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Anscombe's theorem 60 years later

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Abstract

The point of departure of the present paper is Anscombe's seminal 1952-paper on limit theorems for randomly indexed processes. We discuss the importance of this result and mention some of its impact, mainly on stopped random walks. The main aim of the paper is to illustrate the beauty and efficiency of, what will be called, the *Stopped Random Walk-method* (the SRW-method).

1 Introduction

The typical or standard procedure for estimating a parameter or to test some hypothesis concerning the parameter is to take a sample and perform the necessary analysis. Now, the first obvious (polemic) remark against this procedure is that one might have taken an unnecessarily large sample; a smaller one would have been sufficient, and this would also have saved lives. Alternatively, the sample was not large enough in order to allow for a (sufficiently) significant conclusion.

A natural suggestion thus would be to take an appropriately defined *random size* sample, where the (random) size typically would be defined by stopping when something particular occurs.

The first obvious task that then suggests itself would be to check, that is, to prove or disprove, certain (standard) results that hold for processes with fixed index or time for the setting with a random index or time.

A first example illustrating that things may go wrong is the following.

Example 1.1. Let $X, X_1, X_2, \ldots$ be independent, identically distributed (i.i.d.) coin-tossing random variables, that is, $P(X = 1) = P(X = -1) = 1/2$, set $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, and let

$$N = \min\{n : S_n = 1\}.$$

Since $\{S_n, n \geq 1\}$ is a centered random walk, we know that

$$E\,S_n = 0 \quad \text{for all} \quad n.$$

However, we immediately observe that, since $S_N = 1$ a.s., we must have

$$E\,S_N = 1 \neq 0.$$

So, the natural guess that $E\,S_n = 0$ might be replaced by

$$E\,S_N = E\,N \cdot E\,X,$$

does not seem to be true.

Or, ... is it true "sometimes"?

The answer to this one is "yes". Sometimes. In the present example the problem is that $E\,N = +\infty$, which implies that the RHS equals $\infty \cdot 0$. □

Hmmm, ... but, with regard to Anscombe's theorem, what about the central limit theorem?

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Example 1.2. Consider the same example with

$$N(n) = \text{the index of the actual partial sum at the time of the } n\text{th visit to } 0, \quad n \geq 1.$$

Now, from random walk theory we know that $P(S_n = 0 \text{ i.o.}) = 1$, so that $N(n) \stackrel{a.s.}{\to} \infty$ as $n \to \infty$. However

$$\frac{S_{N(n)}}{\sqrt{N(n)}} = 0 \quad \text{for all} \quad n,$$

which is far from asymptotic normality. Thus, something more than $N(n) \stackrel{a.s.}{\to} +\infty$ as $n \to \infty$ seems to be necessary in order to ensure a positive result. □

Example 1.3. I toss a coin until the first head appears, after which you toss a coin the same number of times. Clearly the outcomes for your coin are independent of the number of tosses required for me to succeed. □

Although this is not a particularly interesting example, it illustrates how the outcomes of some process under investigation is independent of the number of performances of the process. However, a natural context with this kind of independence is the Galton–Watson process, where "the size of next generation" is determined by a random sum in which the summands are the children of "the current generation" and the upper summation index equals "the number of sisters and brothers of the current generation". Thus, in this important example the number of terms in the sum is indeed independent of the summands.

The mathematically most interesting case is when the family of indices constitutes a family of stopping times, in particular, relative to the random walk at hand. Formally, (cf. [20]) if $\{S_n,\, n \geq 1\}$ is a random walk and $\{\tau(t),\, t \geq 0\}$ is a family of random indices, such that

$$\{\tau(t) \leq n\} \quad \text{is} \quad \sigma\{S_1, S_2, \ldots, S_n\}\text{-measurable},$$

we call the family

$$\{S_{\tau(t)},\, t \geq 0\} \quad \text{a } \textit{Stopped Random Walk.}$$

The central point of this paper is to show how one can take an ordinary limit theorem, such as the law of large numbers and the central limit theorem, as point of departure, and then, via a random index version, obtain some desired result. In several instances it is, in fact, not necessary for the indices to be stopping times. The two limit theorems just mentioned are such examples; the stopping time property is essential when martingale methods come into play, for example in results concerning existence of moments. We shall nevertheless call the approach the "Stopped random walk method", the *SRW-method* for short. As we shall see the method leads to efficient and neat proofs.

And, in order to illustrate all of this, Anscombe's theorem is a beautiful point of departure and source of inspiration.

In Section 2 we present a random-sum-SLLN and a random-sum-CLT. The latter is a special case of Anscombe's theorem, which, in this form with a direct proof, is due to Rényi [33]. We also state and prove an extension of his result to weighted sums for later use. After this, Section 3 is devoted to renewal theory for random walks, Section 4 to a two-dimensional extension, after which we include a section containing some applications to probabilistic models in various contexts where random sums are the key object. Continuing down the road, Section 6 is devoted to perturbed random walks, followed by a section on repeated significance tests. We close with a section on records, which, on the one hand is not immediately related to random walks, but, on the other, illustrates how certain results can be obtained with the aid of an interesting generalization of Anscombe's theorem to a non-i.i.d. setting.

2 Anscombe's theorem

As mentioned in the introduction, it might, sometimes, in practice, be more natural to study random processes during fixed time intervals, which means that the number of observations is random.

Following is the celebrated result due to Anscombe [2], which was established as "recently" as in 1952.

Theorem 2.1. *Suppose that* $Y_1, Y_2, \dots$ *are random variables, such that*

$$Y_n \xrightarrow{d} Y \quad \text{as} \quad n \to \infty,$$

and that $\{\tau(t),\, t \geq 0\}$ *is a family of positive, integer valued random variables, such that, for some family of positive reals* $\{b(t),\, t \geq 0\}$*, where* $b(t) \nearrow \infty$ *as* $t \to \infty$,

$$\frac{\tau(t)}{b(t)} \xrightarrow{p} 1 \quad \text{as} \quad t \to \infty. \tag{2.1}$$

Finally, suppose that, given $\varepsilon > 0$*, there exist* $\eta > 0$ *and* n_0*, such that, for all* $n > n_0$,

$$P\Big(\max_{\{k:|k-n|<n\delta\}} |Y_k - Y_n| > \varepsilon\Big) < \eta. \tag{2.2}$$

Then

$$Y_{\tau(t)} \xrightarrow{d} Y \quad \text{as} \quad t \to \infty.$$

Remark 2.1. Condition (2.2) is called the *Anscombe condition*; Anscombe calls the condition *uniform continuity in probability.*

Remark 2.2. The important feature of the theorem is that *nothing is assumed about independence between the random sequence* $\{Y_n,\, n \geq 1\}$ *and the index family.*

Remark 2.3. It is no restriction to assume that the limit in (2.1) equals 1, since any other value could be absorbed into the normalizing sequence.

Remark 2.4. The limit in (2.1) may, in fact, be replaced by a positive random variable; see, e.g., [3] and [38]. □

In order to keep ourselves within reasonable bounds we shall in the remainder of the paper (basically) confine ourselves to randomly indexed partial sums of i.i.d. random variables, in which case Anscombe's theorem turns into a "random sum central limit theorem". The following version was first given with a direct proof by Rényi [33]. The essence is that, instead of verifying the Anscombe condition, Rényi provides a direct proof (which essentially amounts to the same work).

For completeness (and since we shall need it later) we begin with a "random sum strong law", which is a consequence of the Kolmogorov strong law and the fact that the union of two null sets is, again, a null set.

Theorem 2.2. *Let* $X_1, X_2, \dots$ *be i.i.d. random variables with finite mean* μ*, set* $S_n = \sum_{k=1}^n X_k$, $n \geq 1$*, and suppose that* $\{\tau(t),\, t \geq 0\}$ *is a family of positive, integer valued random variables, such that* $\tau(t) \xrightarrow{a.s.} +\infty$ *as* $t \to \infty$*. Then*

$$\frac{S_{\tau(t)}}{\tau(t)} \xrightarrow{a.s.} \mu \quad \text{and} \quad \frac{X_{\tau(t)}}{\tau(t)} \xrightarrow{a.s.} 0 \quad \text{as} \quad t \to \infty.$$

If, in addition, $\tau(t)/t \xrightarrow{a.s.} \theta$ *as* $t \to \infty$ *for some* $\theta \in (0, \infty)$*, then*

$$\frac{S_{\tau(t)}}{t} \xrightarrow{a.s.} \mu\theta \quad \text{as} \quad t \to \infty.$$

Here is now Rényi's adaptation of Anscombe's theorem to random walks.

Theorem 2.3. *Let* $X_1, X_2, \dots$ *be i.i.d. random variables with mean* 0 *and positive, finite, variance* σ^2*, set* $S_n = \sum_{k=1}^n X_k$, $n \geq 1$*, and suppose that* $\{\tau(t),\, t \geq 0\}$ *is a family of positive, integer valued random variables, such that*

$$\frac{\tau(t)}{t} \xrightarrow{p} \theta \quad (0 < \theta < \infty) \quad \text{as} \quad t \to \infty. \tag{2.3}$$

Then

$$\frac{S_{\tau(t)}}{\sigma\sqrt{\tau(t)}} \xrightarrow{d} N(0,1) \quad \text{and} \quad \frac{S_{\tau(t)}}{\sigma\sqrt{\theta t}} \xrightarrow{d} N(0,1) \quad \text{as} \quad t \to \infty.$$

Remark 2.5. The normalization with t in (2.3) can be replaced by more general increasing functions of t, such as t raised to some power. This influences only the second assertion. □

Instead of providing Rényi's direct proof of this landmark result, we shall, in the following subsection, adapt it to a generalization to weighted sums.

2.1 An Anscombe-Rényi theorem for weighted sums

Theorem 2.4. *Let X_1, X_2, ... be i.i.d. random variables with mean 0 and positive, finite, variance σ^2, let $\gamma > 0$, and set $S_n = \sum_{k=1}^{n} k^\gamma X_k$, $n \geq 1$. Suppose that $\{\tau(t),\, t \geq 0\}$ is a family of positive, integer valued random variables, such that*

$$\frac{\tau(t)}{t^\beta} \xrightarrow{p} \theta \quad (0 < \theta < \infty) \quad as \quad t \to \infty, \tag{2.4}$$

for some $\beta > 0$. Then

$$\frac{S_{\tau(t)}}{(\tau(t))^{\gamma+(1/2)}} \xrightarrow{d} N\big(0, \frac{\sigma^2}{2\gamma+1}\big) \qquad and \qquad \frac{S_{\tau(t)}}{t^{\beta(2\gamma+1)/2}} \xrightarrow{d} N\big(0, \frac{\sigma^2\theta^{2\gamma+1}}{2\gamma+1}\big) \quad as \quad t \to \infty.$$

Proof. First of all, for weighted sums it is well known (and/or easily checked with the aid of characteristic functions) that

$$\frac{S_n}{n^{\gamma+(1/2)}} \xrightarrow{d} N\big(0, \frac{\sigma^2}{2\gamma+1}\big) \quad \text{as} \quad n \to \infty. \tag{2.5}$$

In the remainder of the proof we assume w.l.o.g. that $\sigma^2 = \theta = 1$. With $n_0 = [t^\beta]$ we then obtain

$$\frac{S_{\tau(t)}}{\tau(t)^{\gamma+(1/2)}} = \Big(\frac{S_{n_0}}{n_0^{\gamma+(1/2)}} + \frac{S_{\tau(t)} - S_{n_0}}{n_0^{\gamma+(1/2)}}\Big)\Big(\frac{n_0}{\tau(t)}\Big)^{\gamma+(1/2)},$$

so that, in view of (2.4) and (2.5), it remains to show that

$$\frac{S_{\tau(t)} - S_{n_0}}{n_0^{\gamma+(1/2)}} \xrightarrow{p} 0 \quad \text{as} \quad t \to \infty$$

for the first claim, which, in turn, yields the second one.

Toward that end, let $\varepsilon \in (0, 1/3)$, and set $n_1 = [n_0(1-\varepsilon^3)] + 1$ and $n_2 = [n_0(1+\varepsilon^3)]$. Then, by exploiting the Kolmogorov inequality, we obtain

$$\begin{aligned}
P(|S_{\tau(t)} - S_{n_0}| > \varepsilon n_0^{\gamma+(1/2)}) &= P\big(\{|S_{\tau(t)} - S_{n_0}| > \varepsilon n_0^{\gamma+(1/2)}\} \cap \{\tau(t) \in [n_1, n_2]\}\big) \\
&\quad + P\big(\{|S_{\tau(t)} - S_{n_0}| > \varepsilon n_0^{\gamma+(1/2)}\} \cap \{\tau(t) \notin [n_1, n_2]\}\big) \\
\leq\ & P(\max_{n_1 \leq k \leq n_0} |S_k - S_{n_0}| > \varepsilon n_0^{\gamma+(1/2)}) + P(\max_{n_0 \leq k \leq n_2} |S_k - S_{n_0}| > \varepsilon n_0^{\gamma+(1/2)}) \\
&\quad + P(\tau(t) \notin [n_1, n_2]) \\
\leq\ & \frac{\sum_{k=n_1+1}^{n_0} k^{2\gamma}}{\varepsilon^2 n_0^{2\gamma+1}} + \frac{\sum_{k=n_0+1}^{n_2} k^{2\gamma}}{\varepsilon^2 n_0^{2\gamma+1}} + P(\tau(t) \notin [n_1, n_2]) \leq \frac{(n_2 - n_1) n_2^{2\gamma}}{\varepsilon^2 n_0^{2\gamma+1}} + P(\tau(t) \notin [n_1, n_2]) \\
\leq\ & \frac{2 n_0 \varepsilon^3 (n_0(1+\varepsilon^3))^{2\gamma}}{\varepsilon^2 n_0^{2\gamma+1}} + P(\tau(t) \notin [n_1, n_2]) = 2\varepsilon(1+\varepsilon^3)^{2\gamma} + P(\tau(t) \notin [n_1, n_2]),
\end{aligned}$$

so that, recalling (2.4),

$$\limsup_{t \to \infty} P(|S_{\tau(t)} - S_{n_0}| > \varepsilon n_0^{\gamma+(1/2)}) \leq 2\varepsilon(1+\varepsilon^3)^{2\gamma},$$

which, due to the arbitrariness of ε, proves the conclusion. □

2.2 A generalized Anscombe-Rényi theorem

There also exist versions for more general sums of non-i.i.d. distributed random variables based on the Lindeberg conditions, at times under *Generalized Anscombe Conditions*. Since Anscombe's theorem is the main focus of this paper, and since, in fact, we shall, in our final section on records, apply the following generalization due to Csörgő and Rychlik [5, 6], we present it here.

Toward this end the authors need the following generalized Anscombe condition: A sequence Y_1, Y_2, ... satisfies the generalized Anscombe condition with norming sequence $\{k_n,\, n \geq 1\}$ if, for every $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\limsup_{n \to \infty} P(\max_{\{j : |k_j^2 - k_n^2| \leq \delta k_n^2\}} |Y_j - Y_n| > \varepsilon) < \varepsilon). \tag{2.6}$$

Theorem 2.5. *Let $X_1, X_2, \ldots$ be independent random variables with finite variances, and set, for $k \geq 1$, $E\,X_k = \mu_k$, $\operatorname{Var} X_k = \sigma_k^2$, and, for $n \geq 1$, $S_n = \sum_{k=1}^n X_k$, and $s_n^2 = \sum_{k=1}^n \sigma_k^2$. Suppose that the Lindeberg conditions are satisfied, that $\{(S_n - \sum_{k=1}^n \mu_k)/s_n,\, n \geq 1\}$ satisfies the generalized Anscombe condition for some normalizing sequence $\{k_n,\, n \geq 1\}$, and that $\{\tau_n,\, n \geq 1\}$ is a sequence of positive, integer valued random variables, such that*

$$\frac{k_{\tau_n}}{k_{a_n}} \xrightarrow{p} 1 \quad \text{as} \quad n \to \infty, \tag{2.7}$$

for some sequence $\{a_n,\, n \geq 1\}$ of positive integers increasing to $+\infty$. Then,

$$\frac{S_{\tau_n} - \sum_{k=1}^{\tau_n} \mu_k}{s_{\tau_n}} \xrightarrow{d} N(0,1) \quad \text{as} \quad n \to \infty.$$

3 Renewal theory

A *random walk* $\{S_n,\, n \geq 0\}$ is a sequence of random variables starting at $S_0 = 0$ with i.i.d. increments $X_1, X_2, \ldots$. A *renewal process* is a random walk with nonnegative increments. The canonical example is a lightbulb, (more generally, some machine), that whenever it (some component) fails is instantly replaced by a new, identical one, which, upon failure is replaced by another one, and so on.

The central object of interest is the *(renewal) counting process*,

$$N(t) = \max\{n : S_n \leq t\}, \quad t \geq 0\,,$$

which counts the number of replacements during the time interval $(0, t]$.

A discrete example is the *binomial process*, in which the durations are independent, Be(p)-distributed random variables. This means that with probability p there is a new occurrence after one time unit and with probability $1 - p$ after zero time (an instant occurrence). The number of occurrences $N(t)$ up to time t follows a (translated) *negative binomial* distribution; some references are [8, 9, 32, 20].

A related topic is that of recurrent events, for which we refer to Feller's classic [7], see also [8], Chapter XIII, [32], Chapter 5.

Limit theorems, such as the strong law and the central limit theorem for the counting process, were originally established via inversion, technically via the relation

$$\{S_n \leq t\} = \{N(t) > n\}. \tag{3.1}$$

In addition, the lattice case and the nonlattice case were treated separately. Furthermore, the inversion method relies heavily on the fact that the summands are nonnegative. We refer to the above sources for details.

Before closing this short introduction to renewal theory we mention that the elements $N(t)$ of the counting process are *not* stopping times, whereas the *first passage times*

$$\tau(t) = \min\{n : S_n > t\}, \quad t \geq 0,$$

indeed are stopping times. We also note that for practical purposes, say, if one observes some random process it seems more reasonable to take action the first time some strange event occurs, rather than the last time it does not.

Next, we turn our attention to the case when the summands are not necessarily nonnegative, although having positive expectation. But first some pieces on notation.

A random variable without index is interpreted as a generic random variable for the corresponding i.i.d. sequence, $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$ for $x \in \mathbb{R}$.

3.1 Renewal theory for random walks

Let $X_1, X_2, \ldots$ be i.i.d. random variables with positive, finite, mean μ, partial sums S_n, $n \geq 1$, and the associated first passage process, $\{\tau(t),\, t \geq 0\}$ as above.

Now, whereas $\tau(t) = N(t)+1$ for all t for renewal processes this is no longer true here. Moreover, the inversion relation (3.1) breaks down in the random walk case, so one has to seek other methods of proof. In addition one can show that, for $r \geq 1$,

$$E(\tau(t))^r < \infty \quad \Longleftrightarrow \quad E(X^-)^r < \infty,$$

whereas

$$E(N(t))^r < \infty \quad \Longleftrightarrow \quad E(X^-)^{r+1} < \infty;$$

cf. [29], [20], Chapter 3. The "price" for lacking the stopping time property for the counting process is additional integrability.

The important point is that all proofs to follow will be based on the SRW-method. In particular, Anscombe's theorem will be the decisive tool for the central limit theorem.

Before we step into results and proofs here is one fundamental piece involved in the SRW-method, namely "the sandwich lemma".

Lemma 3.1. *We have*

$$t < S_{\tau(t)} \leq t + X_{\tau(t)} = t + X^+_{\tau(t)}.$$

Proof. The result is an immediate consequence of the facts that

$$S_{\tau(t)-1} \leq t < S_{\tau(t)},$$

and that the final jump is necessarily positive. □

Here is now the strong law for first passage times.

Theorem 3.1. *In the above setup,*

$$\frac{\tau(t)}{t} \stackrel{a.s.}{\to} \frac{1}{\mu} \quad \textit{as} \quad t \to \infty.$$

Proof. First of all, $\tau(t) \stackrel{p}{\to} \infty$ as $t \to \infty$, and is nondecreasing, so that, in fact, $\tau(t) \stackrel{a.s.}{\to} \infty$ as $t \to \infty$, which, via Theorem 2.2, tells us that

$$\frac{S_{\tau(t)}}{\tau(t)} \stackrel{a.s.}{\to} \mu \quad \text{and that} \quad \frac{X_{\tau(t)}}{\tau(t)} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad t \to \infty. \tag{3.2}$$

An application of the sandwich lemma concludes the proof. □

Next is the corresponding central limit theorem.

Theorem 3.2. *If, in addition,* $\operatorname{Var} X = \sigma^2 < \infty$, *then*

$$\frac{\tau(t) - t/\mu}{\sqrt{\frac{\sigma^2 t}{\mu^3}}} \stackrel{d}{\to} N(0,1) \quad \textit{as} \quad t \to \infty.$$

Proof. The central limit theorem and Anscombe's theorem (notably Theorem 2.3) together yield

$$\frac{S_{\tau(t)} - \mu\tau(t)}{\sqrt{\sigma^2\tau(t)}} \stackrel{d}{\to} N(0,1) \quad \text{as} \quad t \to \infty.$$

By Theorem 2.2 and the sandwich formula we next obtain

$$\frac{t - \mu\tau(t)}{\sqrt{\sigma^2\tau(t)}} \stackrel{d}{\to} N(0,1) \quad \text{as} \quad t \to \infty,$$

which, via the strong law Theorem 3.1, applied to the denominator, and the symmetry of the normal distribution finishes the proof. □

3.2 A short intermediate summary

We observe that the proofs above cover all cases; lattice, nonlattice, pure renewal, as well as random walks. Because of its efficiency and usefulness we call it, as mentioned in the introduction, "the SRW-method".

To summarize we observe that the ingredients of the SRW-method are:

♠ An ordinary limit theorem, such as the strong law or the central limit theorem;

♠ A transitory theorem that tells us that the ordinary result is also valid for random sums, such as Theorem 2.2 and Anscombe's theorem (for our purposes Rényi's version);

♠ A sandwich inequality, typically Lemma 3.1.

3.3 A remark on additional results

As mentioned earlier our main focus is on the central limit theorem. However, let us, in passing and for completeness, briefly mention that there also exist

♯ Marcinkiewicz-Zygmund type moment inequalities, cf. [10, 20];

♯ Marcinkiewicz-Zygmund laws, cf. [10, 20];

♯ LIL results, cf. [37, 15, 20];

♯ Stable analogs, cf. [10, 19, 20];

♯ Weak invariance principles, viz., Anscombe-Donsker results, cf. [20], Chapter 5, and further references given there;

♯ Strong invariance principles, cf. [25, 26, 27, 28, 4, 36];

♯ Analogs for curved barriers, typically $\tau(t) = \min\{n : S_n > tn^\alpha\}$, where $0 < \alpha < 1$, cf. most of the above sources;

♯ Results for random processes with i.i.d. increments, cf. [11, 17, 18].

3.4 Renewal theory with a trend

In a recent paper [21] the following situation was considered.

Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with finite mean 0 and set $X_k = Y_k + k^\gamma\mu$ for $k \geq 1$, $\gamma \in \mathbb{R}$ and some $\mu > 0$. Further, set $T_n = \sum_{k=1}^n Y_k$ and $S_n = \sum_{k=1}^n X_k$, $n \geq 1$, and

$$\tau(t) = \min\{n : S_n > t\}, \quad t \geq 0.$$

For $\gamma = 0$ the problem reduces to "Renewal theory for random walks". The case of interest here is $\gamma \in (0, 1]$. By comparing with the case $\gamma = 0$ one easily finds that $\tau(t) < \infty$ almost surely, and, via the sandwich inequality (Lemma 3.1), that $\tau(t) \nearrow +\infty$ as $t \to \infty$.

Here is the corresponding strong law, followed by the central limit theorem with hints to the proofs, which in the latter case (of course) involves Anscombe's theorem.

Theorem 3.3. *For $0 < \gamma \leq 1$, we have*

$$\frac{\tau(t)}{t^{1/(\gamma+1)}} \overset{a.s.}{\to} \Big(\frac{\gamma+1}{\mu}\Big)^{1/(\gamma+1)} \quad \text{as} \quad t \to \infty.$$

Proof. Upon noticing that $\sum_{k=1}^n k^\gamma \sim \frac{1}{\gamma+1} n^{\gamma+1}$ as $n \to \infty$, the (ordinary) strong law becomes

$$\frac{S_n - \frac{\mu}{\gamma+1} n^{\gamma+1}}{n} = \frac{T_n}{n} + \frac{\frac{\mu}{\gamma+1} n^{\gamma+1} - \mu \sum_{k=1}^n k^\gamma}{n} \overset{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty,$$

from which it follows that

$$\frac{S_n}{n^{\gamma+1}} \overset{a.s.}{\to} \frac{\mu}{\gamma+1} \quad \text{and that} \quad \frac{X_n}{n^{\gamma+1}} \overset{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty. \tag{3.3}$$

Combining this with Theorem 2.2 and Lemma 3.1 we conclude that

$$\frac{S_{\tau(t)}}{(\tau(t))^{\gamma+1}} \overset{a.s.}{\to} \frac{\mu}{\gamma+1}, \qquad \frac{X_{\tau(t)}}{(\tau(t))^{\gamma+1}} \overset{a.s.}{\to} 0, \qquad \frac{t}{(\tau(t))^{\gamma+1}} \overset{a.s.}{\to} \frac{\mu}{\gamma+1} \quad \text{as} \quad t \to \infty. \qquad \square$$

Theorem 3.4. *Let* $\gamma \in (0, 1/2)$. *If, in addition,* $\operatorname{Var} Y = \sigma^2 < \infty$, *then*

$$\frac{\tau(t) - \left(\frac{(\gamma+1)t}{\mu}\right)^{1/(\gamma+1)}}{t^{(1-2\gamma)/(2(\gamma+1))}} \overset{d}{\to} N\Big(0,\, \sigma^2 \cdot \frac{(\gamma+1)^{(1-2\gamma)/(\gamma+1)}}{\mu^{3/(\gamma+1)}}\Big) \quad \textit{as} \quad t \to \infty.$$

Proof. By the ordinary central limit theorem (and the fact that $\gamma \in (0, 1/2)$), we first have

$$\frac{S_n - \frac{\mu}{\gamma+1} n^{\gamma+1}}{\sigma\sqrt{n}} = \frac{T_n}{\sigma\sqrt{n}} + \frac{\frac{\mu}{\gamma+1} n^{\gamma+1} - \mu \sum_{k=1}^{n} k^\gamma}{\sigma\sqrt{n}} \overset{d}{\to} N(0,1) \quad \text{as} \quad n \to \infty,$$

so that, by Anscombe's theorem and Theorem 3.3,

$$\frac{S_{\tau(t)} - \frac{\mu}{\gamma+1}(\tau(t))^{\gamma+1}}{\sigma\left(\frac{(\gamma+1)t}{\mu}\right)^{1/(2(\gamma+1))}} \overset{d}{\to} N(0,1) \quad \text{as} \quad t \to \infty. \tag{3.4}$$

Next we note that

$$\frac{X_n}{\sqrt{n}} = \frac{X_n - n^\gamma \mu}{\sqrt{n}} + \frac{n^\gamma \mu}{\sqrt{n}} = \frac{Y_n}{\sqrt{n}} + n^{\gamma-(1/2)}\mu \overset{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty,$$

since $\operatorname{Var} Y < \infty$ (and $0 < \gamma < 1/2$), so that, by Theorem 2.2,

$$\frac{X_{\tau(t)}}{\sqrt{\tau(t)}} \overset{a.s.}{\to} 0 \quad \text{as} \quad t \to \infty. \tag{3.5}$$

Combining (3.4), (3.5) and the sandwich lemma leads (after some reshuffling) to

$$\Big(\frac{\mu}{\gamma+1}\Big)^{(2\gamma+3)/(2(\gamma+1))} \cdot \frac{(\tau(t))^{\gamma+1} - (\gamma+1)t/\mu}{\sigma t^{1/(2(\gamma+1))}} \overset{d}{\to} N(0,1) \quad \text{as} \quad t \to \infty. \tag{3.6}$$

The proof is now completed by exploiting the delta-method (cf. e.g. [19], Section 7.4.1) applied to the function $g(x) = x^{1/(\gamma+1)}$, the details of which we omit (since they are not of interest here). $\square$

3.5 Alternating renewal theory

A more general model, which allows for repair times, is the *alternating renewal process.* Here the lifetimes can be considered as the time periods during which some device functions, and an additional random sequence that may be interpreted as repair times is introduced. In, for example, queueing theory, lifetimes might correspond to busy times and repair times to idle times.

A natural problem in this context would be to find expressions for the availability, i.e. the relative amount of time that the device is functioning, or the relative amount of time that the server is busy.

This problem can be modeled within a more general framework, namely a special kind of two-dimensional random walk that is stopped when the second component reaches a given level, after which the first component is evaluated at that particular time point. This is our next topic, which is followed by a brief return to the alternating renewal process.

4 Stopped two-dimensional random walks

Motivated by a problem in chromatograpy [22], the following topic emerged as joint work with Svante Janson [23], see also [20], Section 4.2.

Let $\{(U_n^{(1)}, U_n^{(2)}),\, n \geq 1\}$ be a two-dimensional random walk with i.i.d. increments $(X_k^{(1)}, X_k^{(2)})$, $k \geq 1$, such that $\mu_2 = E\,X^{(2)} > 0$ and $\mu_1 = E\,X^{(1)}$ exists, finite. *Nothing is assumed about independence between the components* $X_k^{(1)}$ *and* $X_k^{(2)}$, *which, typically, is an essential point in many applications.* Furthermore, set $\mathcal{F}_n = \sigma\{(X_k^{(1)}, X_k^{(2)}) : k \leq n\}$ for $n \geq 1$, and define the first passage time process

$$\tau(t) = \min\{n : U_n^{(2)} > t\}, \quad t \geq 0\,.$$

We observe immediately that everything we know about renewal theory for random walks applies to $\{\tau(t),\, t \geq 0\}$ as well as to $\{U_{\tau(t)}^{(2)},\, t \geq 0\}$ since $\mu_2 > 0$.

The process of our concern is the stopped random walk

$$\{U_{\tau(t)}^{(1)}, \quad t \geq 0\}. \tag{4.1}$$

In the sources cited above one finds a variety of results for this process. Here we confine ourselves to the usual strong law and central limit theorem, where, once again, Anscombe's theorem does the main job.

Theorem 4.1.

$$\frac{U_{\tau(t)}^{(1)}}{t} \overset{a.s.}{\to} \frac{\mu_1}{\mu_2} \quad \textit{as} \quad t \to \infty.$$

Proof. We have

$$\frac{U_{\tau(t)}^{(1)}}{t} = \frac{U_{\tau(t)}^{(1)}}{\tau(t)} \cdot \frac{\tau(t)}{t} \overset{a.s.}{\to} \mu_1 \cdot \frac{1}{\mu_2} \quad \text{as} \quad t \to \infty.$$

The convergence of the first factor is justified by Theorem 2.2, and that of the second one by Theorem 3.1. □

Theorem 4.2. *Suppose, in addition, that* $\sigma_1^2 = \operatorname{Var} X^{(1)} < \infty$, $\sigma_2^2 = \operatorname{Var} X^{(2)} < \infty$ *and that*

$$v^2 = \operatorname{Var}(\mu_2 X^{(1)} - \mu_1 X^{(2)}) > 0.$$

Then

$$\frac{U_{\tau(t)}^{(1)} - \frac{\mu_1}{\mu_2} t}{v \mu_2^{-3/2} \sqrt{t}} \overset{d}{\to} N(0,1) \quad \textit{as} \quad t \to \infty.$$

Proof. Using a device originating in [33] we set

$$S_n = \mu_2 U_n^{(1)} - \mu_1 U_n^{(2)}, \quad n \geq 1, \tag{4.2}$$

thus fabricating a random walk $\{S_n,\, n \geq 1\}$ whose increments have mean 0 and positive, finite variance v^2.

The ordinary central limit theorem, together with Theorem 4.1, Theorem 2.2 and Anscombe's theorem, now tells us that

$$\frac{S_{\tau(t)}}{v\sqrt{\mu_2^{-1} t}} \overset{d}{\to} N(0,1) \quad \text{as} \quad t \to \infty,$$

which, rewritten, is the same as

$$\frac{\mu_2 U_{\tau(t)}^{(1)} - \mu_1 U_{\tau(t)}^{(2)}}{v\sqrt{\mu_2^{-1} t}} \overset{d}{\to} N(0,1) \quad \text{as} \quad t \to \infty.$$

Sandwiching $U^{(2)}_{\tau(t)}$, that is, noticing that

$$0 \leq \frac{U^{(2)}_{\tau(t)} - t}{\sqrt{t}} \leq \frac{X^{(2)}_{\tau(t)}}{\sqrt{t}} \overset{a.s.}{\to} 0 \quad \text{as} \quad t \to \infty,$$

and some rearranging finishes the proof. □

As promised above, here is a quick return to the alternating renewal process.

Let $T^{(b)}_k$ and $T^{(i)}_k$, $k \geq 1$ be the busy and idle periods in a queueing system or the periods when a device functions or is being repaired, respectively. Then, with

$$U^{(1)}_n = \sum_{k=1}^{n} T^{(b)}_k \quad \text{and} \quad U^{(2)}_n = \sum_{k=1}^{n} (T^{(b)}_k + T^{(i)}_k), \quad n \geq 1, \tag{4.3}$$

we note that $\{U^{(2)}_n,\, n \geq 1\}$ measures time in general and that $\{U^{(1)}_n,\, n \geq 1\}$ measures busy time/the time the device is functioning. Stopping $\{U^{(2)}_n,\, n \geq 1\}$ and checking $\{U^{(1)}_n,\, n \geq 1\}$ then should provide availability, that is, $U^{(1)}_{\tau(t)}$ should model availability during the time interval $(0, t]$.

Apart from some sandwiching.

We shall return to this example and to some further applications in Section 5.

4.1 Stopped two-dimensional random walks with a trend

This subsection is devoted to two-dimensional versions of the random walk with a trend from Subsection 3.4. We shall mainly consider the cases when there is trend in the stopping (second) component, but none in the first one, and when there is the same trend in both components.

We thus let $\{(U^{(1)}_n, U^{(2)}_n),\, n \geq 1\}$ be a two-dimensional random walk with i.i.d. increments $(X^{(1)}_k, X^{(2)}_k)$, $k \geq 1$, where, in turn, for $i = 1, 2$, $X^{(i)}_k = Y^{(i)}_k + k^{\gamma_i}\mu_i$, with $\mu_1 \in \mathbb{R}$, $\mu_2 > 0$, and $\gamma_i \in [0, 1]$; zero is included in order to cover the case when there is no trend in the first components.

As before we define

$$\tau(t) = \min\{n : U^{(2)}_n > t\}, \quad t \geq 0,$$

and wish to establish results for

$$\{U^{(1)}_{\tau(t)}, \quad t \geq 0\}. \tag{4.4}$$

From Subsection 3.4 we know that

$$\frac{\tau(t)}{t^{1/(\gamma_2+1)}} \overset{a.s.}{\to} \Big(\frac{\gamma_2 + 1}{\mu_2}\Big)^{1/(\gamma_2+1)} \quad \text{as} \quad t \to \infty, \tag{4.5}$$

so that, by arguing as there, we immediately obtain

$$\frac{U^{(1)}_{\tau(t)}}{t^{(\gamma_1+1)/(\gamma_2+1)}} = \frac{U^{(1)}_{\tau(t)}}{(\tau(t))^{\gamma_1+1}} \cdot \Big(\frac{\tau(t)}{t^{1/(\gamma_2+1)}}\Big)^{\gamma_1+1} \overset{a.s.}{\to} \frac{\mu_1}{\gamma_1 + 1} \cdot \Big(\frac{\gamma_2 + 1}{\mu_2}\Big)^{(\gamma_1+1)/(\gamma_2+1)} \quad \text{as} \quad t \to \infty,$$

which establishes the following strong law.

Theorem 4.3.

$$\frac{U^{(1)}_{\tau(t)}}{t^{(\gamma_1+1)/(\gamma_2+1)}} \overset{a.s.}{\to} \frac{\mu_1}{\gamma_1 + 1} \cdot \Big(\frac{\gamma_2 + 1}{\mu_2}\Big)^{(\gamma_1+1)/(\gamma_2+1)} \quad \textit{as} \quad t \to \infty.$$

As for a corresponding central limit theorem the procedure is the analogous one, except for the fact that the expression for the variance v^2 emerging from the special mean zero random walk $\{S_n,\, n \geq 1\}$ constructed in the proof (recall (4.2)) becomes more or less tractable depending on the trends.

Here we shall consider only two cases. In the first one we assume that the trend is the same in both components. If, for example, both components represent the same kind of measurement, and one seeks some kind of availability (cf. Subsection 3.5), then this might be reasonable.

In the second example we assume that there is no trend in the first component. This might be relevant if, for example, one "fears" that the assumption $\gamma_2 = 0$ is violated, in which case the "reward" $U^{(1)}_{\tau(t)}$ turns into the cost for a possible disaster.

Thus, let us turn to the first case, in which the trends are the same, viz., $\gamma_1 = \gamma_2 = \gamma$. Recalling the proof of Theorem 4.2 we find that the appropriate random walk is

$$\begin{aligned} S_n &= \mu_2 U_n^{(1)} - \mu_1 U_n^{(2)} = \sum_{k=1}^{n} (\mu_2 X_k^{(1)} - \mu_1 X_k^{(2)}) \\ &= \sum_{k=1}^{n} \big(\mu_2 (Y_k^{(1)} - k^\gamma \mu_1) - \mu_1 (Y_k^{(2)} - k^\gamma \mu_2)\big) = \sum_{k=1}^{n} (\mu_2 Y_k^{(1)} - \mu_1 Y_k^{(2)}), \quad n \geq 1, \end{aligned}$$

where the summands are i.i.d. with mean 0 and variance $v^2 = \mathrm{Var}\,(\mu_2 Y^{(1)} - \mu_1 Y^{(2)})$.

By combining the proofs of Theorems 4.2 and 3.3 we first obtain

$$\frac{\mu_2 U^{(1)}_{\tau(t)} - \mu_1 U^{(2)}_{\tau(t)}}{v(\mu_2^{-1}(\gamma+1)t)^{1/(2(\gamma+1)}} \stackrel{d}{\to} N(0,1) \quad \text{as} \quad t \to \infty,$$

and after sandwiching $U^{(2)}_{\tau(t)}$ the following result emerges.

Theorem 4.4. *If, in addition,* $\mathrm{Var}\, Y^{(1)} < \infty$, $\mathrm{Var}\, Y^{(2)} < \infty$, $\gamma_1 = \gamma_2 = \gamma \in (0, 1/2)$, *and*

$$v^2 = \mathrm{Var}\,(\mu_2 Y^{(1)} - \mu_1 Y^{(2)}) > 0,$$

then

$$\frac{U^{(1)}_{\tau(t)} - \frac{\mu_1}{\mu_2} t}{t^{1/(2(\gamma+1)}} \stackrel{d}{\to} N\big(0, v^2 \mu_2^{(2\gamma+3)/(\gamma+1)} (\gamma+1)^{1/(\gamma+1)}\big) \quad \textit{as} \quad t \to \infty.$$

In the second case we thus assume (fear) that the second, running, component has some trend ($\gamma_2 = \gamma$), and that the first one has no trend ($\gamma_1 = 0$).

However, we redefine the first component in that we introduce the trend of the second component as a kind of discount factor; viz.,

$$U_n^{(1)} = \sum_{k=1}^{n} k^\gamma X_k^{(1)} = \sum_{k=1}^{n} k^\gamma (Y_k^{(1)} + \mu_1) \quad \text{for} \quad n \geq 1.$$

This means that "the reward" in the k th step has a discount factor k^γ.

The corresponding centered random walk then is

$$\begin{aligned} S_n &= \mu_2 U_n^{(1)} - \mu_1 U_n^{(2)} = \sum_{k=1}^{n} (\mu_2 k^\gamma X_k^{(1)} - \mu_1 X_k^{(2)}) \\ &= \sum_{k=1}^{n} \big(\mu_2 k^\gamma (Y_k^{(1)} + \mu_1) - \mu_1 (Y_k^{(2)} + k^\gamma \mu_2)\big) = \sum_{k=1}^{n} (\mu_2 k^\gamma Y_k^{(1)} - \mu_1 Y_k^{(2)}), \quad n \geq 1. \end{aligned}$$

Since we have redefined the first component we first need a corresponding strong law.

Theorem 4.5.

$$\frac{U^{(1)}_{\tau(t)}}{t} \stackrel{a.s.}{\to} \frac{\mu_1}{\mu_2} \quad \textit{as} \quad t \to \infty.$$

Proof. Recalling that $\sum_{k=1}^n k^\gamma \sim \frac{n^{\gamma+1}}{\gamma+1}$ as $n \to \infty$, an application of the wellknown strong law of large numbers for weighted sums yields

$$\frac{U_n^{(1)}}{n^{\gamma+1}} = \frac{\sum_{k=1}^n k^\gamma Y_k^{(1)}}{n^{\gamma+1}} + \mu_1 \frac{\sum_{k=1}^n k^\gamma}{n^{\gamma+1}} \stackrel{a.s.}{\to} 0 + \frac{\mu_1}{\gamma+1} = \frac{\mu_1}{\gamma+1} \quad \text{as} \quad n \to \infty,$$

after which the remaining piece of the proof runs as that of Theorem 4.3. □

In order to prove a central limit theorem, the first step is to establish asymptotic normality for S_n as $n \to \infty$. Toward that end we first consider $S_n^{(1)} = \sum_{k=1}^n \mu_2 k^\gamma Y_k^{(1)}$, $n \geq 1$, for which (2.5) tells us that

$$\frac{S_n^{(1)}}{n^{\gamma+(1/2)}} \xrightarrow{d} N\big(0, \frac{\mu_2^2\sigma_1^2}{2\gamma+1}\big) \quad \text{as} \quad n \to \infty.$$

Next, since asymptotic normality for $\sum_{k=1}^n \mu_1 Y_k^{(2)}$ requires normalization with $\sqrt{n}$, it follows that

$$\frac{\sum_{k=1}^n \mu_1 Y_k^{(2)}}{n^{\gamma+(1/2)}} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty,$$

so that, by joining the last two conclusions, we obtain

$$\frac{S_n}{n^{\gamma+(1/2)}} \xrightarrow{d} N\big(0, \frac{\mu_2^2\sigma_1^2}{2\gamma+1}\big) \quad \text{as} \quad n \to \infty. \tag{4.6}$$

After this we are in the position to apply Theorem 2.4 (with $\beta = 1/(\gamma+1)$) to conclude that

$$\frac{S_{\tau(t)}}{(\tau(t))^{\gamma+(1/2)}} \xrightarrow{d} N\big(0, \frac{\mu_2^2\sigma_1^2}{2\gamma+1}\big) \quad \text{as} \quad t \to \infty,$$

which is the same as

$$\frac{\mu_2 U_{\tau(t)}^{(1)} - \mu_1 U_{\tau(t)}^{(2)}}{(\tau(t))^{\gamma+(1/2)}} \xrightarrow{d} N\big(0, \frac{\mu_2^2\sigma_1^2}{2\gamma+1}\big) \quad \text{as} \quad t \to \infty,$$

which, in turn, in view of (4.5) (remember Theorem 3.3), yields

$$\frac{\mu_2 U_{\tau(t)}^{(1)} - \mu_1 U_{\tau(t)}^{(2)}}{t^{\frac{2\gamma+1}{2(\gamma+1)}}} \xrightarrow{d} N\Big(0, \frac{((\gamma+1)/\mu_2)^{\frac{2\gamma+1}{2(\gamma+1)}}\mu_2^2\sigma_1^2}{2\gamma+1}\Big) \quad \text{as} \quad t \to \infty.$$

Sandwiching $U_{\tau(t)}^{(2)}$ and rearranging, finally, establishes the following result.

Theorem 4.6. *If, in addition,* $\operatorname{Var} Y^{(1)} < \infty$, $\operatorname{Var} Y^{(2)} < \infty$, $\gamma_1 = 0$ *and* $\gamma_2 = \gamma \in (0, 1/2)$, *then*

$$\frac{U_{\tau(t)}^{(1)} - \frac{\mu_1}{\mu_2} t}{t^{\frac{2\gamma+1}{2(\gamma+1)}}} \xrightarrow{d} N\Big(0, \frac{\sigma_1^2}{2\gamma+1} \cdot \Big(\frac{\gamma+1}{\mu_2}\Big)^{\frac{2\gamma+1}{2(\gamma+1)}}\Big) \quad \text{as} \quad t \to \infty.$$

5 Some applications

After these theoretical findings we provide some contexts where stopped random walks naturally enter into the probabilistic models, and, in particular, illustrate the usefulness of our results concerning the quantity $U_{\tau(t)}^{(1)}$ from Section 4.

Chromatography

In May 1979 I received a telephone call from a friend of a friend who wanted help with a problem in chromatography. This, in turn, led to [22] and, later, to the model of Section 4 (for more on this we refer once more to [23]; cf. also [20], Chapter 4.

The basis for chromatographic separation is a sample of molecules that is injected onto a column and, during its transport along the column, oscillates between a *mobile* phase and a *stationary* phase (where the molecules do not move) in order to separate the compounds.

By identifying the two phases with the busy periods (the functioning of some component) and the idle periods (the repair times), respectively, in the language of Subsection 3.5, we realize that we are faced with an alternating renewal process. The relative time spent in the mobile phase thus corresponds to availability, and assuming constant velocity v in the mobile phase, we easily obtain the distance travelled at time t with the aid of the results from Section 4.

In addition, by letting $\{(X_k^{(1)}, X_k^{2)}), k \geq 1\}$ be the times in the mobile and stationary phases, respectively, then $U_n^{(1)} = \sum_{k=1}^n (X_k^{(1)} + X_k^{2)})$ and $U_n^{(2)} = \sum_{k=1}^n v X_k^{(1)}$, $n \geq 1$, represent time and distance travelled, respectively, so that, with $\tau(L) = \min\{n : U_n^{(2)} > L\}$, where $L =$ the length of a column, $U_{\tau(L)}^{(1)}$ provides information about the elution time.

Markov renewal theory

In [1] some of the results above are generalized to Markov renewal processes, which (i.a.) allows the mobile phase in the previous example to be split into several layers, which makes the model more realistic.

Queuing theory

This was already hinted at in Subsection 3.5. On the other hand, if $X_k^{(2)}$ are the times between customers arriving at a cash register, and $X_k^{(1)}$ are the amounts of their purchases, then, in the usual notation, $U_{\tau(t)}^{(1)}$ equals the amount of money in the cash register at time t. Or, if $X_k^{(1)} = 1$ whenever a customer makes a purchase and 0 otherwise, then $U_{\tau(t)}^{(1)}$ equals the number of customers that did purchase something before time t.

Replacement policies

In replacement based on age one replaces an object or component upon failure or at som prescribed age whichever occurs first (death or retirement for humans). Comparing with the queueing system we immediately see how to model the number of components replaced because of failure during the time interval $(0, t]$.

Shock models

Shock models are systems that at random times are subject to shocks of random magnitudes. In cumulative shock models systems break down because of a cumulative effect (and in extreme shock models systems break down because of one single large shock).

If $\{(X_k^{(1)}, X_k^{2)}),\, k \geq 0\}$ are (nonnegative) i.i.d. two-dimensional random vectors, where $X_k^{(1)}$ represents the time between the $(k-1)$st and the kth shock, and $X_k^{(2)}$ the magnitude of the kth shock, then the number of shocks until failure can be described by

$$\tau(t) = \min\{n : \sum_{k=1}^{n} X_k^{(2)} > t\},$$

and the failure time by $\sum_{k=1}^{\tau(t)} X_k^{(1)}$, and Section 4 is in action again.

Remark 5.1. Note how, obviously, the two components of the various random walks above are *not* independent. □

Insurance risk theory

The number of claims as well as the claim sizes during a given time period are random, so that the total amount claimed is a random sum, typically, a compound Poisson process. We refer to the abundance of books and papers in the area.

6 Renewal theory for perturbed random walks

Throughout this section $X_1, X_2, \ldots$ are i.i.d. random variables with positive, finite mean μ and partial sums $\{S_n,\, n \geq 1\}$. In addition we let $\{\xi_n,\, n \geq 1\}$, with increments $\{\eta_k,\, k \geq 1\}$, be a sequence of random variables, such that

$$\frac{\xi_n}{n} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty\,. \tag{6.1}$$

Definition 6.1. *A process $\{Z_n,\, n \geq 1\}$, such that*

$$Z_n = S_n + \xi_n, \quad n \geq 1,$$

where $\{S_n,\, n \geq 1\}$and $\{\xi_n,\, n \geq 1\}$ are as above, is called a perturbed random walk. □

A main reference here is [16]; see also [20], Chapter 6.

Remark 6.1. This definition is more general than that of nonlinear renewal theory as introduced in [30, 31] and further developed in [39, 35], in that we do not assume that $\operatorname{Var} X < \infty$, and neither that the elements of the perturbing process are independent of the future of the random walk nor that the perturbing process satisfies the Anscombe condition. □

Once again we define the first passage times

$$\tau(t) = \min\{n : S_n > t\}, \quad t \geq 0.$$

Following are the strong law and central limit theorem in this setting.

Theorem 6.1.

$$\frac{\tau(t)}{t} \overset{a.s.}{\to} \frac{1}{\mu} \quad \text{as} \quad t \to \infty.$$

In order to formulate the central limit theorems to follow we need the following condition.

Definition 6.2. *The sequence* $\{\xi_n, n \geq 1\}$ *satisfies* Condition AP *if*

$$\frac{\xi_n}{\sqrt{n}} \overset{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty \qquad \text{or if}$$

$$\frac{\xi_n}{\sqrt{n}} \overset{p}{\to} 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad \left\{\frac{\xi_n}{\sqrt{n}}, n \geq 1\right\} \quad \text{satisfies the Anscombe condition.}$$ □

Theorem 6.2. *Suppose, in addition, that* $\sigma^2 = \operatorname{Var} X < \infty$. *If* $\{\xi_n, n \geq 1\}$ *satisfies Condition AP, then*

$$\frac{\tau(t) - t/\mu}{\sigma\mu^{-3/2}\sqrt{t}} \overset{d}{\to} N(0,1) \quad \text{as} \quad t \to \infty.$$

The proofs are based on the SRW-method along the lines of the proof of Theorem 4.2, the point being that the assumptions are exactly those needed for the additional perturbing contribution to vanish asymptotically. In addition one needs the following sandwich inequality:

$$t < Z_{\tau(t)} \leq t + X_{\tau(t)} + \eta_{\tau(t)} \leq t + X^+_{\tau(t)} + \eta^+_{\tau(t)}\,. \tag{6.2}$$

6.1 The case $Z_n = n \cdot g(\bar{Y}_n)$

Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with positive finite mean, θ, and finite variance, ν^2, and suppose that g is a positive function, that is twice continuously differentiable in some neighborhood of θ. Finally, set

$$Z_n = n \cdot g(\bar{Y}_n), \quad n \geq 1, \tag{6.3}$$

where $\bar{Y}_n = \frac{1}{n}\sum_{k=1}^n Y_k$, $n \geq 1$.

Although this case is less general it covers many important applications, in particular various sequential testing procedures; we shall provide a hint on this in Subsection 7.1 below.

To see that $\{Z_n, n \geq 1\}$ defines a perturbed random walk we make a Taylor expansion of g at θ to obtain

$$Z_n = n \cdot g(\theta) + n \cdot g'(\theta)(\bar{Y}_n - \theta) + n \cdot \frac{g''(\rho_n)}{2}(\bar{Y}_n - \theta)^2, \tag{6.4}$$

where $\rho_n = \rho_n(\omega)$ lies between θ and $\bar{Y}_n$.

By setting $X_k = g(\theta) + g'(\theta)(Y_k - \theta)$, $k \geq 1$, we obtain an i.i.d. sequence of random variables with mean $\mu = g(\theta) + g'(\theta) \cdot 0 = g(\theta) > 0$ and variance $\sigma^2 = \nu^2(g'(\theta))^2$. Thus, with

$$S_n = \sum_{k=1}^n X_k = \sum_{k=1}^n \big(g(\theta) + g'(\theta)(Y_k - \theta)\big) \quad \text{and} \quad \xi_n = \frac{ng''(\rho_n)}{2}(\bar{Y}_n - \theta)^2, \quad n \geq 1,$$

the former sequence defines a random walk with positive mean, and the second one a perturbing component, since

$$\frac{\xi_n}{n} = \frac{g''(\rho_n)}{2}(\bar{Y}_n - \theta)^2 \overset{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty,$$

in view of the continuity of g'' and the strong law of large numbers.

The strong law and central limit theorem turn into

$$\frac{\tau(t)}{t} \overset{a.s.}{\to} \frac{1}{g(\theta)} \quad \text{as} \quad t \to \infty,$$

and

$$\frac{\tau(t) - t/g(\theta)}{\nu g'(\theta)(g(\theta))^{-3/2}\sqrt{t}} \overset{d}{\to} \frac{1}{g(\theta)} \quad \text{as} \quad t \to \infty,$$

respectively.

Remark 6.2. One can in fact even verify that this case defines a nonlinear renewal process as treated in the sources cited above. However, weakening the differentiability and integrability assumptions, still yields a perturbed random walk. But no longer a nonlinear renewal process. □

6.2 Renewal theory for perturbed random walks with a trend

Let, as in Subsection 4.1, $Y_1, Y_2, \ldots$ be i.i.d. random variables with mean 0, let $\xi_1, \xi_2, \ldots$ be the perturbations, let $\gamma \in (0, 1]$, and set, for $k \geq 1$, $X_k = Y_k + k^\gamma \mu$, with $S_n = \sum_{k=1}^n X_k$, $n \geq 1$, and, finally, $Z_n = S_n + \xi_n$, $n \geq 1$.

In order to complete the setup we introduce the family of first passage times

$$\tau(t) = \min\{n : Z_n > t\}, \quad t \geq 0.$$

Combining the arguments from Subsection 4.1, together with an additional caretaking of the perturbing part, leads to the following results.

Theorem 6.3. *For $0 < \gamma \leq 1$, we have*

$$\frac{\tau(t)}{t^{1/(\gamma+1)}} \overset{a.s.}{\to} \Big(\frac{\gamma+1}{\mu}\Big)^{1/(\gamma+1)} \quad \textit{as} \quad t \to \infty.$$

Proof. Recalling that $\sum_{k=1}^n k^\gamma \sim \frac{1}{\gamma+1} n^{\gamma+1}$ as $n \to \infty$, and invoking the (ordinary) strong law we obtain

$$\frac{Z_n - \frac{\mu}{\gamma+1} n^{\gamma+1}}{n^{\gamma+1}} = \frac{T_n}{n^{\gamma+1}} + \frac{\frac{\mu}{\gamma+1} n^{\gamma+1} - \mu \sum_{k=1}^n k^\gamma}{n^{\gamma+1}} + \frac{\xi_n}{n^{\gamma+1}} \overset{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty.$$

By copying the proof of Theorem 3.3 it then follows that

$$\frac{Z_n}{n^{\gamma+1}} \overset{a.s.}{\to} \frac{\mu}{\gamma+1} \quad \text{and that} \quad \frac{X_n}{n^{\gamma+1}} \overset{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty,$$

and in this case also that $\eta_n / n^{\gamma+1} \overset{a.s.}{\to} 0$.

An application of Theorem 2.2 and sandwiching, recall (6.2), concludes the proof. □

Theorem 6.4. *Let $\gamma \in (0, 1/2)$. If, in addition,* $\operatorname{Var} Y = \sigma^2 < \infty$, *and Condition AP is satisfied, then*

$$\frac{\tau(t) - \big(\frac{(\gamma+1)t}{\mu}\big)^{1/(\gamma+1)}}{t^{(1-2\gamma)/(2(\gamma+1))}} \overset{d}{\to} N\Big(0, \sigma^2 \cdot \frac{(\gamma+1)^{(1-2\gamma)/(\gamma+1)}}{\mu^{3/(\gamma+1)}}\Big) \quad \textit{as} \quad t \to \infty.$$

The proof consists of a modification of the proof of Theorem 3.4 along the lines of the previous proof, the details of which we leave to the reader(s).

6.3 Stopped two-dimensional perturbed random walks

Just as the results in Section 4 are extensions from renewal theory to a two-dimensional case, one can obtain corresponding analogs for perturbed random walks. This is interesting in its own right, but, more importantly, the results are useful in certain multiple testing procedures, as will shall soon see.

Thus, let, as before, $\{(U_n^{(1)}, U_n^{(2)}), n \geq 1\}$ be a two-dimensional random walk with i.i.d. increments $\{(X_k^{(1)}, X_k^{(2)}), k \geq 1\}$, and suppose that $\mu_2 = E\,X^{(2)} > 0$ and that $\mu_1 = E\,X^{(1)}$ exists, finite. Furthermore, $\{\xi_n^{(1)}, n \geq 1\}$ and $\{\xi_n^{(2)}, n \geq 1\}$ are perturbing sequences in the sense of (6.1).

Given this, we define the two-dimensional perturbed random walk

$$(Z_n^{(1)}, Z_n^{(2)}) = (U_n^{(1)} + \xi_n^{(1)}, U_n^{(2)} + \xi_n^{(2)}), \quad n \geq 1,$$

and the first passage time process

$$\tau(t) = \min\{n : Z_n^{(2)} > t\}, \quad t \geq 0.$$

Clearly, the first passage times are stopping times (relative to the sequence of σ-algebras generated by the perturbed random walk). Moreover, since $\mu_2 > 0$ the results from the early part of the present section apply to the second component.

We are thus set in order to investigate stopped perturbed random walk

$$\{Z_{\tau(t)}^{(1)}, \quad t \geq 0\}.$$

And, no surprise, we end up as follows:

Theorem 6.5. *We have*

$$\frac{Z_{\tau(t)}^{(1)}}{t} \overset{a.s.}{\to} \frac{\mu_1}{\mu_2} \quad \textit{as} \quad t \to \infty.$$

Theorem 6.6. *Suppose, in addition, that* $\sigma_1^2 = \operatorname{Var} X^{(1)} < \infty$*, that* $\sigma_2^2 = \operatorname{Var} X^{(1)} < \infty$ *and that*

$$v^2 = \operatorname{Var}(\mu_2 X^{(1)} - \mu_1 X^{(2)}) > 0.$$

If $\{\xi_n^{(1)}, n \geq 1\}$ *and* $\{\xi_n^{(2)}, n \geq 1\}$ *satisfy Condition AP, then*

$$\frac{Z_{\tau(t)}^{(1)} - \frac{\mu_1}{\mu_2} t}{v \mu_2^{-3/2} \sqrt{t}} \overset{d}{\to} N(0,1) \quad \textit{as} \quad t \to \infty.$$

6.4 The case $(Z_n^{(1)}, Z_n^{(2)}) = \big(n \cdot g_1(\bar{Y}_n^{(1)}), n \cdot g_2(\bar{Y}_n^{(2,1)}, \bar{Y}_n^{(2,2)})\big)$

Without further ado we just mention that the special case from the one-dimensional setting carries over also to this situation. For completeness we state the two usual results; the notation is selfexplanatory. Besides, this is the variant we shall exploit later.

A glance at the heading tells us that we consider the two-dimensional perturbed random walk $(Z_n^{(1)}, Z_n^{(2)}) = (n \cdot g_1(\bar{Y}_n^{(1)}), n \cdot g_2(\bar{Y}_n^{(2,1)}, \bar{Y}_n^{(2,2)}))$, $n \geq 1$, and the first passage time process

$$\tau(t) = \min\{n : Z_n^{(2)} > t\}, \quad t \geq 0,$$

with focus on the stopped family

$$\{Z_{\tau(t)}^{(1)}, \quad t \geq 0\}.$$

Theorem 6.7. *We have*

$$\frac{Z_{\tau(t)}^{(1)}}{t} \overset{a.s.}{\to} \frac{g_1(\theta_1)}{g_2(\theta_2^{(2,1)}, \theta_2^{(2,2)})} \quad \textit{as} \quad t \to \infty.$$

Theorem 6.8. *Suppose, in addition, that* $\operatorname{Var} Y^{(1)} < \infty$, $\operatorname{Cov} \mathbf{Y}^{(2)}$ *is positive definite, and that* g_1', $\frac{\partial g_2}{\partial y_1^{(2)}}$ *and* $\frac{\partial g_2}{\partial y_2^{(2)}}$ *are continuous at* θ_1 *and* $(\theta_2^{(2,1)}, \theta_2^{(2,2)})$*, respectively. Then*

$$\frac{Z_{\tau(t)}^{(1)} - \frac{g_1(\theta_1)}{g_2(\theta_2^{(2,1)}, \theta_2^{(2,2)})} t}{v\big(g_2(\theta_2^{(2,1)}, \theta_2^{(2,2)})\big)^{-3/2} \sqrt{t}} \overset{d}{\to} N(0,1) \quad \textit{as} \quad t \to \infty,$$

where

$$\begin{aligned} v^2 &= \operatorname{Var}\Big(g_2(\theta_2^{(2,1)},\theta_2^{(2,2)})g_1'(\theta_1)Y^{(1)} \\ &\quad -g_1(\theta_1)\Big\{\frac{\partial g_2}{\partial y_1^{(2)}}(\theta_2^{(2,1)},\theta_2^{(2,2)})\cdot Y^{(2,1)} + \frac{\partial g_2}{\partial y_2^{(2)}}(\theta_2^{(2,1)},\theta_2^{(2,2)})\cdot Y^{(2,2)}\Big\}\Big) \end{aligned}$$

is assumed to be positive.

6.5 Stopped two-dimensional perturbed random walks with a trend

Consider the (obvious) two-dimensional version of the perturbed random walk with a trend from Subsection 6.2, or, else, the perturbed version of the two-dimensional random walk from Subsection 4.1, that is, the two-dimensional perturbed random walk

$$(Z_n^{(1)},\, Z_n^{(2)}) = (U_n^{(1)} + \xi_n^{(1)},\, U_n^{(2)} + \xi_n^{(2)}), \quad n \geq 1,$$

where we now assume that $X_k^{(i)} = Y_k^{(i)} + k^{\gamma_i}\mu_i$, with $\mu_1 \in \mathbb{R}$, $\mu_2 > 0$ and $\gamma_i \in [0,1]$, where, in turn, $Y_1^{(i)}, Y_2^{(i)}, \ldots$ are sequences of i.i.d. random variables with mean 0, $i = 1, 2$.

The first passage time process is the usual one, namely

$$\tau(t) = \min\{n : Z_n^{(2)} > t\}, \quad t \geq 0,$$

and the object in focus is (of course)

$$\{Z_{\tau(t)}^{(1)}, \quad t \geq 0\}. \tag{6.5}$$

The following three results are the same as those for the two-dimensional random walk in Subsection 4.1, and the proofs, which we omit, follow the same lines as there. The perturbation is throughout of an asymptotically vanishing order.

Theorem 6.9.

$$\frac{Z_{\tau(t)}^{(1)}}{t^{(\gamma_1+1)/(\gamma_2+1)}} \stackrel{a.s.}{\to} \frac{\mu_1}{\gamma_1+1}\cdot\Big(\frac{\gamma_2+1}{\mu_2}\Big)^{(\gamma_1+1)/(\gamma_2+1)} \quad \textit{as} \quad t \to \infty.$$

Theorem 6.10. *Suppose, in addition, that* $\operatorname{Var} Y^{(1)} < \infty$, $\operatorname{Var} Y^{(2)} < \infty$, $\gamma_1 = \gamma_2 = \gamma \in (0, 1/2)$, *and that*

$$v^2 = \operatorname{Var}(\mu_2 Y^{(1)} - \mu_1 Y^{(2)}) > 0.$$

If $\{\xi_n^{(1)},\, n \geq 1\}$ *and* $\{\xi_n^{(2)},\, n \geq 1\}$ *satisfy Condition AP, then*

$$\frac{Z_{\tau(t)}^{(1)} - \frac{\mu_1}{\mu_2}t}{t^{1/(2(\gamma+1))}} \stackrel{d}{\to} N\big(0, v^2\mu_2^{(2\gamma+3)/(\gamma+1)}(\gamma+1)^{1/(\gamma+1)}\big) \quad \textit{as} \quad t \to \infty.$$

For the second case we recall from Subsection 4.1 that the first component of the random walk contribution is defined as

$$U_n^{(1)} = \sum_{k=1}^{n} k^\gamma X_k^{(1)} = \sum_{k=1}^{n} k^\gamma (Y_k^{(1)} + \mu_1) \quad \text{for} \quad n \geq 1.$$

Theorem 6.11. *If, in addition,* $\operatorname{Var} Y^{(1)} < \infty$, $\operatorname{Var} Y^{(2)} < \infty$, $\gamma_1 = 0$ *and* $\gamma_2 = \gamma \in (0, 1/2)$, *and* $\{\xi_n^{(1)},\, n \geq 1\}$ *and* $\{\xi_n^{(2)},\, n \geq 1\}$ *satisfy Condition AP, then*

$$\frac{U_{\tau(t)}^{(1)} - \frac{\mu_1}{\mu_2}t}{t^{\frac{2\gamma+1}{2(\gamma+1)}}} \stackrel{d}{\to} N\Big(0, \frac{\sigma_1^2}{2\gamma+1}\cdot\Big(\frac{\gamma+1}{\mu_2}\Big)^{\frac{2\gamma+1}{2(\gamma+1)}}\Big) \quad \textit{as} \quad t \to \infty.$$

7 Repeated significance tests

This is an important topic in the theory of sequential analysis. In the following we shall see how some procedures depend on results for perturbed random walks, in particular, from the model from Subsections 6.1 and 6.4.

7.1 Repeated significance tests in one-parameter exponential families

Consider the family of distributions

$$G_\theta(dx) = \exp\{\theta x - \psi(\theta)\}\lambda(dx), \quad \theta \in \Theta\,,$$

where λ is a nondegenerate, σ-finite measure on $(-\infty, \infty)$ and Θ a nondegenerate real interval, let $Y_1, Y_2, \ldots$ be i.i.d. random variables with distribution function G_θ for some $\theta \in \Theta$, where θ is unknown, and suppose that we wish to test the hypothesis

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0,$$

where we w.l.o.g. assume that $\theta_0 = 0$ and that $\psi(0) = \psi'(0) = 0$.

For this model it is well-known that the moment generating function $E_\theta \exp\{tY\}$ exists for $t + \theta \in \Theta$, that ψ is convex, and that $E_\theta Y = \psi'(\theta)$ and $\mathrm{Var}_\theta Y = \psi''(\theta) > 0$, under the assumption that ψ is twice differentiable in a neighborhood of $\theta_0 = 0$.

Moreover, the log-likelihood ratio is

$$T_n = \sup_{\theta\in\Theta} \log \prod_{k=1}^{n} \exp\{\theta Y_k - \psi(\theta)\} = n \cdot \sup_{\theta\in\Theta}\{\theta \bar{Y}_n - \psi(\theta)\} = n \cdot g\big(\bar{Y}_n\big),$$

where $g(x) = \sup_\theta \big(\theta x - \psi(\theta)\big)$, $x \in \mathbb{R}$, is the convex (Fenchel) conjugate of ψ. It follows that $\{T_n,\, n \geq 1\}$ is a perturbed random walk in the special sense of Subsection 6.1.

Since the corresponding sequential test procedure amounts to rejecting H_0 as soon as T_n exceeds some critical value, the stopping time of interest is

$$\tau(t) = \min\{n : T_n > t\}, \quad t > 0,$$

which shows that we are, indeed, in the realm of Subsection 6.1.

Example 7.1. For $Y \in N(\theta, 1)$ we have $\psi(\theta) = \frac{1}{2}\theta^2$, $g(x) = \frac{1}{2}x^2$, and $T_n = \frac{1}{2}n(\bar{Y}_n)^2$, that is, we rediscover the classical square root boundary problem.

Example 7.2. If Y is exponential with mean $1/(1-\theta)$, then $g(x) = x - 1 - \log x$ and

$$\tau(t) = \min\{n : n\big(\bar{Y}_n - 1 - \log \bar{Y}_n\big) > t\}, \quad t \geq 0. \qquad \square$$

Further details can be found in [16] and/or [20], Section 6.9. For additional material we refer to [39, 35].

7.2 Repeated significance tests in two-parameter exponential families

This is more than just an extension from the previous subsection, in that the two-parameter model, i.a., provides relations between marginal one-parameter tests and joint tests. The special scenario we have in mind is when the two-dimensional test statistic falls into its (two-dimensional) critical region, whereas none of the (one-dimensional) marginal test statistics fall into theirs, which means that one can only conclude that "something is wrong somewhere" but not where or what.

The example to follow is taken from [24], where further details and background can be found (cf. also [20], Section 6.12).

In order to put this into mathematics, consider the two-dimensional version of the previous subsection, namely, the family of distributions

$$G_{\theta_1,\theta_2}(dy_1, dy_2) = \exp\{\theta_1 y_1 + \theta_2 y_2 - \psi(\theta_1, \theta_2)\}\lambda(dy_1, dy_2), \quad (\theta_1, \theta_2) \in \Theta,$$

where λ is a nondegenerate, σ-finite measure on $\mathbb{R}^2$, Θ a convex subset of $\mathbb{R}^2$ and ψ is, for simplicity, strictly convex and twice differentiable.

Now, let $(Y_k^{(1)}, Y_k^{(2)})$, $k \geq 1$, be i.i.d. two-dimensional random variables with distribution function G_{θ_1,θ_2}, where the parameters are unknown, and suppose that we wish to test the hypothesis

$$H_0 : \theta_1 = \theta_{01},\ \theta_2 = \theta_{02} \quad \text{vs.} \quad H_1 : \theta_1 \neq \theta_{01} \text{ or } \theta_2 \neq \theta_{02},$$

where w.l.o.g. we assume that $(\theta_{01}, \theta_{02}) = (0, 0) \in \Theta$ and that $\psi(0, 0) = \frac{\partial\psi}{\partial\theta_1}(0, 0) = \frac{\partial\psi}{\partial\theta_2}(0, 0) = 0$.

The log-likelihood ratio then turns out as $T_n = n \cdot g\big(\bar{Y}_n^{(1)}, \bar{Y}_n^{(2)}\big)$, where, for $-\infty < y_1, y_2 < \infty$, $g(y_1, y_2) = \sup_{\theta_1,\theta_2}\big(\theta_1 y_1 + \theta_2 y_2 - \psi(\theta_1, \theta_2)\big)$ is the convex conjugate of ψ with the usual properties, so that $\{T_n,\ n \geq 1\}$ is a two-dimensional perturbed random walk in the sense of Subsection 6.4.

However, we may carry this one step further. Namely, consider T_n, $n \geq 1$, as a second component of a two-dimensional perturbed random walk as treated in Subsection 6.4, and suppose, for the sake of illustration, that $(Y_k^{(1)}, Y_k^{(2)})'$, $k \geq 1$, are i.i.d. normal random vectors with mean $(\theta_1, \theta_2)'$ and common variance 1. Then $\psi(\theta_1, \theta_2) = \frac{1}{2}(\theta_1^2 + \theta_2^2)$ and $g(y_1, y_2) = \frac{1}{2}(y_1^2 + y_2^2)$, from which it follows that

$$T_n = \frac{n}{2}\Big(\big(\bar{Y}_n^{(1)}\big)^2 + \big(\bar{Y}_n^{(2)}\big)^2\Big) = \frac{1}{2n}\Big(\Big(\sum_{k=1}^n Y_k^{(1)}\Big)^2 + \Big(\sum_{k=1}^n Y_k^{(2)}\Big)^2\Big) = \frac{1}{2n}\big((\Sigma_n^{(1)})^2 + (\Sigma_n^{(2)})^2\big),$$

which compares naturally with the one-dimensional case above.

With $\mathbf{\Sigma}_n = (\Sigma_n^{(1)}, \Sigma_n^{(2)})'$ and $\|\cdot\|$ denoting Euclidean distance in $\mathbb{R}^2$, the appropriate stopping time becomes

$$\tau(t) = \min\{n : \|\mathbf{\Sigma}_n\| > \sqrt{2tn}\}, \quad t \geq 0,$$

which might be interpreted as a generalization of the square root boundary problem.

Given this setup, here are two conclusions under relevant alternatives.

♣ Theorem 6.7 with $g_1(x) \equiv 1$ and $g_2(y_1, y_2) = g(y_1, y_2)$, yields

$$\frac{\tau(t)}{t} \overset{a.s.}{\to} \frac{2}{\theta_1^2 + \theta_2^2} \quad \text{as} \quad t \to \infty.$$

Since the corresponding strong laws for the marginal tests are

$$\frac{\tau_i(t)}{t} \overset{a.s.}{\to} \frac{2}{\theta_i^2} \quad \text{as} \quad t \to \infty, \quad i = 1, 2.$$

it follows that, under the alternative, we would, at stopping, encounter a two-dimensional rejection, but, possibly not (yet?) a one-dimensional rejection (i.e., someting is wrong but nothing more).

♣ If $T_{i,n}$, $i = 1, 2$, denote the marginal log-likelihood ratios, and $g_1(x) = \frac{1}{2}x^2$, then, for $\theta_1\theta_2 \neq 0$,

$$\frac{T_{i,\tau(t)} - \frac{\theta_i^2}{\theta_1^2+\theta_2^2}t}{\frac{\theta_1\theta_2}{\theta_1^2+\theta_2^2}\sqrt{2t}} \overset{d}{\to} N(0, 1) \quad \text{as} \quad t \to \infty, \quad i = 1, 2.$$

This result provides information about the size of the marginal likelihood at the moment of rejection of the joint null hypothesis. Once again we might arrive at the point "something seems wrong somewhere", but, again, without any hint on where.

8 Records

In this, final, section we turn our attention to a somewhat different kind of problem, the motivation being that the SRW-method provides a nice alternative method of proof.

Let $X_1, X_2, \ldots$ be i.i.d. continuous random variables, with record times $L(1) = 1$ and, recursively, $L(n) = \min\{k : X_k > X_{L(n-1)}\}$, $n \geq 2$.

The associated counting process, $\{\mu(n),\ n \geq 1\}$, is defined by

$$\mu(n) = \#\,\text{records among } X_1, X_2, \ldots, X_n = \max\{k : L(k) \leq n\} = \sum_{k=1}^n I_k,$$

where $I_k = 1$ when X_k is a record, and $I_k = 0$ otherwise, i.e., $I_k \in \text{Be}(1/k)$, $k \geq 1$. Moreover, one can show that the indicators are independent. The standard background reference is [34].

Since the indicators are independent (although not identically distributed) and bounded it is an easy task to prove the following limit theorem.

Theorem 8.1. *We have*

$$\frac{\mu(n)}{\log n} \overset{a.s.}{\to} 1 \quad \text{as} \quad n \to \infty;$$
$$\frac{\mu(n) - \log n}{\sqrt{\log n}} \overset{d}{\to} N(0,1) \quad \text{as} \quad n \to \infty.$$

In order to prove the corresponding result for the record times, Rényi exploited the inversion formula

$$\{L(n) \geq k\} = \{\mu(k) \leq n\}.$$

Now, with the inversion formula (3.1), the renewal counting process, and the SRW-type proofs for first passage times processes in mind, one might guess that this kind of SRW-approach would work here too.

And, indeed, it does; cf. [13] (also [20], Section 6.13). The only obstacle is that Anscombe's theorem is not applicable, and here Theorem 2.5 comes in handy. A compensatory relief is that the boundary is hit exactly, viz.,

$$\mu(L(n)) = n, \tag{8.1}$$

so that no sandwich inequality is needed.

The following result emerges.

Theorem 8.2. *We have*

$$\frac{\log L(n)}{n} \overset{a.s.}{\to} 1 \quad \text{as} \quad n \to \infty;$$
$$\frac{\log L(n) - n}{\sqrt{n}} \overset{d}{\to} N(0,1) \quad \text{as} \quad n \to \infty.$$

Proof. To prove the strong law, we first apply Theorem 2.2, to obtain

$$\frac{\mu(L(n))}{\log L(n)} \overset{a.s.}{\to} 1 \quad \text{as} \quad n \to \infty,$$

so that, recalling (8.1), we are done after turning the conclusion upside down.

As for asymptotic normality, the usual procedure with Theorem 2.5 replacing Anscombe's theorem does it. □

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