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ANTI-COMMUTING REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS

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Abstract

In this paper we give a nonexistence theorem for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with anti-commuting shape operator.

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0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_m(c)$ or in quaternionic space forms there have been many characterizations of model hypersurfaces of type A_1, A_2, B, C, D and E in complex projective space $P_m(\mathbb{C})$, of type A_0, A_1, A_2 and B in complex hyperbolic space $H_m(\mathbb{C})$, or of type A_1, A_2 and B in quaternionic projective space $\mathbb{H}P^m$, which are completely classified by Cecil and Ryan [5], Kimura [6], Berndt [1], and Martinez and Pérez [7], respectively. Among them there have been only a few characterizations of homogeneous hypersurfaces of type B in complex projective space $P_m(\mathbb{C})$. For example, the condition that $A\phi + \phi A = k\phi$, for nonzero constant k, is a model characterization of this kind of type B, which is a tube over a real projective space $\mathbb{R}P^n$ in $P_m(\mathbb{C})$, $m = 2n$ (see Yano and Kon [9]).

Let M be a $(4m - 1)$-dimensional Riemannian manifold with an almost contact structure (ϕ, ξ, η) and an associated Riemannian metric g. Write

$$\omega(X, Y) = g(\phi X, Y), \tag{0.1}$$

where ω defines a 2-form on M and rank ω = rank $\phi = 4m - 2$.

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If there is a nonzero-valued function ρ such that

$$\rho g(\phi X, Y) = \rho\omega(X, Y) = d\eta(X, Y), \tag{0.2}$$

the rank of the matrix (ω) being $4m-2$,

$$\eta \wedge \overbrace{\omega \wedge \cdots \wedge \omega}^{2m-1 \text{ times}} = \eta \wedge \rho^{-(2m-1)} \overbrace{d\eta \wedge \cdots \wedge d\eta}^{2m-1 \text{ times}} \neq 0.$$

Let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces of $\mathbb{C}^{m+2}$. We call such a set $G_2(\mathbb{C}^{m+2})$ complex two-plane Grassmannians. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometry that is equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J} = \text{Span}\{J_1, J_2, J_3\}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold (see Berndt and Suh [3, 4]).

Now we consider a $(4m-1)$-dimensional real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. Then from the Kähler structure of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact structure ϕ on M. If the nonzero function ρ satisfies (0.2), we call M a *contact* hypersurface of the Kähler manifold. Moreover, it can easily be proved that a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is *contact* if and only if there exists a nonzero constant function ρ defined on M such that

$$\phi A + A\phi = k\phi, \quad k = 2\rho. \tag{0.3}$$

This means that

$$g((\phi A + A\phi)X, Y) = 2d\eta(X, Y),$$

where the exterior derivative $d\eta$ of the 1-form η is defined by

$$d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X$$

for any vector fields X, Y on M in $G_2(\mathbb{C}^{m+2})$.

On the other hand, in $G_2(\mathbb{C}^{m+2})$ we are able to consider two kinds of natural geometric condition for real hypersurfaces M that

$$[\xi] = \text{Span}\{\xi\} \quad \text{or} \quad \mathfrak{D}^{\perp} = \text{Span}\{\xi_1, \xi_2, \xi_3\}, \quad \xi_i = -J_i N, \quad i = 1, 2, 3,$$

where N denotes a unit normal to M, is invariant under the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. The first result in this direction is the classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying both conditions. Namely, Berndt and Suh [3] have proved the following.

THEOREM A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if:*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$; or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In Theorem A the vector ξ contained in the one-dimensional distribution $[\xi]$ is said to be a *Hopf* vector when it becomes a principal vector for the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. Moreover, in such a situation M is said to be a *Hopf* hypersurface. Besides this, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ also admits the three-dimensional distribution $\mathfrak{D}^\perp$, which is spanned by *almost contact three-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$. Also Berndt and Suh [4] have given a characterization of real hypersurfaces of type A when the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ, which is equivalent to the condition that the Reeb flow on M is isometric, as follows.

THEOREM B. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

On the other hand, as a characterization of real hypersurfaces of type B in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ in Theorem A, Suh [8], asserted the following fact in terms of the *contact* hypersurface.

THEOREM C. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature satisfying*

$$A\phi + \phi A = k\phi,$$

where the function k is nonzero and constant. Then M is congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.

Now in this paper let us consider a real hypersurface M in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ satisfying $A\phi + \phi A = 0$. When the function k mentioned in Theorem C identically vanishes, the shape operator is said to be *anti-commuting*, that is, the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ satisfies

$$A\phi + \phi A = 0. \tag{*}$$

In such a case we call a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfying (*) an *anti-commuting* hypersurface. We give a nonexistence property of hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with anti-commuting shape operator as follows.

THEOREM. *There exist no anti-commuting real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature.*

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$; for details we refer to [2–4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is 8. Since $G_2(\mathbb{C}^3)$ is isometric to the three-dimensional complex projective space $\mathbb{C}P^3$ with constant holomorphic sectional curvature 8, we shall assume that $m \geq 2$ from now on. Note that the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo 3. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local 1-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Let $p \in G_2(\mathbb{C}^{m+2})$ and W be a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that W is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (or totally complex) submanifold of

$G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (or totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ & - g(JX, Z)JY - 2g(JX, Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y, Z)J_\nu X \\ & - g(J_\nu X, Z)J_\nu Y - 2g(J_\nu X, Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\}, \end{aligned} \tag{1.2}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas

In this section let us give some basic formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ which will be used later.

The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthemore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then expression (1.2) for the curvature tensor $\bar{R}$, the Gauss and the Codazzi equations are respectively given by

$$\begin{aligned}R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + \sum_{\nu=1}^{3}\{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(\phi_\nu \phi Y, Z)\phi_\nu \phi X - g(\phi_\nu \phi X, Z)\phi_\nu \phi Y\} \\ & - \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu \phi X - \eta(X)\eta_\nu(Z)\phi_\nu \phi Y\} \\ & - \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu \phi Y, Z) - \eta(Y)g(\phi_\nu \phi X, Z)\}\xi_\nu \\ & + g(AY, Z)AX - g(AX, Z)AY \end{aligned}$$

and

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \end{aligned}$$

$$+\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\}$$
$$+\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\}\xi_\nu,$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

The following identities can be proved straightforwardly and will be used frequently in subsequent calculations:

$$\begin{aligned}\phi_{\nu+1}\xi_\nu=-\xi_{\nu+2},\quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\ \phi\xi_\nu=\phi_\nu\xi,\quad \eta_\nu(\phi X)=\eta(\phi_\nu X),\\ \phi_\nu\phi_{\nu+1}X=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\ \phi_{\nu+1}\phi_\nu X=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}.\end{aligned}\tag{2.1}$$

Now let us put

$$JX=\phi X+\eta(X)N,\quad J_\nu X=\phi_\nu X+\eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and formulas (1.1) and (2.1),

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi,\quad \nabla_X\xi=\phi AX,\tag{2.2}$$
$$\nabla_X\xi_\nu=q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX,\tag{2.3}$$
$$(\nabla_X\phi_\nu)Y=-q_{\nu+1}(X)\phi_{\nu+2}Y+q_{\nu+2}(X)\phi_{\nu+1}Y+\eta_\nu(Y)AX-g(AX,Y)\xi_\nu.\tag{2.4}$$

Summing up these formulas, we obtain

$$\begin{aligned}\nabla_X(\phi_\nu\xi)&=\nabla_X(\phi\xi_\nu)\\ &=(\nabla_X\phi)\xi_\nu+\phi(\nabla_X\xi_\nu)\\ &=q_{\nu+2}(X)\phi_{\nu+1}\xi-q_{\nu+1}(X)\phi_{\nu+2}\xi+\phi_\nu\phi AX\\ &\quad -g(AX,\xi)\xi_\nu+\eta(\xi_\nu)AX.\end{aligned}\tag{2.5}$$

Moreover, from $JJ_\nu=J_\nu J$, $\nu=1,2,3$, it follows that

$$\phi\phi_\nu X=\phi_\nu\phi X+\eta_\nu(X)\xi-\eta(X)\xi_\nu.\tag{2.6}$$

3. Some key propositions

Now let us take an inner product to Codazzi's equation with ξ and use (2.1) and (2.2). Then

$$\begin{aligned}g((\nabla_XA)Y,\xi)-g((\nabla_YA)X,\xi)=-2g(\phi X,Y)\\ +2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y)-\eta_\nu(Y)\eta_\nu(\phi X)-g(\phi_\nu X,Y)\eta_\nu(\xi)\}.\end{aligned}$$

On the other hand, from formula (*) in the introduction, $A\xi = \alpha\xi$ where $\alpha = \eta(A\xi)$. From this, by taking the covariant derivative and using (2.2),

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX.$$

It follows that

$$g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) - 2g(A\phi AX, Y).$$

Combining the above two equations,

$$\begin{aligned} -2g(\phi X, Y) + 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\ = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) - 2g(A\phi AX, Y). \end{aligned} \tag{3.1}$$

Putting $X = \xi$ in (3.1),

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y), \tag{3.2}$$

$$\text{grad}\,\alpha = (\xi\alpha)\xi + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi\xi_\nu. \tag{3.3}$$

Now substituting (3.2) into (3.1) gives

$$\begin{aligned} g(A\phi AX, Y) - g(\phi X, Y) = 2\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\eta_\nu(\xi) \\ - \sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \end{aligned} \tag{3.4}$$

for any tangent vector fields X and Y on M.

LEMMA 3.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with anti-commuting shape operator. Then* Tr $A = \alpha$.

PROOF. From (*) and (2.2) it follows that

$$AX - \phi A\phi X - \alpha\eta(X)\xi = 0,$$

where we have put $\alpha = \eta(A\xi)$. If we take an orthonormal basis for M in such a way that

$$\{e_i \mid i = 1, 2, \ldots, 4m-1\},$$

then

$$\sum_{i=1}^{4m-1}\{g(Ae_i, e_i) - g(\phi A\phi e_i, e_i) - \alpha\eta(e_i)g(\xi, e_i)\} = 0,$$

that is, Tr A − Tr $\phi A\phi - \alpha = 0$.

On the other hand, we see that $\operatorname{Tr}\phi A\phi = \operatorname{Tr} A\phi^2 = -\operatorname{Tr} A + \alpha$. Therefore, $\operatorname{Tr} A = \alpha$. □

LEMMA 3.2. *Let M be an anti-commuting real hypersurface in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature. Then ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

PROOF. By Lemma 3.1 and the assumption we know that α is constant. And from (3.2) we get

$$\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y) = 0.$$

Now let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$. Then

$$\eta_1(\xi)\eta_1(\phi Y) = 0.$$

First, if $\eta_1(\xi) = 0$, then obviously $\xi \in \mathfrak{D}$.

Next let us consider the case where $\eta_1(\phi Y) = 0$. By putting $\phi_1\xi$ in Y we know $\eta(X_0) = 0$, which gives $\xi \in \mathfrak{D}^\perp$. This proves our assertion. □

Now let us denote by $\mathfrak{h}$ the orthogonal complement of the Reeb vector field ξ in the tangent space of M in $G_2(\mathbb{C}^{m+2})$.

LEMMA 3.3. *If $A\phi + \phi A = 0$, $X \in \mathfrak{h}$ with $AX = \lambda X$, then*

$$\lambda A\phi X - \phi X + \sum_{\nu=1}^{3}\{2\eta_\nu(\xi)\eta_\nu(\phi X)\xi - \eta_\nu(X)\phi_\nu\xi - \eta_\nu(\phi X)\xi_\nu - \eta_\nu(\xi)\phi_\nu X\} = 0. \tag{3.5}$$

PROOF. From (3.4) it follows that

$$A\phi AX - \phi X + 2\sum_{\nu=1}^{3}\{\eta(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi\}\eta_\nu(\xi)$$
$$- \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X\} = 0.$$

And using the assumption, for $X \in \mathfrak{h}$ such that $AX = \lambda X$, leads to the above formula. □

PROPOSITION 3.4. *There exist no anti-commuting real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature for $\xi \in \mathfrak{D}^\perp$.*

PROOF. By (3.5) and (*), for any $X \in \mathfrak{h}$,

$$(\lambda^2 + 1)X - \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi\phi_\nu\xi + \eta_\nu(\phi X)\phi\xi_\nu + \eta_\nu(\xi)\phi\phi_\nu X\} = 0.$$

Since $\xi \in \mathfrak{D}^{\perp}$, we can put $\xi = \xi_1$. Then

$$(\lambda^2 + 1)X + 2\eta_2(X)\xi_2 + 2\eta_3(X)\xi_3 - \phi\phi_1 X = 0.$$

Since $X \in \mathfrak{h}$, we suppose that $X = \mathfrak{D}X + \eta_2(X)\xi_2 + \eta_3(X)\xi_3$. This implies that

$$(\lambda^2 + 1)\mathfrak{D}X + (\lambda^2 + 2)\eta_2(X)\xi_2 + (\lambda^2 + 2)\eta_3(X)\xi_3 - \phi\phi_1\mathfrak{D}X = 0. \tag{3.6}$$

Putting $X = \xi_2$ and $X = \xi_3$ in (3.6), we obtain $(\lambda^2 + 2)\xi_2 = 0$ and $(\lambda^2 + 2)\xi_3 = 0$, respectively. From these facts, we see that $\lambda^2 + 2 = 0$. Therefore we get a contradiction, which gives the proof of our proposition. □

4. Anti-commuting hypersurfaces in $G_2(\mathbb{C}^{m+2})$ for $\xi \in \mathfrak{D}^{\perp}$

In this section we wish to show that there exist no hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with anti-commuting shape operator for $\xi \in \mathfrak{D}$. In order to do this we assert the following result.

LEMMA 4.1. *Let M be an anti-commuting real hypersurface in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature for $\xi \in \mathfrak{D}$. Then $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.*

PROOF. From the assumption we know that the function α is constant. Then for $\xi \in \mathfrak{D}$ and from (3.1), for any tangent vector field X on M,

$$\phi X - A\phi AX + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi + \eta_\nu(\phi X)\xi_\nu\} = 0. \tag{4.1}$$

To prove this lemma it suffices to show that $g(A\mathfrak{D}, \xi_\nu) = 0$, $\nu = 1, 2, 3$. In order to do this, we put

$$\mathfrak{D} = [\xi] \oplus [\phi_1\xi, \phi_2\xi, \phi_3\xi] \oplus \mathfrak{D}_0,$$

where the distribution $\mathfrak{D}_0$ is an orthogonal complement of $[\xi] \oplus [\phi_1\xi, \phi_2\xi, \phi_3\xi]$ in the distribution $\mathfrak{D}$.

First, from the assumption $\xi \in \mathfrak{D}$ we know that $g(A\xi, \xi_\nu) = 0$, $\nu = 1, 2, 3$, because $A\xi = \alpha\xi$.

Next, we also get the conclusion $g(A\phi_i\xi, \xi_\nu) = 0$, for $i, \nu = 1, 2, 3$. In fact, using (2.3) and $\xi \in \mathfrak{D}$,

$$\begin{aligned}
g(A\phi_i\xi, \xi_\nu) &= g(A\xi_\nu, \phi_i\xi) \\
&= g(A\xi_\nu, \phi\xi_i) \\
&= -g(\phi A\xi_\nu, \xi_i) \\
&= -g(\nabla_{\xi_\nu}\xi, \xi_i) \\
&= g(\xi, \nabla_{\xi_\nu}\xi_i) \\
&= g(\xi, q_{i+2}(\xi_\nu)\xi_{i+1} - q_{i+1}(\xi_\nu)\xi_{i+2} + \phi_i A\xi_\nu) \\
&= g(\xi, \phi_i A\xi_\nu) \\
&= -g(A\phi_i\xi, \xi_\nu),
\end{aligned}$$

that is, $g(A\phi_i\xi, \xi_\nu) = 0$, $\nu = 1, 2, 3$.

Finally, we consider the case $X \in \mathfrak{D}_0$, where the distribution $\mathfrak{D}_0$ is denoted by

$$\mathfrak{D}_0 = \{X \in \mathfrak{D} \mid X \perp \xi \text{ and } \phi_i \xi,\ i = 1, 2, 3\}.$$

In order to show this, let us replace X by ξ_μ in (4.1). Then it follows that

$$2\phi\xi_\mu = A\phi A\xi_\mu.$$

From this, together with the assumption (*),

$$A^2\phi\xi_\mu = -2\phi\xi_\mu.$$

Then multiplying both sides by ϕ and also using the formula $A\phi + \phi A = 0$,

$$A^2(-\xi_\mu + \eta(\xi_\mu)\xi) = -2(-\xi_\mu + \eta(\xi_\mu)\xi).$$

This implies that

$$A^2\xi_\mu = -2\xi_\mu, \quad \mu = 1, 2, 3. \tag{4.2}$$

On the other hand, if we consider the case where $X \in \mathfrak{D}_0$ in (3.4), then

$$\phi X = A\phi A X.$$

From $A\phi + \phi A = 0$, this becomes $-A^2\phi X = \phi X$. Then from this, replacing X by ϕX leads, for any $X \in \mathfrak{D}_0$, to

$$A^2 X = -X. \tag{4.3}$$

Using (4.2) and (4.3),

$$\begin{aligned} g(AX, \xi_\mu) &= g(A(-A^2X), \xi_\mu) \\ &= -g(A^3X, \xi_\mu) = -g(AX, A^2\xi_\mu) \\ &= -g(AX, -2\xi_\mu) = 2g(AX, \xi_\mu), \end{aligned}$$

for any vector fields X in $\mathfrak{D}_0$. Then for any $X \in \mathfrak{D}_0$, $g(AX, \xi_\mu) = 0$, $\mu = 1, 2, 3$. This completes the proof. □

For a tube of type B in Theorem A let us recall a proposition given in Berndt and Suh [3] as follows.

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r),$$

with some $r \in (0, \pi/4)$. *The corresponding multiplicities are*

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu),$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathfrak{J}J\xi, \quad T_\gamma = \mathfrak{J}\xi, \quad T_\lambda, \quad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

Now by using Proposition A let us check whether a tube of type B in Theorem A, that is, a tube over a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$ cannot satisfy the formula (*).

In fact, for any $\xi_\nu \in T_\beta$, $\beta = 2 \cot 2r$, the eigenspace $T_\gamma = \mathfrak{J}\xi$ gives $\phi\xi_\nu \in T_\gamma$. This implies that $A\phi\xi_\nu = 0$ for any $\nu = 1, 2, 3$. From this,

$$A\phi\xi_\nu + \phi A\xi_\nu = 2 \cot 2r\phi\xi_\nu = 0.$$

For any $X \in T_\lambda$, $\lambda = \cot r$, we know that $JT_\lambda = T_\mu$ gives

$$A\phi X + \phi AX = -\tan r\phi X + \cot r\phi X = 2 \cot 2r\phi X = 0.$$

From this, we get $\cot 2r = 0$, giving a contradiction. So real hypersurfaces of type B cannot satisfy formula (*).

PROPOSITION 4.2. *There exist no anti-commuting real hypersurfaces in* $G_2(\mathbb{C}^{m+2})$ *with constant mean curvature for* $\xi \in \mathfrak{D}$.

Taking this Proposition 4.2 together with Proposition 3.4 gives a complete proof of our main theorem in the introduction.

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