation shows that

$$ch(E)=r-xy+\frac{1}{2}xy(x-y)+\frac{1}{12}x^2y^2-\frac{1}{24}x^2y^2(x-y),$$

$$ch(\mathcal{O}(p, q))=1+(px+qy)+\frac{1}{2}(px+qy)^2+\frac{1}{6}(px+qy)^3+\frac{1}{24}(px+qy)^4$$
$$+\frac{1}{120}(\mathrm{px}+\mathrm{qy})^5,$$

$$td(F^5)=1+\frac{3}{2}(x+y)+\left\{(x+y)^2+\frac{1}{3}xy\right\}+\left\{\frac{3}{8}(x+y)^3+\frac{1}{2}xy(x+y)\right\}$$
$$+\frac{11}{6}x^2y^2+\frac{1}{2}x^2y^2(x+y),$$

where ch means the Chern character and td means the Todd class. Then the Hirzebruch-Riemann-Roch theorem and our relations (2.1) yield our desired result. □

COROLLARY 3.2. *Under the same notation as in Lemma* 3.1, *we have*

$$\chi(E(p, q))=\begin{cases}0 & \text{if } p+q=-3,\\ \frac{1}{12}(q+1)(q+2)\{rq(q+1)+6\} & \text{if } p+q=-2,\\ q(q+2)\{\frac{r}{6}(q+1)(q-1)+1\} & \text{if } p+q=-1,\\ r-3+\frac{q}{4}\{rq(q^2-5)+6(q+1)\} & \text{if } p+q=0,\end{cases}$$

and so

$$\chi(E(-1, -1))=\chi(E(0, -2))=\chi(E(-1, 0))=\chi(E(1, -2))=0$$

and $\chi(E)=r-3$.

We also consider the dual bundle E^*. Note that we can obtain a similar formulae for the bundle E^*, if we interchange the roles of p and q.

Next, we introduce two divisors using the holomorphic fibrations $p_1 : F^5 \to \mathbf{P}^3$ and $p_2 : F^5 \to \mathbf{P}^{3*}$. We fix linear subspaces $\mathbf{P}^2$ and $\mathbf{P}^{2*}$ in $\mathbf{P}^3$ and $\mathbf{P}^{3*}$ respectively, in such a way that the intersection $p_1^{-1}(\mathbf{P}^2) \cap p_2^{-1}(\mathbf{P}^{2*})$ is the twistor space F^3 of $Gr_2(\mathbf{C}^3)$. A divisor $p_1^{-1}(\mathbf{P}^2)$ is denoted by Y_1 and $p_2^{-1}(\mathbf{P}^{2*})$ is denoted by Y_2. From our definition, we get exact sequences of sheaves:

$$0 \to \mathcal{O}(-1, 0) \to \mathcal{O} \to \mathcal{O}_{Y_1} \to 0\,, \tag{3.1}$$

$$0 \to \mathcal{O}_{Y_1}(0, -1) \to \mathcal{O}_{Y_1} \to \mathcal{O}_{F^3} \to 0\,, \tag{3.2}$$

$$0 \to \mathcal{O}(0, -1) \to \mathcal{O} \to \mathcal{O}_{Y_2} \to 0\,, \tag{3.3}$$

$$0 \to \mathcal{O}_{Y_2}(-1, 0) \to \mathcal{O}_{Y_2} \to \mathcal{O}_{F^3} \to 0\,, \tag{3.4}$$

where $\mathcal{O}_{Y_i}(p, q)$ denotes a restriction of $\mathcal{O}(p, q)$ to Y_i for $i=1, 2$.

The next lemma has been shown implicitly in [N-N2]. Buchdahl's vanishing theorems [Bu] and Theorem 2.7 and the exact sequences (3.1–3.4) imply the desired result. (For more details, see the paragraph before Theorem 4.10 in [N-N2].)

LEMMA 3.3. *Let E be an arbitrary anti-self-dual bundle on $Gr_2(\mathbf{C}^4)$. Then we have the following vanishing theorems*:

$$H^0(Y_i, E(p, q))=0 \quad \textit{if } p+q \leqq -1\,, \qquad H^1(Y_i, E(p, q))=0 \quad \textit{if } p+q \leqq -2\,,$$

$$H^3(Y_i, E(p, q))=0 \quad \textit{if } p+q \geqq -3\,, \qquad H^4(Y_i, E(p, q))=0 \quad \textit{if } p+q \geqq -4\,,$$

where $i=1$ or 2.

We denote by $h^i(E(p, q))$ the dimension of $H^i(F^5, E(p, q))$.

PROPOSITION 3.4. *Let E be an anti-self-dual bundle on $Gr_2(\mathbf{C}^4)$ satisfying the hypothesis of Main Theorem* 1. *Then we have*

$$h^2(E(p, q))=0 \quad \textit{if} \quad p=0, -1 \quad \textit{or} \quad q=-1, -2 \quad \textit{or} \quad p \geqq 1 \textit{ and } q \geqq 0\,,$$

$$h^2(E^*(p, q))=0 \quad \textit{if} \quad p=-1, -2 \quad \textit{or} \quad q=0, -1 \quad \textit{or} \quad p \geqq 0 \textit{ and } q \geqq 1\,,$$

$$h^3(E(p, q))=0 \quad \textit{if} \quad p=-1, -2 \quad \textit{or} \quad q=-2, -3 \quad \textit{or} \quad p \leqq -3 \textit{ and } q \leqq -4\,,$$

$$h^3(E^*(p, q))=0 \quad \textit{if} \quad p=-2, -3 \quad \textit{or} \quad q=-1, -2 \quad \textit{or} \quad p \leqq -4 \textit{ and } q \leqq -3\,.$$

PROOF. When we make use of Serre duality and the isomorphism $K_{F^5} \cong \mathcal{O}(-3, -3)$ (K_{F^5} is the canonical bundle on F^5), it suffices to prove vanishing theorems about h^2.

Step 1. From Theorem 2.7 and Corollary 3.2, we obtain

$$h^2(E(-1, -1))=h^2(E(0, -2))=h^2(E^*(-1, -1))=h^2(E^*(-2, 0))=0\,.$$

We tensor the sequence (3.1) with $E(p, q)$ and take the long exact sequence of cohomology groups

$$\cdots \to H^1(Y_1, E(p, q)) \to H^2(F^5, E(p-1, q)) \to H^2(F^5, E(p, q)) \to \cdots .$$

This, together with Lemma 3.3 and $h^2(E(-1, -1))=h^2(E(0, -2))=0$, implies that $h^2(E(-2, -1))=h^2(E(-1, -2))=0$. In a similar way, we get $h^2(E^*(-2, -1))=h^2(E^*(-3, 0))=0$. Applying Serre duality, we obtain $h^3(E(0, -3))=h^3(E^*(-1, -2))=0$. These vanishing theorems, Theorem 2.7 and Corollary 3.2 yield that $h^2(E(0, -3))=h^2(E^*(-1, -2))=0$.

Step 2. If we use Serre duality, the results in Step 1 imply

$$h^3(E(-2, -2))=h^3(E(-1, -3))=h^3(E(-1, -2))=h^3(E(0, -3))=0 ,$$

$$h^3(E^*(-2, -2))=h^3(E^*(-3, -1))=h^3(E^*(-1, -2))=h^3(E^*(-2, -1))=0 .$$

The same argument as in the last part of Step 1 gives that $h^3(E(-2, -1))=h^3(E^*(-3, 0))=0$. Next, we take the same long exact sequence as in Step 1:

$$\cdots \to H^2(F^5, E(p, q)) \to H^2(Y_1, E(p, q)) \to H^3(F^5, E(p-1, q)) \to \cdots .$$

When we substitute $(-1, -1)$, $(-1, -2)$, $(0, -2)$ and $(0, -3)$ into (p, q), our vanishing theorems in Steps 1 and 2 yield

$$h^2(Y_1, E(-1, -1))=h^2(Y_1, E(-1, -2))=h^2(Y_1, E(0, -2))=h^2(Y_1, E(0, -3))=0 ,$$

where $h^i(Y_1, E(p, q))=\dim H^i(Y_1, E(p, q))$. By a similar method, we have

$$h^2(Y_1, E^*(-2, 0))=h^2(Y_1, E^*(-2, -1))=h^2(Y_1, E^*(-1, -1))$$
$$=h^2(Y_1, E^*(-1, -2))=0 .$$

Buchdahl's vanishing theorems $H^1(F^3, E(p, q))=0$ for $p+q \leqq -2$ and the exact sequence (3.2) imply the injectivity of $H^2(Y_1, E(p, q-1)) \to H^2(Y_1, E(p, q))$, if $p+q \leqq -2$. Consequently, we obtain

$$h^2(Y_1, E(0, q))=0 \quad \text{if } q \leqq -1 , \qquad h^2(Y_1, E(-1, q))=0 \quad \text{if } q \leqq -2 ,$$

$$h^2(Y_1, E^*(-1, q))=0 \quad \text{if } q \leqq -1 , \qquad h^2(Y_1, E^*(-2, q)=0 \quad \text{if } q \leqq 0 .$$

By definition, Y_1 is smooth and the adjunction formula yields the isomorphism between the canonical bundle K_{Y_1} and $\mathcal{O}_{Y_1}(-2, -3)$. Combined with the above vanishing, Serre duality implies

$$h^2(Y_1, E(0, q))=h^2(Y_1, E(-1, q))=0 \qquad \text{if} \quad q \in \mathbf{Z} ,$$

$$h^2(Y_1, E^*(-1, q))=h^2(Y_1, E^*(-2, q))=0 \qquad \text{if} \quad q \in \mathbf{Z} .$$

Using vanishing theorems $h^2(E(-1, 0))=h^2(E^*(-2, 0))=0$ and $h^2(Y_1, E)=h^2(Y_1, E^*(-1, 0))=0$, we obtain from the exact sequence (3.1) that $h^2(E(0, -1))=$

$h^2(E^*(-1,0))=0$.

Step 3. Since Y_1 is $p_1^{-1}(\mathbf{P}^2)$, the dual Euler sequence on $\mathbf{P}^2$ gives

$$0 \to p_1^*\Omega^1_{\mathbf{P}^2} \to \mathcal{O}_Y(-1,0)^{\oplus 3} \to \mathcal{O}_Y \to 0 .$$

This, together with Lemma 3.3, implies that $H^1(Y_1, p_1^*\Omega^1_{\mathbf{P}^2} \otimes E(p,q))=0$ if $p+q \leqq -1$. Dualizing this sequence and using $T\mathbf{P}^2 \cong \Omega^1_{\mathbf{P}^2}(3)$, we get from the above vanishing theorem $H^2(Y_1, E(p,q)) \to H^2(Y_1, E(p+1,q))^{\oplus 3}$ are injective if $p+q \leqq -4$. Hence, vanishing theorems $h^2(Y_1, E(-1,q))=h^2(Y_1, E^*(-2,q))=0$ $(q \in \mathbf{Z})$ yield that

$$h^2(Y_1, E(-2,q))=0 \quad \text{if } q \leqq -2, \qquad h^2(Y_1, E^*(-3,q))=0 \quad \text{if } q \leqq -1 .$$

By induction with respect to p and Serre duality, we have

$$h^2(Y_1, E(p,q))=0 \quad \text{if} \quad p \leqq -2 \text{ and } q \leqq -2 \quad \text{or} \quad p \geqq 1 \text{ and } q \geqq -2 ,$$

$$h^2(Y_1, E^*(p,q))=0 \quad \text{if} \quad p \leqq -3 \text{ and } q \leqq -1 \quad \text{or} \quad p \geqq 0 \text{ and } q \geqq -1 .$$

These vanishing and the long exact sequence associated with the sequence (3.1) show inductively that

$$h^2(E(p,-1))=h^2(E(p,-2))=0 \quad \text{if} \quad q \in \mathbf{Z} ,$$

$$h^2(E^*(p,0))=h^2(E^*(p,-1))=0 \quad \text{if} \quad q \in \mathbf{Z} .$$

Using Serre duality, we have

$$h^3(E(p,-2))=h^3(E(p,-3))=0 \quad \text{if} \quad q \in \mathbf{Z} ,$$

$$h^3(E^*(p,-1))=h^3(E^*(p,-2))=0 \quad \text{if} \quad q \in \mathbf{Z} .$$

Step 4. In this final step, we make use of the other divisor Y_2 of F^5. Vanishing theorems in Step 3 and the sequence (3.3) imply that

$$h^2(Y_2, E(p,-1))=h^2(Y_2, E(p,-2))=0 \quad \text{if} \quad p \in \mathbf{Z} ,$$

$$h^2(Y_2, E^*(p,0))=h^2(Y_2, E^*(p,-1))=0 \quad \text{if} \quad p \in \mathbf{Z} .$$

It is shown in a similar way as in Step 3 that $H^2(Y_2, E(p,q)) \to H^2(Y_2, E(p,q+1))^{\oplus 3}$ are injective, if $p+q \leqq -4$. Consequently, we have

$$h^2(Y_2, E(p,q))=0 \quad \text{if} \quad p \leqq -1 \text{ and } q \leqq -3 \quad \text{or} \quad p \geqq -1 \text{ and } q \geqq 0 ,$$

$$h^2(Y_2, E^*(p,q))=0 \quad \text{if} \quad p \leqq -2 \text{ and } q \leqq -2 \quad \text{or} \quad p \geqq -2 \text{ and } q \geqq 1 .$$

Using again the sequence (3.3), we obtain inductively that

$$h^2(E(p,q))=0 \quad \text{if} \quad p \geqq -2 \text{ and } q \geqq 1 ,$$

$$h^2(E^*(p,q))=0 \quad \text{if} \quad p \geqq -2 \text{ and } q \geqq 1 ,$$

completing the proof. □

PROPOSITION 3.5. *Let E be an anti-self-dual bundle on $Gr_2(\mathbf{C}^4)$ satisfying the hypothesis of Main Theorem* 1. *Then we have*

$$h^1(E(0,-1))=1\,,\qquad h^1(E(-1,0))=h^1(E(1,-2))=0\,,$$
$$h^1(E^*(-1,0))=1\,,\qquad h^1(E^*(0,-1))=h^1(E^*(-2,1))=0\,,$$
$$h^4(E(-2,-3))=1\,,\qquad h^4(E(-3,-2))=h^4(E(-1,-4))=0\,,$$
$$h^4(E^*(-3,-2))=1\,,\qquad h^4(E^*(-2,-3))=h^4(E^*(-4,-1))=0\,.$$

PROOF. Theorem 2.7 and Corollary 3.2 imply

$$0=\chi(E(-1,0))=-h^1(E(-1,0))+h^2(E(-1,0))\,.$$

By Proposition 3.4, we know $h^2(E(-1,0))=0$. Hence, $h^1(E(-1,0))=0$. In a similar way, we get desired results about h^1 and applying Serre duality, we also obtain results about h^4. □

THEOREM 3.6. *Let E be an anti-self-dual bundle on $Gr_2(\mathbf{C}^4)$ satisfying the hypothesis of Main Theorem* 1. *Then we have $h^0(E)=r-3$ and $h^1(E)=0$ on the twistor space F^5. Moreover, E decomposes into a direct sum $E=E'\oplus T$, where E' is an $SU(3)$ anti-self-dual bundle and T is a flat bundle of rank $r-3$.*

PROOF. Theorem 2.7, Corollary 3.2 and Proposition 3.4 yield $h^0(E)-h^1(E)=r-3$ and so $h^0(E)\geqq r-3$.

In general, a holomorphic section $s\in H^0(Z,E)$ corresponds to a covariant constant section of E over M, where Z is the twistor space of a quaternion-Kähler manifold M and E is an anti-self-dual bundle on M. (See, for example, [W-W: p. 422] or [Na-2]. A direct computation in [Na-2] shows this fact.) Consequently, there exists a holomorphically trivial bundle T such that $E\cong T\oplus E'$ and the rank of T is greater than or equal to $r-3$, where E' is a subbundle of E. Then we have the rank of E' is less than or equal to 3. However, if the rank of E' is less than 3, $c_3(E)=c_3(E')$ vanishes. This is a contradiction and we have $rk(T)=r-3$ and $rk(E')=3$, where $rk(T)$ means the rank of T and so on. The same argument implies that E' has no section. Hence we get $h^0(E)=h^0(T)=rk(T)=r-3$ and so $h^1(E)=0$. □

Due to Theorem 3.6, from now on, we assume that the rank of E is 3 and so, $h^0(E)=h^1(E)=0$. Then we can apply the same method for the dual bundle E^* and we have $h^0(E^*)=h^1(E^*)=0$.

LEMMA 3.7. *Let E be an $SU(3)$ anti-self-dual bundle on $Gr_2(\mathbf{C}^4)$ satisfying the hypothesis of Main Theorem* 1. *Then we have $h^0(E(p,-p))=0$, $h^0(E^*(p,-p))=0$, $h^5(E(-p-3,p-3)=0$ and $h^5(E^*(-p-3,p-3))=0$ where $p\neq\pm 1$.*

PROOF. The homogeneous bundle $\mathcal{O}(p,-p)$ on F^5 is the pull-back bundle of an anti-self-dual bundle on $Gr_2(\mathbf{C}^4)$ from [Na-3; Theorem 3.4]. Consequently, $E(p,-p)$

is also the pull-back of an anti-self-dual bundle. A direct computation shows that the third Chern class $c_3(E(p, -p)) = -(p-1)(p+1)^2xy(x-y)$ under the relation (2.1). The same argument as in Theorem 3.6 implies that if $h^0(E(p, -p))$ does not vanish, $E(p, -p)$ has a trivial summand and $c_3(E(p, -p))$ vanishes. This is a contradiction when $p \neq \pm 1$. A similar way gives vanishing about the dual bundle E^* and Serre duality gives vanishing about h^5. □

LEMMA 3.8. *Under the same assumption as in Lemma* 3.7, *we have* $h^1(E(2, -2)) = h^1(E^*(-2, 2)) = h^4(E(-1, -5)) = h^4(E^*(-5, -1)) = 0$.

PROOF. From Corollary 3.2, we have $\chi(E(2, -2)) = 0$. Theorem 2.7, Proposition 3.4 and Lemma 3.7 give the desired result. □

We define

$$W_1 = H^1(F^5, E(0, -1))^* \quad \text{and} \quad W_2 = H^1(F^5, E^*(-1, 0))^* .$$

By Proposition 3.5, we have $\dim W_1 = \dim W_2 = 1$. On the other hand, we obtain from our definition that $H^1(W_1 \otimes E(0, -1)) \cong W_1 \otimes H^1(E(0, -1)) \cong \mathrm{End}(H^1(E(0, -1)))$ and $H^1(W_2 \otimes E^*(-1, 0)) \cong \mathrm{End}(H^1(E^*(-1, 0)))$, where we regard W_i as trivial bundles on F^5 with fibres W_i for $i = 1, 2$. (For brevity, we will omit F^5 in cohomology groups.) Consequently, using the identity elements of $\mathrm{End}(H^1(E(0, -1)))$ and $\mathrm{End}(H^1(E^*(-1, 0)))$ respectively, we have the extensions S_1 and S_2 such that

$$0 \to E \to S_1 \to W_1^*(0, 1) \to 0 , \tag{3.5}$$

$$0 \to E^* \to S_2 \to W_2^*(1, 0) \to 0 . \tag{3.6}$$

Dualizing (3.6), tensoring with $W_1(0, -1)$ and taking the associated cohomology sequence, we get from Theorem 2.8

$$H^1(S_2^* \otimes W_1(0, -1)) \cong H^1(E \otimes W_1(0, -1)) \cong \mathrm{End}(H^1(E(0, -1))) .$$

Hence, there is the compatible extension V such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_2^* & \longrightarrow & V & \longrightarrow & W_1^*(0, 1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & S_1 & \longrightarrow & W_1^*(0, 1) & \longrightarrow & 0 . \end{array}$$

Therefore, we have the display of a monad

$$W_2(-1, 0) \to V \to W_1^*(0, 1)$$

such that

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_2(-1,0) & \longrightarrow & S_2^* & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_2(-1,0) & \longrightarrow & V & \longrightarrow & S_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W_1^*(0,1) & = & W_1^*(0,1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.7}
$$

To prove Main Theorem 1, we must determine the bundle V. We will apply Proposition 2.6 for the bundle $V(0,-1)$ and consequently, we consider cohomology groups $H^i(V(p,q))$ and $H^1(V\otimes Q_1^*(p,q))$.

LEMMA 3.9. *Let S_1 be the bundle defined as above. Then we have $h^i(S_1(p,q))=0$ if*

$i=0$ *for* $p+q\leqq 0$ *and* $(p,q)\neq(-1,1),(0,0),(0,-1),(1,-1)$,

$i=1$ *for* $p+q\leqq -2$ *or* $(p,q)=(-1,0),(0,0),(1,-2),(2,-2)$,

$i=2$ *for* $p+q\leqq -4$ *or* $p\geqq -1$ *and* $q\geqq -2$ *or* $(p,q)=(-2,-1),(0,-3)$,

$i=3$ *for* $p+q\geqq -2$ *or* $p\leqq -1$ *and* $q\leqq -2$ *or* $(p,q)=(-2,-1),(0,-3)$,

$i=4$ *for* $p+q\geqq -4$ *or* $(p,q)=(-3,-2),(-3,-3),(-1,-4),(-1,-5)$,

$i=5$ *for* $p+q\geqq -6$ *and* $(p,q)\neq(-4,-2),(-2,-4)$.

PROOF. From the exact sequence (3.5), we get a long exact sequence:

$$\cdots\to H^i(E(p,q))\to H^i(S_1(p,q))\to W_1^*\otimes H^i(\mathcal{O}(p,q+1))\to\cdots.$$

Our vanishing theorems for $H^i(E(p,q))$ (Theorems 2.7 and 3.6, Propositions 3.4 and 3.5, Lemmas 3.7 and 3.8) and the Bott-Borel-Weil theorem for $H^i(\mathcal{O}(p,q+1))$ (Theorem 2.8) yield our results. □

LEMMA 3.10. *Let V be the bundle defined as above. Then we have $h^i(V(p,q))=0$ if*

$i=0$ *for* $p+q\leqq 0$ *and* $(p,q)\neq(-1,1),(0,0),(0,-1),(1,-1)$,

$i=1$ *for* $p+q\leqq -2$ *or* $(p,q)=(-1,0),(0,0),(1,-2),(2,-2)$,

$i=2$ *for* $p+q\leqq -4$ *or* $p\geqq -1$ *and* $q\geqq -2$ *or* $(p,q)=(-2,-1),(0,-3)$,

$i=3$ for $p+q \geqq -2$ or $p \leqq -1$ and $q \leqq -2$ or $(p, q)=(-2, -1), (0, -3)$,

$i=4$ for $p+q \geqq -4$ or $(p, q)=(-3, -2), (-3, -3), (-1, -4), (-1, -5)$,

$i=5$ for $p+q \geqq -6$ and $(p, q) \neq (-4, -2), (-3, -3), (-2, -3), (-2, -4)$.

PROOF. From the second row of (3.7), we get a long exact sequence:

$$\cdots \to W_2 \otimes H^i(\mathcal{O}(p-1, q)) \to H^i(V(p, q)) \to H^i(S_1(p, q)) \to \cdots$$

Our vanishing theorems for $H^i(S_1(p, q))$ (Lemma 3.9) and the Bott-Borel-Weil theorem for $H^i(\mathcal{O}(p-1, q))$ (Theorem 2.8) yield our results. □

LEMMA 3.11. *Let V be the bundle defined as above. For brevity, we denote by T the pull-back bundle $p_1^* T\mathbf{P}^3$, where $T\mathbf{P}^3$ is the holomorphic tangent bundle of $\mathbf{P}^3$. Then we have $h^i(V \otimes T(p, q))=0$ if*

$i=0$ for $p+q \leqq -2$ and $(p, q) \neq (-1, -1)$ or $(p, q)=(1, -2)$,

$i=1$ for $p+q \leqq -4$ or $(p, q)=(-2, -1), (-1, 0), (0, -2), (0, -3), (1, -2)$,

$i=2$ for $p \leqq -1$ and $q \leqq -2$ or $p \geqq -2$ and $q \geqq -2$,

$i=3$ for $p+q \geqq -3$ or $(p, q)=(-3, -1), (-3, -2), (-3, -3), (-1, -3)$,

$i=4$ for $p+q \geqq -5$ and $(p, q) \neq (-2, -3)$,

$i=5$ for $p+q \geqq -7$ and $(p, q) \neq (-5, -2), (-4, -3), (-3, -3), (-3, -4)$.

PROOF. Using the Euler sequence on $\mathbf{P}^3$ and the holomorphic fibration p_1, we obtain

$$0 \to \mathcal{O}_{F^5} \to \mathcal{O}(1, 0)^{\oplus 4} \to T \to 0 . \tag{3.8}$$

Taking the associated long exact sequence, we have

$$\cdots \to H^i(V(p+1, q))^{\oplus 4} \to H^i(V \otimes T(p, q)) \to H^{i+1}(V(p, q)) \to \cdots$$

Our vanishing theorems for $H^i(V(p, q))$ (Lemma 3.10) yield our results. □

LEMMA 3.12. *Under the same assumption and the notation as in Lemma 3.11, we have $h^i(V \otimes Q_1^*(p, q))=0$ if*

$i=0$ for $p+q \leqq -2$ and $(p, q) \neq (-1, -1)$ or $(p, q)=(1, -2)$,

$i=1$ for $p+q \leqq -4$ or $(p, q)=(-2, -1), (0, -3)$,

$i=2$ for $p \leqq -1$ and $q \leqq -1$ and $(p, q) \neq (-1, -2)$,

$i=3$ for $p \geqq -2$ and $q \geqq -3$ and $(p, q) \neq (-2, -2), (-2, -3)$ and $(p, q)=(-3, -2), (-3, -3)$,

$i=4$ for $p+q \geqq -4$ or $(p, q)=(-3, -2), (-1, -4)$,

$i=5$ *for* $p+q \geqq -6$ *and* $(p, q) \neq (-3, -3)$ *or* $(p, q)=(-2, -5)$.

PROOF. Dualizing the exact sequence in Definition 2.5, we use Lemmas 3.10 and 3.11. □

LEMMA 3.13. *Under the same hypothesis as in Lemma* 3.10, *we have* $h^0(V(0, -1))= h^1(V(0, -1))=0$.

PROOF. From the second row of (3.7) and the Bott-Berel-Weil theorem (Theorem 2.8), we obtain $H^i(V(0, -1)) \cong H^i(S_1(0, -1))$ for $i=0, 1, \cdots, 5$. Making use of (3.5) and vanishing theorems $h^0(E(0, -1))=0$ (Theorem 2.7) and $h^1(\mathcal{O})=0$ (Theorem 2.8), we get an exact sequence:

$$0 \to H^0(S_1(0, -1)) \to W_1^* \otimes H^0(\mathcal{O}) \to H^1(E(0, -1)) \to H^1(S_1(0, -1)) \to 0.$$

From our definition of the extension of E by $W_1^*(0, 1)$, the Bockstein operator $W_1^* \otimes H^0(\mathcal{O}) \to H^1(E(0, -1))= W_1^*$ is the identity. Hence, $h^0(S_1(0, -1))=h^1(S_1(0, -1))= 0$. □

LEMMA 3.14. *Under the same hypothesis as in Lemma* 3.10, $h^3(V(-3, -1))=1$.

PROOF. By Lemma 3.10, we get $h^i(V(-3, -1))=0$ for $i \neq 3$. A direct computation and our definition of E and V give $ch(V)=ch(\mathcal{O}^{\oplus 4} \oplus \mathcal{O}(-1, 1))$. Consequently, the Hirzeburch-Riemann-Roch theorem implies $\chi(V(-3, -1))=4\chi(\mathcal{O}(-3, -1))+ \chi(\mathcal{O}(-4, 0))$. The Bott-Borel-Weil theorem (Theorem 2.8) yields $\chi(\mathcal{O}(-3, -1))=0$ and $\chi(\mathcal{O}(-4, 0))= -1$ and so, $h^3(V(-3, -1))=1$. □

LEMMA 3.15. *Under the same hypothesis as in Lemma* 3.10, *we have* $h^0(V\otimes Q_1^*(-1, -1))=0$ *and the identification* $H^1(V\otimes Q_1^*(-1, -1)) \cong W_1^* \otimes \mathbf{C}^4$, *where* $\mathbf{C}^4$ *is the standard representation space of* $SU(4)$.

PROOF. The exact sequence (3.8), Lemmas 3.10 and 3.13 yield $h^0(V\otimes T(-1, -1))=h^1(V\otimes T(-1, -1))=0$. These vanishing, combined with the dualized exact sequence in Definition 2.5 and Lemma 3.10, imply that $h^0(V\otimes Q_1^*(-1, -1))=0$ and $H^0(V) \cong H^1(V\otimes Q_1^*(-1, -1))$. From the second row of (3.7) and Theorem 2.8, we get $H^0(V) \cong H^0(S_1)$. Theorem 3.6 and (3.5) yield that $H^0(S_1) \cong W_1^* \otimes H^0(\mathcal{O}(0, 1))$. The Bott-Borel-Weil theorem implies the identification $H^0(\mathcal{O}(0, 1)) \cong \mathbf{C}^4$ as the representation space of $SU(4)$. □

LEMMA 3.16. *Under the same hypothesis as in Lemma* 3.10, $h^3(V\otimes Q_1^*(-3, -1)) =0$.

PROOF. In the same way as in Lemma 3.14, we obtain $\chi(V\otimes Q_1^*(-3, -1))= 4\chi(Q_1^*(-3, -1))+\chi(Q_1^*(-4, 0))$. Since the bundle Q_1^* is homogeneous, the Bott-Borel-Weil theorem implies that $\chi(Q_1^*(-3, -1))=0$ and $\chi(Q_1^*(-4, 0))=0$. Hence, Lemma 3.12 yields $h^3(V\otimes Q_1^*(-3, -1))=0$. □

LEMMA 3.17. *Under the same hypothesis as in Lemma* 3.10, $h^3(V\otimes Q_1^*(-4,-1))=h^4(V\otimes Q_1^*(-4,-1))=0$.

PROOF. Using the homogeneity of bundles Q_1 and Q_1^*, we get an isomorphism between Q_1^* and $Q_1(3,-1)$. Consequently, $h^i(V\otimes Q_i^*(-4,-1))=h^i(V\otimes Q_1(-1,-2))$ for $i=0,1,\cdots,5$.

Serre duality implies that $h^4(V(-2,-3))=h^1(V^*(-1,0))$ and $h^5(V(-2,-3))=h^0(V^*(-1,0))$. Dualizing the first column of (3.7), we obtain from Theorem 2.8 that $H^i(V^*(-1,0))\cong H^i(S_2(-1,0))$, for $i=0,1,\cdots,5$. Making use of (3.6) and vanishing theorems $h^0(E^*(-1,0))=0$ (Theorem 2.7) and $h^1(\mathcal{O})=0$ (Theorem 2.8), we get an exact sequence:

$$0\to H^0(S_2(-1,0))\to W_2^*\otimes H^0(\mathcal{O})\to H^1(E^*(-1,0))\to H^1(S_2(-1,0))\to 0\,.$$

From our definition of the extension of E^* by $W_2^*(1,0)$, the Bockstein operator $W_2^*\otimes H^0(\mathcal{O})\to H^1(E^*(-1,0))=W_2^*$ is the identity. Hence, $h^0(S_2(-1,0))=h^1(S_2(-1,0))=0$ and so, $h^4(V(-2,-3))=h^5(V(-2,-3))=0$.

The exact sequence in Definition 2.5, Lemma 3.10 and the above vanishing theorem yield that $H^i(V\otimes T^*(-1,-2))\cong H^i(V\otimes Q_1(-1,-2))$ for $i=3,4$. Next, dualizing (3.8), we obtain from Lemma 3.10 that $H^i(V\otimes T^*(-1,-2))=0$ for $i=3,4$. □

THEOREM 3.18. *Under the same hypothesis as in Lemma* 3.10, *we have an isomorphism between V and $W_1^*\otimes\mathbf{C}^4\oplus\mathcal{O}(-1,1)$, where $W_1=H^1(E(0,-1))^*$ and $\mathbf{C}^4$ is the standard representation space of $SU(4)$.*

PROOF. We apply our spectral sequence (Proposition 2.6) to the vector bundle $V(0,-1)$. Our vanishing theorems (Lemmas 3.10, 3.12–3.17) imply that

$$E_1^{-1,1}\cong W_1^*\otimes\mathbf{C}^4\otimes\mathcal{O}(0,-1)\,,\qquad E_1^{-3,3}\cong\mathcal{O}(-1,0)\,,$$

and the other E_1-terms vanish. Hence, by Proposition 2.6, we have

$$0\to W_1^*\otimes\mathbf{C}^4\otimes\mathcal{O}(0,-1)\to V(0,-1)\to\mathcal{O}(-1,0)\to 0\,.$$

However, the Bott-Borel-Weil theorem (Theorem 2.8) yields that $H^1(\mathcal{O}(1,0)\otimes\mathcal{O}(0,-1))\cong H^1(\mathcal{O}(1,-1))=0$ and so, the above exact sequence splits. Consequently, we obtain $V(0,-1)\cong W_1^*\otimes\mathbf{C}^4\otimes\mathcal{O}(0,-1)\oplus\mathcal{O}(-1,0)$. □

Theorem 3.18, together with Theorem 3.6, yields Main Theorem 1 in the case $n=2$.

4. Classification II.

In this section, we also use the same symbol E for an anti-self-dual bundle on $Gr_2(\mathbf{C}^{n+2})$ and its pull-back on F^{2n+1}.

We apply Theorem 2.4 to an anti-self-dual bundle on $Gr_2(\mathbf{C}^{n+2})$.

PROPOSITION 4.1. *Let E be an anti-self-dual bundle with a hermitian structure on $Gr_2(\mathbf{C}^{n+2})$. We assume that $p+q$ is an even number. Then we have for $2 \leqq i \leqq n$*

$$H^1(F^{2n+1}, E(p, q))=0 \quad \text{if} \quad p+q \leqq -2,$$

$$H^i(F^{2n+1}, E(p, q))=0 \quad \text{if} \quad \begin{cases} i: \text{odd and } p+q \leqq -i-3, \\ i: \text{even and } p+q \leqq -i-2. \end{cases}$$

REMARK. The assumption that $p+q$ is even is caused by the non-existence of the line bundle L ($\mathcal{O}(1)$) in [S]. Under the notation in Theorem 2.4, we have an identification $\mathcal{O}(2) \cong \mathcal{O}(1, 1)$. However, this assumption is not needed from the viewpoint of the Penrose transform ([Ba] and [M-S]). We will give an elementary proof for this fact in Proposition 4.3.

For $H^0(F^{2n+1}, E(p, q))$, we have the next lemma (see also [N-N2; Lemma 3.3]).

LEMMA 4.2. *Let E be an anti-self-dual bundle with a hermitian structure on $Gr_2(\mathbf{C}^{n+2})$. Then we have $H^0(F^{2n+1}, E(p, q))=0$ if $p+q \leqq -1$.*

PROOF. Let s be a section of $E(p, q)$. We denote by $\mathbf{P}_x$ the twistor fibre ($x \in Gr_2(\mathbf{C}^{n+2})$). From [Na-3; Lemma 3.3], we have $E(p, q)|_{\mathbf{P}_x} \cong E_x \otimes \mathcal{O}_{\mathbf{P}^1}(p+q)$. If $p+q \leqq -1$, $H^0(\mathbf{P}^1, \mathcal{O}(p+q))=0$. Hence, the restricted section $s|_{\mathbf{P}_x}$ vanishes. □

PROPOSITION 4.3. *The assumption in Proposition 4.1 that $p+q$ is an even number is unnecessary.*

PROOF. We employ an induction. When $n=1$, this result is obtained in [Bu] and when $n=2$, this is nothing but Theorem 2.7.

Using the holomorphic fibration $p_1 : F^{2n+1} \to \mathbf{P}^{n+1}$, we denote by Y_1 the divisor $p_1^{-1}(\mathbf{P}^n)$ in a similar way as in §3. Hence we have exact sequences of sheaves:

$$0 \to \mathcal{O}(-1, 0) \to \mathcal{O} \to \mathcal{O}_{Y_1} \to 0, \tag{4.1}$$

$$0 \to \mathcal{O}_{Y_1}(0, -1) \to \mathcal{O}_{Y_1} \to \mathcal{O}_{F^{2n-1}} \to 0. \tag{4.2}$$

From Lemma 4.2, the sequence (4.2) and the hypothesis of induction, we obtain that for $2 \leqq i \leqq n-1$,

$$\begin{aligned} &H^0(Y_1, E(p, q-1)) \cong H^0(Y_1, E(p, q)) \quad \text{if} \quad p+q \leqq -1, \\ &H^1(Y_1, E(p, q-1)) \cong H^1(Y_1, E(p, q)) \quad \text{if} \quad p+q \leqq -2, \\ &H^i(Y_1, E(p, q-1)) \cong H^i(Y_1, E(p, q)) \quad \text{if} \quad \begin{cases} i: \text{odd and } p+q \leqq -i-3, \\ i: \text{even and } p+q \leqq -i-2. \end{cases} \end{aligned} \tag{4.3}$$

The sequence (4.1), Proposition 4.1 and Lemma 4.2 imply that

$$H^0(Y_1, E(p, q))=0 \quad \text{if} \quad p+q \text{ is odd and } p+q \leqq -1.$$

This, together with (4.3), yields $H^0(Y_1, E(p,q))=0$ if $p+q\leqq -1$. Using again (4.1) and Proposition 4.1, we get $H^1(F^{2n+1}, E(p-1,q))=0$ if $p+q$ is even and is less than or equal to -2. Consequently, we have $H^1(F^{2n+1}, E(p,q))=0$ if $p+q\leqq -2$.

The similar way implies the desired result. □

REMARK. In this proof, we also obtain for $2\leqq i\leqq n-1$

$$(4.4)\quad \begin{aligned} &H^0(Y_1, E(p,q))=0 \quad \text{if} \quad p+q\leqq -1\,,\\ &H^1(Y_1, E(p,q))=0 \quad \text{if} \quad p+q\leqq -2\,,\\ &H^i(Y_1, E(p,q))=0 \quad \text{if} \quad \begin{cases} i:\text{odd and } p+q\leqq -i-3\,,\\ i:\text{even and } p+q\leqq -i-2\,.\end{cases}\end{aligned}$$

From now on, we give a proof of Main Theorem 1 in the case $n\geqq 3$. To do so, we will prove a slightly stronger theorem.

MAIN THEOREM 1′. *Let E be a holomorphic vector bundle of rank r on F^{2n+1} ($n\geqq 2$) which satisfies the following condition.*

(1) *The bundle E (resp. E^*) and the restricted bundle $E|_{F^{2m+1}}$ (resp. $E^*|_{F^{2m+1}}$) to an arbitrary $F^{2m+1}\subset F^{2n+1}$ satisfies the same type of vanishing theorems as in Lemma 4.2 and Proposition 4.3, where $1\leqq m\leqq n$.*

(2) *$c_1(E)=0$, $c_2(E)=xy$, $c_3(E)=xy(x-y)$ and $c_4(E)=x^3y-x^2y^2+xy^3$.*

Then, E is the cohomology bundle of the following monad,

$$(\mathrm{M})\qquad \mathcal{O}(-1,0)\rightarrow \underline{V}\oplus\mathcal{O}(-1,1)\rightarrow\mathcal{O}(0,1)\,,$$

where $\underline{V}$ is a trivial bundle $F\times V$ of rank $r+1$.

REMARK 1. In §3, we mainly use vanishing theorems and a spectral sequence for holomorphic vector bundles. Moreover, F^{2m+1} can be regarded as the twistor space of $Gr_2(\mathbf{C}^{m+2})$. Consequently, we also have proved Main Theorem 1′ in the case $n=2$. Hence, we employ an induction with respect to the dimension of the base space F^{2n+1}. We assume that Main Theorem 1′ is true on F^{2n-1} ($n\geqq 3$).

REMARK 2. For brevity, we refer to the above condition (1) as Lemma 4.2 and Proposition 4.3. Note that the condition (1) implies (4.4) from a similar argument as in the proof of Proposition 4.3.

LEMMA 4.4. *Let E be a holomorphic vector bundle on F^{2n+1} satisfying the hypothesis of Main Theorem 1′. Then we have*

$$\dim H^1(F^{2n+1}, E(0,-1))=1 \quad \textit{and} \quad \dim H^1(F^{2n+1}, E^*(-1,0))=1\,.$$

PROOF. Fix $F^{2n-1}\subset Y_1\subset F^{2n+1}$ and we restrict the bundle E to F^{2n+1}. The restricted bundle satisfies the conditions in Main Theorem 1′, because $n\geqq 3$. (The third Chern class $c_3(E)$ does not vanish.) From the hypothesis of induction, the restricted bundle $E|_{F^{2n-1}}$ is the cohomology bundle of the monad (M). Now the Bott-Borel-Weil theorem implies that $H^i(F^{2n-1}, \mathcal{O}(-1, -1))$, $H^i(F^{2n-1}, \mathcal{O}(0, -1))$ and $H^i(F^{2n-1}, \mathcal{O}(-1, 0))$ vanish for $i=0, 1$. Combined with the first row and the first column of the display of (M), these vanishing yields that $H^0(\mathcal{O}) \cong H^1(F^{2n-1}, E(0, -1))$. Consequently, we have $\dim H^1(F^{2n-1}, E(0, -1))=1$ and $\dim H^1(F^{2n-1}, E^*(-1, 0))=1$ in a similar way.

If $n \geqq 3$, the Bott-Borel-Weil theorem yields that

$$H^2(F^{2n-1}, \mathcal{O}(p, q))=0 \quad \text{if} \quad \begin{cases} p \geqq -2 \text{ and } q \geqq -2, \\ \text{or } p+q \leqq -3 \end{cases}$$

$$H^3(F^{2n-1}, \mathcal{O}(p, q))=0 \quad \text{if} \quad \begin{cases} p \geqq -3 \text{ and } q \geqq -3, \\ \text{or } p \leqq -1 \text{ and } q \leqq -1 . \end{cases}$$

This, together with the display of the monad (M), implies that $H^2(F^{2n-1}, E(p, q))=0$ if $p \leqq 0$ and $q \leqq -1$. Hence, Proposition 4.3 and the exact sequence (4.2) yield that $H^2(Y_1, E(p, q-1)) \cong H^2(Y_1, E(p, q))$ if $p \leqq 0$, $q \leqq -1$ and $(p, q) \neq (0, -1)$. From (4.4), we obtain

$$H^2(Y_1, E(p, q))=0 \quad \text{if} \quad p \leqq 0, \ q \leqq -1 \text{ and } (p, q) \neq (0, -1). \tag{4.5}$$

Since $H^1(Y_1, E(0, -2))=0$ from (4.4) and $H^2(Y_1, E(0, -2))=0$ from (4.5), (4.2) implies that $H^1(Y, E(0, -1)) \cong H^1(F^{2n-1}, E(0, -1))$.

On the other hand, (4.1), (4.4) and (4.5) yield $H^2(F^{2n+1}, E(p-1, q)) \cong H^2(F^{2n+1}, E(p, q))$ if $p \leqq 0$, $q \leqq -1$ and $(p, q) \neq (0, -1)$. From Proposition 4.3, we get

$$H^2(F^{2n+1}, E(p, q))=0 \quad \text{if} \quad p \leqq 0, \ q \leqq -1 \text{ and } (p, q) \neq (0, -1). \tag{4.6}$$

Since $H^1(F^{2n+1}, E(-1, -1))=0$ from Proposition 4.3 and $H^2(F^{2n+1}, E(-1, -1))=0$ from (4.6), (4.1) implies that $H^1(F^{2n+1}, E(0, -1)) \cong H^1(Y_1, E(0, -1))$. Consequently, we have $\dim H^1(F^{2n+1}, E(0, -1)) = \dim H^1(F^{2n-1}, E(0, -1))=1$.

In a similar way, we obtain $\dim H^1(F^{2n+1}, E^*(-1, 0)) = \dim H^1(F^{2n-1}, E^*(-1, 0)) =1$. □

We define

$$W_1 = H^1(F^{2n+1}, E(0, -1))^* \quad \text{and} \quad W_2 = H^1(F^{2n+1}, E^*(-1, 0))^* .$$

By Lemma 4.4, we have $\dim W_1 = \dim W_2 = 1$. Using the identity elements of $\mathrm{End}(H^1(E(0, -1))$ and $\mathrm{End}(H^1(E^*(-1, 0))$ respectively, we have the extensions S_1 and S_2 such that

$$0 \to E \to S_1 \to W_1^*(0, 1) \to 0, \tag{4.7}$$

$$0 \to E^* \to S_2 \to W_2^*(1, 0) \to 0, \tag{4.8}$$

in a similar way as in §3. Since $n \geqq 3$, the Bott-Borel-Weil theorem implies that $H^1(F^{2n+1}, \mathcal{O}(p, q)) = H^2(F^{2n+1}, \mathcal{O}(p, q))=0$. Dualizing (4.8), tensoring with $W_1(0, -1)$

and taking the associated cohomology sequence, we get

$$\begin{aligned} H^1(F^{2n+1}, S_2^*\otimes W_1(0,-1)) &\cong H^1(F^{2n+1}, E\otimes W_1(0,-1)) \\ &\cong \mathrm{End}(H^1(F^{2n+1}, E(0,-1)) . \end{aligned}$$

Hence, there is the compatible extension V_1 such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_2^* & \longrightarrow & V_1 & \longrightarrow & W_1^*(0,1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & S_1 & \longrightarrow & W_1^*(0,1) & \longrightarrow & 0 . \end{array}$$

Therefore, we have the display of a monad

(M1) $$W_2(-1,0)\to V_1\to W_1^*(0,1)$$

such that

(4.9)
$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W_2(-1,0) & \longrightarrow & S_2^* & \longrightarrow & E & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W_2(-1,0) & \longrightarrow & V_1 & \longrightarrow & S_1 & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & W_1^*(0,1) & = & W_1^*(0,1) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

To prove Main Theorem 1′, we must determine the bundle V_1.

Now we introduce another monad on F^{2n+1}:

(M2) $$W_2(-1,0)\to \underline{V}\oplus\mathcal{O}(-1,1)\to W_1^*(0,1),$$

where $\underline{V}$ is a trivial bundle of rank $r+1$. The cohomology bundle of (M2) satisfies the condition (1) and the condition (2) in Main Theorem 1′ by the Bott-Borel-Weil theorem and a direct computation respectively.

PROPOSITION 4.5. *We have an isomorphism on F^{2n-1} that*

$$a:\underline{V}\oplus\mathcal{O}(-1,1)|_{F^{2n-1}}\cong V_1|_{F^{2n-1}}.$$

PROOF. By the hypothesis of induction, there are isomorphisms between the cohomology bundles of the restricted monad (M1) to F^{2n-1} and the cohomology bundles of the restricted monad (M2) to F^{2n-1}. From a theorem of Okonek-Schneider-

Spindler ([O-S-S; Corollary 1, p. 279]), if $H^i(F^{2n-1}, \underline{V}^*(-1,0)\oplus\mathcal{O}(0,-1))$, $H^i(F^{2n-1}, V_1(0,-1))$, $H^j(F^{2n-1}, \mathcal{O}(-1,-1))$, $H^i(F^{2n-1}, V_1^*(-1,0))$ and $H^i(F^{2n-1}, \underline{V}(0,-1)\oplus\mathcal{O}(-1,0))$ vanish for $i=0, 1$ and $j=1, 2$, we obtain the desired isomorphism.

Since $\underline{V}$ is a trivial bundle, the Bott-Borel-Weil theorem implies that $H^i(F^{2n-1}, \underline{V}^*(-1,0)\oplus\mathcal{O}(0,-1))=0$, $H^j(F^{2n-1}, \mathcal{O}(-1,-1))=0$ and $H^i(F^{2n-1}, \underline{V}(0,-1)\oplus\mathcal{O}(-1,0))=0$ for $i=0, 1$ and $j=1, 2$.

Next, since $H^i(F^{2n-1}, \mathcal{O}(-1,-1))=0$ $(i=0,\cdots,2n-1)$ by the Bott-Borel-Weil theorem, if we restrict the second row of the display (4.9) to F^{2n-1}, we have $H^i(F^{2n-1}, V_1(0,-1))\cong H^i(F^{2n-1}, S_1(0,-1))$ for $i=0, 1$.

Making use of the second column of (4.9) and Lemma 4.2, we obtain an exact sequence:

$$0\to H^0(F^{2n+1}, S_1(0,-1))\to W_1^* \to H^1(F^{2n+1}, E(0,-1))\to H^1(F^{2n+1}, S_1(0,-1))\to 0,$$

where $H^1(F^{2n+1}, \mathcal{O})=0$ by the Bott-Borel-Weil theorem. The definition of the extension yields that $W_1^*\to H^1(F^{2n+1}, S_1(0,-1))$ is the identity. Hence $H^i(F^{2n+1}, S_1(0,-1))=0$ for $i=0, 1$. Moreover, from the second column of the display (4.9), Lemma 4.2 and Proposition 4.3, the Bott-Borel-Weil theorem implies that

$$\text{(4.10)}\qquad \begin{aligned} &H^0(F^{2n+1}, S_1(p,q))=0 \quad \text{if } p+q\leqq -1,\\ &H^1(F^{2n+1}, S_1(p,q))=0 \quad \text{if } p+q\leqq -2,\\ &H^2(F^{2n+1}, S_1(p,q))=0 \quad \text{if } p+q\leqq -4.\end{aligned}$$

Then the exact sequence (4.1) and (4.10) yield that

$$\text{(4.11)}\qquad H^0(Y_1, S_1(p,q))=0 \quad \text{if } p+q\leqq -1,$$

$$\text{(4.12)}\qquad H^1(Y_1, S_1(p,q))=0 \quad \text{if } p+q\leqq -3.$$

Using again the second column of the restricted display (4.9) to F^{2n-1}, we obtain from Lemma 4.2 and Proposition 4.3 that $H^0(F^{2n-1}, S_1(p,q))=0$ if $p+q\leqq -1$ and $(p,q)\neq(0,-1)$ and $H^1(F^{2n-1}, S_1(p,q))=0$ if $p+q\leqq -2$. These vanishing, combined with (4.2), shows that

$$\text{(4.13)}\qquad H^1(Y_1, S_1(p,q-1))\cong H^1(Y_1, S_1(p,q)) \quad \text{if } p+q\leqq -2.$$

By (4.12) and (4.13), we have $H^1(Y_1, S_1(p,q))=0$ if $p+q\leqq -2$. These, together with (4.2) and (4.10), yield that $H^0(F^{2n-1}, S_1(p,q))=0$ if $p+q\leqq -1$, in particular, $H^0(F^{2n-1}, S_1(0,-1))=0$. Then, from the second column of the restricted display (4.9), we have

$$0\to W_1^*\to H^1(F^{2n-1}, E(0,-1))\to H^1(F^{2n-1}, S_1(0,-1))\to 0.$$

Lemma 4.4 and its proof implies that $\dim W_1^* = \dim H^1(F^{2n-1}, E(0,-1))=1$ and so,

$H^1(F^{2n-1}, S_1(0, -1))=0$.

As for $H^i(F^{2n-1}, V_1^*(-1, 0))$ $(i=0, 1)$, we may apply a similar method to the dual monad of (M1). □

Proposition 4.6. *There exists a unique element* $A \in H^0(F^{2n+1}, \mathrm{End}(\underline{V} \oplus \mathcal{O}(-1, 1), V_1))$ *such that the restriction* A *to* F^{2n-1} *corresponds to* a *in Proposition* 4.5.

Proof. Since $\mathrm{End}(\underline{V} \oplus \mathcal{O}(-1, 1), V_1) \cong \underline{V}^* \otimes V_1 \oplus V_1(1, -1)$, if $H^i(Y_1, V_1(0, -1))$, $H^i(Y_1, V_1(-1, 0))$, $H^i(F^{2n+1}, V_1(-1, 1))$ and $H^i(F^{2n+1}, V_1(-2, 1))$ vanish for $i=0, 1$, from (4.1) and (4.2), we obtain that $H^0(F^{2n-1}, \mathrm{End}(V \oplus \mathcal{O}(-1, 1), V_1)) \cong H^0(F^{2n+1}, \mathrm{End}(V \oplus \mathcal{O}(-1, 1), V_1))$ and so, we have the desired A.

First Proposition 4.5 and the Bott-Borel-Weil theorem imply that

$$\begin{aligned} &H^0(F^{2n-1}, V_1(p, q))=0 \quad \text{if} \quad p \leqq -1 \text{ or } q \leqq -2 \text{ or } (p, q)=(0, -1), \\ &H^1(F^{2n-1}, V_1(p, q))=0 . \end{aligned} \tag{4.14}$$

By (4.2) and (4.14), we have for $i=0, 1$.

$$\begin{aligned} &H^i(Y_1, V_1(p, q-1)) \cong H^i(Y_1, V_1(p, q)) \\ &\qquad \text{if} \quad p \leqq -1 \text{ or } q \leqq -2 \text{ or } (p, q)=(0, -1) . \end{aligned} \tag{4.15}$$

The second row of (4.9), (4.10) and the Bott-Borel-Weil theorem yields that

$$\begin{aligned} &H^0(F^{2n+1}, V_1(p, q))=0 \quad \text{if} \quad p+q \leqq -1 , \\ &H^1(F^{2n+1}, V_1(p, q))=0 \quad \text{if} \quad p+q \leqq -2 , \\ &H^2(F^{2n+1}, V_1(p, q))=0 \quad \text{if} \quad p+q \leqq -4 . \end{aligned} \tag{4.16}$$

From (4.1) and (4.16), we obtain that

$$\begin{aligned} &H^0(Y_1, V_1(p, q))=0 \quad \text{if} \quad p+q \leqq -1 , \\ &H^1(Y_1, V_1(p, q))=0 \quad \text{if} \quad p+q \leqq -3 . \end{aligned}$$

These vanishing, together with (4.15), imply that for $i=0, 1$,

$$H^i(Y_1, V_1(p, q))=0 \quad \text{if} \quad p \leqq -1 \text{ or } q \leqq -2 \text{ or } (p, q)=(0, -1) . \tag{4.17}$$

In a similar way, (4.1), (4.16), (4.17) shows that for $i=0, 1$,

$$H^i(F^{2n+1}, V_1(p, q))=0 \quad \text{if} \quad p \leqq -1 \text{ or } q \leqq -2 \text{ or } (p, q)=(0, -1) .$$ □

Theorem 4.7. *The vector bundle* V_1 *in the monad* (M1) *is identified with* $\underline{V} \oplus \mathcal{O}(-1, 1)$, *where* $\underline{V}$ *is a trivial bundle of rank* $r+1$.

Proof. By Proposition 4.6, we obtain a homomorphism $A: \underline{V} \oplus \mathcal{O}(-1, 1) \to V_1$ such that the restriction $A|_{F^{2n-1}}$ is an isomorphism. Hence we also have $\det A: \bigwedge^{r+2}(\underline{V} \oplus \mathcal{O}(-1, 1)) \to \bigwedge^{r+2} V_1$. Since $\underline{V}$ is trivial, $\bigwedge^{r+2}(\underline{V} \oplus \mathcal{O}(-1, 1)) \cong \mathcal{O}(-1, 1)$. On the other hand, from the monad (M1), we get $c(V_1)=c(E)c(\mathcal{O}(-1,0))c(\mathcal{O}(0,1))$ and so, $c_1(\bigwedge^{r+2}V_1)=c_1(\mathcal{O}(-1,1))$. Consequently, we regard $\det A$ as an element of $H^0(F^{2n+1}, \mathcal{O}(-1,1)^*\otimes\mathcal{O}(-1,1))\cong H^0(F^{2n+1},\mathcal{O})\cong\mathbf{C}$. Since $A|_{F^{2n-1}}$ is an isomorphism, $\det A=\det A|_{F^{2n-1}}\neq 0$, and so A is also an isomorphism. □

5. Moduli spaces.

To describe homomorphisms in the monad (M), we make use of the expression of F^{2n+1} as a homogeneous space. For brevity, $SU(n+2)$ is denoted by G and $S(U(1)\times U(n)\times U(1))$ is denoted by K_Z. Let $\mathbf{C}^{n+2}$ be the standard representation space of G with a G-invariant hermitian inner product h. Now we denote by e (resp. f) the highest (resp. lowest) weight vector with the norm 1 in $\mathbf{C}^{n+2}$. Then, by the restriction of the action of G to K_Z, we have two irreducible representation spaces $\mathbf{C}e$ and $\mathbf{C}f$ of K_Z. We also obtain an irreducible representation $\mathbf{C}e\otimes f$ of K_Z by the tensor product. Under this notation, we have

$$\mathcal{O}(-1,0)=G\times_{K_Z}\mathbf{C}e\,,\quad \mathcal{O}(0,1)=G\times_{K_Z}\mathbf{C}f \quad\text{and}\quad \mathcal{O}(-1,1)=G\times_{K_Z}\mathbf{C}e\otimes f\,.$$

Hence, for example, an element of $\mathcal{O}(-1,0)$ is denoted by $[g,ce]$, where c is a complex number and $[g,ce]$ is the coset represented by $(g,ce)\in \mathrm{G}\times\mathbf{C}e$.

PROPOSITION 5.1. *Let α and β be homomorphisms in the monad* (M):

$$\text{(M)}\qquad \mathcal{O}(-1,0)\xrightarrow{\ \alpha\ }\underline{V}\oplus\mathcal{O}(-1,1)\xrightarrow{\ \beta\ }\mathcal{O}(0,1)\,.$$

Then, there exist $A\in\mathrm{Hom}(\mathbf{C}^{n+2},V)$, $B\in\mathrm{Hom}(V,\mathbf{C}^{n+2})$, $z\in\mathbf{C}^{n+2}$ and $w\in\mathbf{C}^{n+2*}$ such that

$$\alpha([g,ce])=(([g],cAge),[g,ch(z,gf)e\otimes f])\,,\tag{5.1}$$

$$\beta(([g],v),[g,c'e\otimes f])=[g,\{h(Bv,gf)+c'w(ge)\}f]\,,\tag{5.2}$$

where $g\in G$, $c,c'\in\mathbf{C}$, and $v\in V$.

PROOF. A homomorphism α is regarded as an element of $H^0(\mathrm{Hom}(\mathcal{O}(-1,0),\underline{V}\oplus\mathcal{O}(-1,1)))\cong V\otimes H^0(\mathcal{O}(1,0))\oplus H^0(\mathcal{O}(0,1))$. (For brevity, we omit F^{2n+1} in cohomology groups.) The Bott-Borel-Weil theorem implies that $H^0(\mathcal{O}(1,0))\cong\mathbf{C}^{n+2*}$ and $H^0(\mathcal{O}(0,1))\cong\mathbf{C}^{n+2}$. Consequently, α is identified with an element of $\mathrm{Hom}(\mathbf{C}^{n+2},V)\oplus\mathbf{C}^{n+2}$. In a similar way, β belongs to $\mathrm{Hom}(V,\mathbf{C}^{n+2})\oplus\mathbf{C}^{n+2*}$. For example, the method of Kostant [K] yields the explicit expressions of α and β. □

Since (M) is a monad, α is an injection, β is a surjection and $\beta\circ\alpha=0$.

LEMMA 5.2. *A homomorphism α in* (M) *is injective if and only if A is injective. A homomorphism β in* (M) *is surjective if and only if B is surjective.*

PROOF. If A is injective, (5.1) implies that α is injective. We assume that A is not injective. Then, $\operatorname{Ker} A$ ($\neq \{0\}$) is a subspace in $\mathbf{C}^{n+2}$. Let u be an element in $\operatorname{Ker} A$ with the norm 1. Since $n \geqq 2$, there exists $v \in \mathbf{C}^{n+2}$ such that $h(v, v)=1$, $h(u, v)=0$ and $h(z, v)=0$. Considering the standard representation of $SU(n+2)$, we obtain $g \in G$ such that $u=ge$ and $v=gf$. Hence, from our choice of u, v and g, (5.1) yields that $\alpha([g, e])=0$. This is a contradiction with the injectivity of α.

As for β and B, the surjectivity of B implies the surjectivity of β by (5.2). If B is not surjective, there exists $g \in G$ such that $h(Bv, gf)=0$ for an arbitrary $v \in V$ and $w(ge)=0$. Consequently, β is not surjective. □

COROLLARY 5.3. *That the rank r of E is greater than or equal to $n+1$ is a necessary condition for the existence of E satisfying the conditions in Main Theorem* 1 (*or* 1′).

PROOF. The vector bundle E is the cohomology bundle of (M) by Main Theorem 1. From Proposition 5.2, the dimension of the vector space V is greater than or equal to $n+2$. □

LEMMA 5.4. *A composition homomorphism $\beta \circ \alpha$ is a* 0*-map if and only if there exists a constant $c \in \mathbf{C}$ such that $BA + z \otimes w = c\,\mathrm{Id}_{\mathbf{C}^{n+2}}$, where we regard $z \otimes w$ as an element of $\mathbf{C}^{n+2} \otimes \mathbf{C}^{n+2*} \cong \operatorname{End}(\mathbf{C}^{n+2})$ and $\mathrm{Id}_{\mathbf{C}^{n+2}}$ is the identity on $\mathbf{C}^{n+2}$.*

PROOF. This proof is a slight modification of [Na-3, Proposition 5.1.2].
From (5.1) and (5.2), $\beta \circ \alpha = 0$ if and only if

$$h(g^{-1}BAge, f) + w(ge)h(g^{-1}z, f) = h(\{g^{-1}(BA + z \otimes w)g\}e, f) = 0\,,$$

for an arbitrary $g \in G$.

As a representation space, $\operatorname{End}(\mathbf{C}^{n+2})$ is decomposed into $\mathfrak{sl}(n+2) \oplus \mathbf{C}\,\mathrm{Id}_{\mathbf{C}^{n+2}}$. According to this decomposition, $BA + z \otimes w$ is assumed to be expressed as $X + c\,\mathrm{Id}_{\mathbf{C}^{n+2}}$, where c is a constant. Then, we have $h(\{g^{-1}(BA + z \otimes w)g\}e, f) = h(g^{-1}Xge, f)$. Combined with the irreducibility of the adjoint representation of G, $X \neq 0$ if and only if there exists $g \in G$ such that $h(g^{-1}Xge, f) \neq 0$. □

THEOREM 5.5. *Let E be a vector bundle satisfying the conditions in Main Theorem* 1. *If E has an irreducible $SU(r)$ anti-self-dual connection, then we have $r = n+1$ and an identification between V and $\mathbf{C}^{n+2}$.*

PROOF. First, we assume that $H^0(E) \neq 0$. The same argument as in the proof of Theorem 3.6 implies that E has a trivial subbundle with a flat connection. This is a contradiction with the irreducibility and so $H^0(E)=0$.

The first row of the display of (M) and the Bott-Borel-Weil theorem yield that $H^i(\operatorname{Ker}\beta) \cong H^i(E)$ for $i=0, \cdots 2n+1$. Consequently, $H^0(\operatorname{Ker}\beta)=0$. Since $n \geqq 2$, the Bott-Borel-Weil theorem implies that $H^1(\mathcal{O}(p, q))=0$ for arbitrary p, q. By the first column of the display of (M), we obtain

(5.3) $$0 \to V \to H^0(\mathcal{O}(0,1)) \to H^1(\mathrm{Ker}\,\beta) \to 0\,.$$

Using again the Bott-Borel-Weil theorem, we have $H^0(\mathcal{O}(0,1)) \cong \mathbf{C}^{n+2}$ as the representation space of G. Hence, $\dim H^1(E) = \dim H^1(\mathrm{Ker}\,\beta) = n+2-\dim V$ and so, $\dim V \leqq n+2$. Since $\mathrm{rank}\,E = \dim V - 1$, we obtain $\mathrm{rank}\,E \leqq n+1$. However, Corollary 5.3 asserts that $\mathrm{rank}\,E \geqq n+1$. Consequently we have $\mathrm{rank}\,E = n+1$ and $H^1(E)=0$. Then, (5.3) yields that $V \cong H^0(\mathcal{O}(0,1)) \cong \mathbf{C}^{n+2}$. □

By Theorem 5.5, we assume that the trivial bundle $\underline{V}$ in the monad (M) is $\underline{\mathbf{C}^{n+2}} = F^{2n+1} \times \mathbf{C}^{n+2}$ and the monad (M) is described as

(MI) $$\mathcal{O}(-1,0) \xrightarrow{\alpha} \underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1,1) \xrightarrow{\beta} \mathcal{O}(0,1)\,.$$

Then, note that A and B are automorphisms on $\mathbf{C}^{n+2}$ by Lemma 5.2.

Proposition 5.6. *Monads* (MI) *and* (MI′) *are isomorphic to each other, in other words, the following diagram is commutative*;

$$\begin{array}{ccccc} (\mathrm{MI}): \mathcal{O}(-1,0) & \xrightarrow{\alpha} & \underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1,1) & \xrightarrow{\beta} & \mathcal{O}(0,1) \\ \downarrow p & & \downarrow F & & \downarrow q \\ (\mathrm{MI}'): \mathcal{O}(-1,0) & \xrightarrow[\alpha']{} & \underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1,1) & \xrightarrow[\beta']{} & \mathcal{O}(0,1) \end{array}$$

where F is an automorphism of $\underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1,1)$, p and q are automorphisms of $\mathcal{O}(-1,0)$ and $\mathcal{O}(0,1)$, respectively, if and only if there exists a non-zero constant a, b and c such that $aB'A' = bBA$, $cz = az'$ and $bw = cw'$ under the notations in (5.1) *and* (5.2).

Proof. The Bott-Borel-Weil theorem implies that $H^0(\mathrm{End}(\mathcal{O}(-1,0))) \cong H^0(\mathrm{End}(\mathcal{O}(0,1))) \cong \mathbf{C}$ and $H^0(\mathrm{End}(\underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1,1))) \cong H^0(\mathrm{End}(\underline{\mathbf{C}^{n+2}})) \oplus H^0(\mathcal{O}) \cong \mathrm{End}(\mathbf{C}^{n+2}) \oplus \mathbf{C}$. Consequently p and q can be regarded as non-zero constants and the automorphism F is expressed as $(C, r) \in \mathrm{Aut}(\mathbf{C}^{n+2}) \oplus \mathbf{C}^*$. Then the commutative diagram, (5.1) and (5.2) yield that

(5.4) $$(([g], CAge), r[g, h(z, gf)e \otimes f]) = p(([g], A'ge), [g, h(z', gf)e \otimes f])\,,$$

(5.5) $$q\{h(Bv, gf) + w(ge)\} = h(B'Cv, gf) + rw'(ge)$$

for arbitrary $g \in G$ and $v \in \mathbf{C}^{n+2}$. Consequently, (5.4) implies that $CAge = pA'ge$ and $rh(z, gf) = ph(z', gf)$ for an arbitrary $g \in G$. From the irreducibility of the standard representation of G, we have $CA = pA'$ and $rz = pz'$. If we put $v=0$ in (5.5), the irreducibility yields $qw = rw'$. Then we obtain $qB = B'C$, using again the irreducibility. Now A and B are automorphisms and so, $C = pA'A^{-1} = qB'^{-1}B$.

Conversely, if $aB'A' = bBA$, $cz = az'$ and $bw = cw'$, we may put $C = aA'A^{-1} = bB'^{-1}B$, $p = a$, $q = b$ and $r = c$. □

Let $\mathcal{M}^{\mathbf{C}}$ be the set consisting of the isomorphism classes of the cohomology bundles of the monad (MI). We call $\mathcal{M}^{\mathbf{C}}$ the *complex moduli space.*

THEOREM 5.7. *The complex moduli space $\mathcal{M}^{\mathbf{C}}$ is identified with*

$$\{(z, w) \in \mathbf{C}^{n+2} \times \mathbf{C}^{n+2*} \mid w(z) \neq 1\}/\mathbf{C}^*,$$

where $\mathbf{C}^*$-action is defined as $p \cdot (z, w) = (pz, \frac{1}{p}w)$ for $p \in \mathbf{C}^*$.

PROOF. Using a theorem of Okonek-Schneider-Spindler ([O-S-S; Corollary 1, p. 279]) and the Bott-Borel-Weil theorem, we have a bijection between the isomorphism classes of the monads (MI) and the isomorphism classes of the cohomology bundles. From Propositions 5.1 and 5.6 and Lemmas 5.2 and 5.4, we obtain

$$\mathcal{M}^{\mathbf{C}} = \{(A, B, z, w, c) \in \mathrm{Aut}(\mathbf{C}^{n+2}) \times \mathrm{Aut}(\mathbf{C}^{n+2}) \times \mathbf{C}^{n+2} \times \mathbf{C}^{n+2*} \times \mathbf{C} \mid BA + z \otimes w = c\, \mathrm{Id}_{\mathbf{C}^{n+2}}\}/\sim ,$$

where $(A, B, z, w, c) \sim (A', B', z', w', c')$ means that there exist non-zero constants p, q and r such that $pB'A' = qBA$, $pz' = rz$, $rw' = qw$ and so, $pc' = qc$.

If $c = 0$, then $BA = -z \otimes w$. Since $z \otimes w$ is not an automorphism, there is a contradiction and so, $c \neq 0$. Using our $\mathbf{C}^*$-action, we may put $c = 1$. Then, if we fix (z, w), BA is uniquely determined and automorphisms A and B are uniquely determined up to the equivalence relation. However, we must consider the condition that $\mathrm{Id}_{\mathbf{C}^{n+2}} - z \otimes w$ is an automorphism, because BA is an automorphism. It is easy to show that $\mathrm{Id}_{\mathbf{C}^{n+2}} - z \otimes w$ is invertible if and only if $w(z) \neq 1$. □

REMARK. Making use of the proof of Theorem 5.7, we obtain another description of $\mathcal{M}^{\mathbf{C}}$:

$$\mathcal{M}^{\mathbf{C}} = \{(z, w, c) \in \mathbf{C}^{n+2} \times \mathbf{C}^{n+2*} \times \mathbf{C}^* \mid w(z) \neq c\}/\mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}^*, \tag{5.6}$$

where the $\mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}^*$-action is defined as $(p, q, r) \cdot (z, w, c) = (\frac{r}{p}z, \frac{q}{r}w, \frac{q}{p}c)$ for $p, q, r \in \mathbf{C}^*$.

To obtain the moduli of anti-self-dual connections, the reality condition (the Ward correspondence in §2) must be taken into account. First, we describe the real structure σ on F^{2n+1}. We define $j \in G$ as $je = f$, $je = -f$ and $ju = u$ for an arbitrary $u \in \mathbf{C}^{n+2}$ which is orthogonal to e and f. Then we have $\sigma([g]) = [gj]$ (for example, [Na-3]). Let $\mathbf{P}_x = \pi^{-1}(x)$ be a twistor fibre, where $x \in Gr_2(\mathbf{C}^{n+2})$.

PROPOSITION 5.8. *Let E be the cohomology bundle of the monad* (MI). *The restricted bundle $E|_{\mathbf{P}_x}$ to the twistor fibre is trivial for each $x \in Gr_2(\mathbf{C}^{n+2})$ if and only if (z, w, c) in* (5.6) *satisfies $w(u)h(z, u) + w(v)h(z, v) \neq c$ for arbitrary u, $v \in \mathbf{C}^{n+2}$ such that $|u| = |v| = 1$ and $h(u, v) = 0$.*

PROOF. This proof is a slight modification of [O-S-S, Lemma 4.2.3, p. 325].

From the theorem of Grothendieck ([O-S-S, Theorem 2.1.1, p. 22]) and $c_1(E) = 0$, $E|_{\mathbf{P}_x}$ is trivial if and only if for an arbitrary non-zero section s of $E|_{\mathbf{P}_x}$ we have $s(z) \neq 0$ for all z in $\mathbf{P}_x$.

Since $\mathcal{O}(-1,0)|_{\mathbf{P}_x}\cong\mathcal{O}(-1)$, $\mathcal{O}(0,1)|_{\mathbf{P}_x}\cong\mathcal{O}(1)$ and $\mathcal{O}(-1,1)|_{\mathbf{P}_x}\cong\mathcal{O}$ by [Na-3, Lemma 3.3], the display of the restricted monad (MI) to the twistor fibre $\mathbf{P}_x$ implies that $I: H^0(\mathbf{P}_x, E|_{\mathbf{P}_x})\cong H^0(\mathbf{P}_x, \mathrm{Ker}\,\beta|_{\mathbf{P}_x})\to\mathbf{C}^{n+2}\oplus\mathbf{C}$ is injective.

If $E|_{\mathbf{P}_x}$ is trivial, the injectivity of $I: H^0(\mathbf{P}_x, E|_{\mathbf{P}_x})\to\mathbf{C}^{n+2}\oplus\mathbf{C}$ yields that there exists a subspace $E_x\subset\mathbf{C}^{n+2}\oplus\mathbf{C}$ such that

$$(5.7)\qquad \bigcap_{[g]\in\mathbf{P}_x}\mathrm{Ker}\,\beta_{[g]}=E_x \qquad\text{and}\qquad \bigcup_{[g]\in\mathbf{P}_x}\mathrm{Im}\,\alpha_{[g]}\cap E_x=\{0\}\,,$$

where we denote $\alpha([g],\cdot)$ by $\alpha_{[g]}: \mathbf{C}e\to\mathbf{C}^{n+2}\oplus\mathbf{C}$ and $\beta([g],(\cdot,\cdot))$ by $\beta_{[g]}: \mathbf{C}^{n+2}\oplus\mathbf{C}\to\mathbf{C}f$, using (5.1) and (5.2). We claim that if $[g_1]$ and $[g_2]$ are different points in $\mathbf{P}_x$, then $\mathrm{Im}\,\alpha_{[g_1]}\cap\mathrm{Im}\,\alpha_{[g_2]}=\{0\}$. Let $Sp(1)$ be the subgroup in G of which the corresponding Lie algebra is generated by the highest root and $U(1)$ be the standard subgroup of $Sp(1)$. Note that $j\in Sp(1)\setminus U(1)$. Then by definition of the twistor space [S], there exists $s\in Sp(1)\setminus U(1)$ such that $g_2=g_1 s$. We assume that there exists a non-zero constant c such that $\alpha([g_1, ce])=\alpha([g_2, e])$. By (5.1), we have $cAg_1e=Ag_1se$. Lemma 5.2, yields that $se=ce$ and so, $s\in U(1)$. This is a contradiction. Hence, $\mathbf{C}^{n+2}\oplus\mathbf{C}$ is decomposed into $\mathrm{Im}\,\alpha_{[g_1]}\oplus\mathrm{Im}\,\alpha_{[g_2]}\oplus E_x$. Then $\beta_{[g_2]}\circ\alpha_{[g_1]}: \mathbf{C}e\to\mathbf{C}f$ is an isomorphism by (5.7).

Next, we assume that the restricted bundle $E|_{\mathbf{P}_x}$ is not trivial and so, there exist a non-zero section $s\in H^0(\mathbf{P}_x, E|_{\mathbf{P}_x})$ and $g_1\in G$ such that $s([g_1])=0$. By the injectivity of I, there exists a unique $(u,c)\in\mathbf{C}^{n+2}\oplus\mathbf{C}$ such that $I(s([g]))=([g],(u,c))$, where $[g]\in\mathbf{P}_x$. From the definition of I, we obtain $\beta([g],(u,c))=0$ for an arbitrary $[g]\in\mathbf{P}_x$. Since $s([g_1])=0$, there exists a constant c' such that $\alpha([g_1, c'e])=([g_1],(u,c))$. Consequently, we have $\beta_{[g]}\circ\alpha_{[g_1]}(c'e)=\beta(([g],(u,c))=0$. Thereby, $E|_{\mathbf{P}_x}$ is trivial if and only if $\beta_{[g_2]}\circ\alpha_{[g_1]}: \mathbf{C}e\to\mathbf{C}f$ is an isomorphism for arbitrary $[g_1]\neq[g_2]\in\mathbf{P}_x$. However, using $j\in Sp(1)\subset G$, we can substitute $g_1 j$ for g_2.

On the other hand, a direct computation and the definition of $j\in G$ imply that $\beta_{[gj]}\circ\alpha_{[g]}(e)=\{-h(BAge, ge)+w(gf)h(z, gf)\}f$. Hence, $E|_{\mathbf{P}_x}$ is trivial if and only if

$$(5.8)\qquad -h(BAge, ge)+w(gf)h(z, gf)\neq 0\,,$$

for an arbitrary $[g]\in\mathbf{P}_x$. Since $BA+z\otimes w=c\,\mathrm{Id}_{\mathbf{C}^{n+2}}$ by Lemma 5.4, (5.8) is equivalent to $w(ge)h(z, ge)+w(gf)h(z, gf)\neq c$. Taking account of the standard representation of G, we obtain the desired result. □

By [Na-3. Lemma 5.1.7], we have isomorphisms $s_1: \mathcal{O}(-1,0)\cong\sigma^*\overline{\mathcal{O}(0,1)}^*$ and $s_2: \mathcal{O}(0,1)\cong\sigma^*\overline{\mathcal{O}(-1,0)}^*$. Since $\mathcal{O}(-1,1)$ is the pull-back of an anti-self-dual bundle from [Na-3, Theorem 3.4], the Ward correspondence implies that $s: \mathcal{O}(-1,1)\cong\sigma^*\overline{\mathcal{O}(-1,1)}$. We call these isomorphisms the standard isomorphisms. More explicitly, the standard isomorphisms are expressed as:

$$s_1([g,e])=([g],[gj,-h(f,\cdot)])\,,\qquad s_2([g,f])=([g],[gj,h(e,\cdot)])\,,$$
$$s([g,e\otimes f])=([g],[gj,h(e\otimes f,\cdot)])\,,$$

where the last h means the induced metric from $h|_{\mathbf{C}e}$ and $h|_{\mathbf{C}f}$.

PROPOSITION 5.9. *Let E be the cohomology bundle of the monad* (MI). *Moreover, the restricted bundle $E|_{\mathbf{P}_x}$ is assumed to be trivial for every x in $Gr_2(\mathbf{C}^{n+2})$. Then, there is an isomorphism $\tau : E \to \sigma^*\bar{E}^*$ with $(\sigma^*\bar{\tau})^* = \tau$ which induces a positive definite hermitian form on sections of $E|_{\mathbf{P}^1_x}$ for every $x \in Gr_2(\mathbf{C}^{n+2})$ if and only if there exist a hermitian metric on $\underline{\mathbf{C}^{n+2}}$, a hermitian metric on $\mathcal{O}(-1, 1)$ and isomorphisms $\mathcal{O}(-1, 0) \cong \sigma^*\overline{\mathcal{O}(0, 1)}^*$ and $\mathcal{O}(0, 1) \cong \sigma^*\overline{\mathcal{O}(-1, 0)}^*$.*

PROOF. From the hypothesis, $\sigma^*\bar{E}^*$ is the cohomology bundle of the monad;

$$\sigma^*\overline{\mathcal{O}(0, 1)}^* \xrightarrow{\sigma^*\bar{\beta}^*} \sigma^*\overline{\underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1, 1)}^* \xrightarrow{\sigma^*\bar{\alpha}^*} \sigma^*\overline{\mathcal{O}(-1, 0)}^* .$$

We can check the conditions in a theorem of Okonek-Schneider-Spindler ([O-S-S, Lemma 4.1.3, p. 276]) using the Bott-Borel-Weil theorem and standard isomorphisms $\mathcal{O}(-1, 0) \cong \sigma^*\overline{\mathcal{O}(0, 1)}^*$, $\mathcal{O}(0, 1) \cong \sigma^*\overline{\mathcal{O}(-1, 0)}^*$ and $\mathcal{O}(-1, 1) \cong \sigma^*\overline{\mathcal{O}(-1, 1)}$. Then we obtain that there is an isomorphism $\tau : E \to \sigma^*\bar{E}^*$ if and only if there exist isomorphisms $\underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1, 1) \cong \sigma^*\overline{\underline{\mathbf{C}^{n+2}}}^* \oplus \sigma^*\overline{\mathcal{O}(-1, 1)}^*$, $\mathcal{O}(-1, 0) \cong \sigma^*\overline{\mathcal{O}(0, 1)}^*$ and $\mathcal{O}(0, 1) \cong \sigma^*\overline{\mathcal{O}(-1, 0)}^*$. Since $H^0(\mathcal{O}(-1, 1)) = H^0(\mathcal{O}(1, -1)) = 0$, we have $\underline{\mathbf{C}^{n+2}} \cong \sigma^*\overline{\underline{\mathbf{C}^{n+2}}}^*$ and $\mathcal{O}(-1, 1) \cong \sigma^*\overline{\mathcal{O}(-1, 1)}^*$. The restricted bundle $\underline{\mathbf{C}^{n+2}}|_{\mathbf{P}_x}$ and $\mathcal{O}(-1, 1)|_{\mathbf{P}_x}$ are trivial by [Na-3, Lemma 3.3]. Hence, these isomorphisms induce non-degenerate hermitian forms on $\underline{\mathbf{C}^{n+2}}$ and $\mathcal{O}(-1, 1)$ respectively.

Next we take the condition imposed upon τ into account. Since $E|_{\mathbf{P}^1_x}$ is trivial, under the notation in the proof of Proposition 5.8, this condition yields E_x has a positive hermitian inner product. If we make use of an identification $\mathcal{O}(-1, 1)|_{\mathbf{P}_x} \cong \mathbf{P}_x \times \mathbf{C}_x$, the proof of Proposition 5.8 yields that

$$\mathrm{Im}\,\alpha_{[g]} \oplus \mathrm{Im}\,\alpha_{[gj]} \oplus E_x = \mathbf{C}^{n+2} \oplus \mathbf{C}_x , \tag{5.9}$$

where $[g] \in \mathbf{P}_x$. Since there exists $g \in G$ such that $h(z, gf) = h(z, gje) = 0$, we have $\mathbf{C}_x \subset E_x$ for this $g \in G$ by (5.1). Then the induced hermitian form on $\mathcal{O}(-1, 1)|_{\mathbf{P}_x}$ is positive, because of the positivity of the hermitian inner product on E_x. The non-degeneracy of the hermitian form implies the positivity of this hermitian form and so, $\mathcal{O}(-1, 1)$ has a hermitian metric. On the other hand, the property of the standard representation of G, Lemma 5.2 and (5.9) yield that the vector space spanned by $\bigcup_{x \in Gr_2(\mathbf{C}^{n+2})} E_x$ has $\mathbf{C}^{n+2}$ as a subspace. Consequently, the induced hermitian form on $\underline{\mathbf{C}^{n+2}}$ is also positive.

Conversely, a hermitian metric on $\underline{\mathbf{C}^{n+2}}$ induces an isomorphism $\underline{\mathbf{C}^{n+2}} \cong \sigma^*\overline{\underline{\mathbf{C}^{n+2}}}^*$ and a hermitian metric on $\mathcal{O}(-1, 1)$ induces an isomorphism $\mathcal{O}(-1, 1) \cong \sigma^*\overline{\mathcal{O}(-1, 1)}^*$. Combined with isomorphisms $\mathcal{O}(-1, 0) \cong \sigma^*\overline{\mathcal{O}(0, 1)}^*$ and $\mathcal{O}(0, 1) \cong \sigma^*\overline{\mathcal{O}(-1, 0)}^*$, these induce the desired $\tau : E \to \sigma^*\bar{E}^*$ under the hypothesis that $E|_{\mathbf{P}_x}$ is trivial. □

Therefore, to describe the moduli space, we fix the G-invariant hermitian inner product h on $\mathbf{C}^{n+2}$ and the standard isomorphisms s, s_1, s_2.

PROPOSITION 5.10. *Under this fixed isomorphisms* $\mathbf{C}^{n+2}\cong\sigma^*\overline{\mathbf{C}^{n+2}}^*$, $s:\mathcal{O}(-1,1)\cong\sigma^*\overline{\mathcal{O}(-1,1)}^*$, $s_1:\mathcal{O}(-1,0)\cong\sigma^*\overline{\mathcal{O}(0,1)}^*$ *and* $s_2:\mathcal{O}(0,1)\cong\sigma^*\overline{\mathcal{O}(-1,0)}^*$, *the following two conditions are equivalent*:

(1) *There exists a commutative diagram*

$$\begin{array}{ccccc} \mathcal{O}(-1,0) & \xrightarrow{\alpha} & \mathbf{C}^{n+2}\oplus\mathcal{O}(-1,1) & \xrightarrow{\beta} & \mathcal{O}(0,1) \\ \downarrow & & \downarrow & & \downarrow \\ \sigma^*\overline{\mathcal{O}(0,1)}^* & \xrightarrow[\sigma^*\bar{\beta}^*]{} & \sigma^*\overline{\mathbf{C}^{n+2}\oplus\mathcal{O}(-1,1)}^* & \xrightarrow[\sigma^*\bar{\alpha}^*]{} & \sigma^*\overline{\mathcal{O}(-1,0)}^* . \end{array}$$

(2) *For all* u *and* v *in* $\mathbf{C}^{n+2}$, $h(Au,v)=h(u,Bv)$ *and* $w=-h(\cdot,z)$.

PROOF. If the diagram is commutative, a direct computation shows that for all u in $\mathbf{C}^{n+2}$ and all g in G, $h(Age,u)=\overline{h(Bu,ge)}$ and $h(z,gf)=\overline{w(gf)}$. The irreducibility of the standard action of G implies the condition (2). Now it is clear that (2) yields (1). □

We denote by A^* the adjoint operator of A with respect to h. Let $\mathcal{M}$ be the moduli space of anti-self-dual connections on E satisfying the hypothesis in Main Theorem 1.

PROOF OF MAIN THEOREM 2. From Lemma 5.4 and Proposition 5.10, we have $A^*Au-h(u,z)z=cu$ for an arbitrary $u\in\mathbf{C}^{n+2}$. In particular, $A^*Az=(|z|^2+c)z$, and if u is orthogonal to z, then $A^*Au=cu$. Hence A is an automorphism on $\mathbf{C}^{n+2}$ if and only if c is a positive real number. On the other hand, Propositions 5.8 and 5.10 imply that $|z|^2\neq -c$. However, this condition is satisfied automatically because of the positivity of c.

By Proposition 5.6, (z,c) and (z',c') induce the isomorphic monads if and only if there exists a non-zero constant $p\in\mathbf{C}^*$ such that $(z',c')=(pz,|p|^2c)$. Consequently, we obtain

$$\mathcal{M}=\{(z,c)\in\mathbf{C}^{n+2}\times\mathbf{R}^+\}/\mathbf{C}^*,$$

where $\mathbf{C}^*$-action is defined as $p\cdot(z,c)=(pz,|p|^2c)$.

Next, we focus our attention on $|z|^2+c$ which is an eigenvalue of A^*A. Since $|pz|^2+|p|^2c=|p|^2(|z|^2+c)$, we can normalize in such a way that $|z|^2+c=1$ using the $\mathbf{C}^*$-action. Then the $\mathbf{C}^*$-action is reduced to S^1-action such that $p\cdot(z,c)=(pz,c)$ where $p\in S^1\subset\mathbf{C}^*$. Therefore, taking account of the positivity of c, we obtain that

$$\mathcal{M}=\{z\in\mathbf{C}^{n+2}\mid |z|^2<1\}/S^1 .$$ □

REMARK. Under the assumption that $|z|^2+c=1$, we may put

$$\begin{cases} Az=z & \\ Au=\sqrt{1-|z|^2}\,u & \text{if } u\in\mathbf{C}^{n+2} \text{ is orthogonal to } z . \end{cases}$$

First we suppose that $z=0$. Then $A=\mathrm{Id}_{\mathbf{C}^{n+2}}$. From Propositions 5.1 and 5.10, the cohomology bundle (MI) is decomposed into $\mathcal{O}(-1,1)$ and the cohomology bundle of the monad:

$$\mathcal{O}(-1, 0) \to \underline{\mathbf{C}^{n+2}} \to \mathcal{O}(0, 1)\,.$$

This monad is the standard monad induced by ϖ_{n+1}. (This terminology is defined in [Na-3, Definition 4.4].) Moreover, the cohomology bundle of the standard monad is homogeneous by [Na-3, Theorem 4.5]. In fact, the cohomology bundle of (MI) is isomorphic to a direst sum $Q_2(0, 1) \oplus \mathcal{O}(-1, 1)$. Consequently, the "vertex" of $\mathcal{M}$ corresponds to a reducible connection. (A $SU(n+1)$-anti-self-dual connection reduces to a $U(n) \times U(1)$ connection.) The centralizer of $U(n) \times U(1)$ in $SU(n+1)$ is $U(1)$. We denote by $\mathbf{Z}_{n+1}$ the center of $SU(n+1)$. The group $U(1)/\mathbf{Z}_{n+1}$ is nothing but S^1 in the description of the moduli in Main Theorem 2.

Finally, we put $|z|^2 = 1$. Then $\mathrm{Ker} A = \{u \in \mathbf{C}^{n+2} \,|\, u \perp z\}$. For brevity, $\mathrm{Ker} A$ is expressed as $z^{\perp}$. The proof of Lemma 5.2 implies that $\alpha([g, e]) = 0$ if and only if $ge \in z^{\perp}$ and $gf \in z^{\perp}$. Combined with Proposition 5.10, the monad (MI) does not define a vector bundle on $Gr_2(z^{\perp})$. In the case $n = 1$, this is a well-known fact, because $Gr_2(z^{\perp})$ is one point.

References

[A] M. F. Atiyah, *Geometry of Yang-Mills Fields*, Lezioni Fermiane, Scuola Norm. Sup. Pisa (1979).

[Ba] R. J. Baston, Quaternionic complexes, J. Geom. Phys. **8** (1992), 29–52.

[B-G-G] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, Schubert cells and the cohomology of the spaces G/P, London Math. Soc. Lecture Note Ser. **69** (1982), Cambridge Univ. Press, 115–140.

[Bu] N. P. Buchdahl, Instantons on CP^2, J. Diff. Geom. **24** (1986), 19–52.

[D] S. K. Donaldson, *Vector Bundles on the Flag Manifolds and the Ward Correspondence in Geometry Today*, Progress in Math. **60** (1985), Birkhäuser.

[G-P] K. Galicki and Y. S. Poon, Duality and Yang-Mills fields on quaternionic Kähler manifold, J. Math. Phys. **32** (1991), 1263–1268.

[K-N] Y. Kametani and Y. Nagatomo, Construction of c_2-self-dual bundles on a quaternionic projective space, Osaka J. Math. **32** (1995), 1023–1033.

[K] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem, Ann. of Math. **74** (1961), 329–387.

[M-S] M. Mamone Capria and S. M. Salamon, Yang-Mills fields on quaternionic spaces, Non-linearity **1** (1988), 517–530.

[Na-1] Y. Nagatomo, Rigidity of c_1-self-dual connections on quaternionic Kähler manifolds, J. Math. Phys. **33** (1992), 4020–4025.

[Na-2] Y. Nagatomo, Vanishing theorem for cohomology groups of c_2-self-dual bundles on quaternionic Kähler manifolds, Diff. Geom. Appl. **5** (1995), 79–95.

[Na-3] Y. Nagatomo, *Representation theory and ADHM-construction on quaternion symmetric spaces*, preprint.

[N-N1] Y. Nagatomo and T. Nitta, Vanishing theorem for quaternionic complexes, Bull. London Math. Soc. **29** (1997), 359–366.

[N-N2] Y. Nagatomo and T. Nitta, k-instantons on $G_2(\mathbf{C}^{n+2})$ and stable vector bundles, to appear in Math. Z.

[N-N3] Y. Nagatomo and T. Nitta, Moduli of 1-instantons on $G_2(\mathbf{C}^{n+2})$, Diff. Geom. Appl. **7** (1997), 115–122.

[Ni] T. NITTA, Vector bundles over quaternionic Kähler manifolds, Tohoku. Math. J. **40** (1988), 425–440.

[O-S-S] C. OKONEK, M. SCHNEIDER and H. SPINDLER, *Vector Bundles on Complex Projective Spaces* Progress in Math. **3** (1980), Birkhäuser.

[S] S. M. SALAMON, Quaternionic Kähler manifolds, Invent. Math. **67** (1982), 143–171.

[W-W] R. S. WARD and R. O. WELLS, JR, *Twistor Geometry and Field Theory*, Cambridge Monographs Math. Physics (1989).

Present Address:
INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA,
TSUKUBA-SHI, IBARAKI, 305–8571, JAPAN.
e-mail: nagatomo@math.tsukuba.ac.jp