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Anti-invariant Riemannian Submersions from Nearly Kaehler Manifolds

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Abstract. We extend the notion of anti-invariant and Langrangian Riemannian submersion to the case when the total manifold is nearly Kaehler. We obtain the integrability conditions for the horizontal distribution while it is noted that the vertical distribution is always integrable. We also investigate the geometry of the foliations of the two distributions and obtain the necessary and sufficient condition for a Langrangian submersion to be totally geodesic. The decomposition theorems for the total manifold of the submersion are obtained.

1. Introduction

The study of Riemannian submersion π of a Riemannian manifold M onto a Riemannian manifold B was initiated by B. O'Neill (cf. [7], [8]) and then A. Gray [5]. A submersion naturally gives rise to two distributions, called the vertical distribution and the horizontal distribution, of which the vertical distribution is always integrable giving rise to the fibres which are closed submanifolds of the ambient space. Later many researchers considered such submersions between manifolds with differentiable structures. Namely, B. Watson [11] defined almost Hermitian submersions between almost Hermitian manifolds and proved that in most cases the base manifold and fibers have the same structures as that of the ambient space. Almost Hermitian submersions were further extended to almost contact manifolds [3] and locally conformal Kaehler manifolds [6].

Let M be a complex m-dimensional almost Hermitian manifold with Hermitian metric g and almost complex structure J and B be a complex n-dimensional almost Hermitian manifold with Hermitian metric g_B and almost complex structures J'. A Riemannian submersion $\pi : M \to B$ is called an almost Hermitian submersion if π is an almost complex mapping , i.e., $\pi_* \circ J = J' \circ \pi_*$. The main outcome of this concept is that the both vertical and horizontal distributions are invariant under J. Escobales [4] studied that the Riemannian submersion from complex projective space onto a Riemannian manifolds by considering the fibers to be connected, complex, totally geodesic submanifold. Which, in fact, also implies that the vertical distribution is invariant under the almost complex structure.

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Recently, B. Sahin [10] has introduced the notion of Riemannian submersion from an almost Hermitian manifold to Riemannian manifold under the assumption that fibers are anti-invariant with respect to the complex structure of the almost Hermitian manifold. Therefore, this assumption implies that the horizontal distribution is not completely invariant under the action of the complex structure of the total space of such a submersion. He called this submersion as the anti-invariant Riemannian submersion. In fact, he studied anti-invariant Riemannian submersions from Kaehler manifolds to Riemannian manifolds. In general, the almost Hermitian submersions are used to describe the geometry of the base manifolds, while the anti-invariant Riemannian submersions, serve to describe the geometry of total space. It is also noted that the geometry of anti-invariant Riemannian submersions completely differs from that of the geometry of almost Hermitian submersions.

It is well known that every Kaehler manifold is a nearly Kaehler manifold but converse is not in general true. For example, the 6-dimensional sphere S^6 with canonical almost complex structure is nearly Kaehler but it is not Kaehler manifold. In fact, for the converse to hold it is necessary that the almost complex structure J should be integrable. A. Gray [5] pointed that like Kaehler manifolds, nearly Kaehler manifolds also have rich geometrical as well as topological properties. It is, therefore, interesting to study the anti-invariant Riemannian submersions from nearly Kaehler manifolds.

In section 2, we recall the basic definitions, notations and some results of submersion which we use in the later sections and define the second fundamental form. In section 3, we study the submersion π and obtain several results. Section 4 is devoted to the study of geodesicness of the foliations of two distributions and obtain several equivalent conditions. In particular, we prove one of our main results:

Theorem 4.3. Let π be an anti-invariant Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then, the following assertions are equivalent to each other

(*a*) $(ker\pi_*)$ defines a totally geodesic foliation on M.
(*b*) $g_B((\nabla\pi_*)(V, JX), \pi_* JW) = -g(\mathcal{P}_W V, CX) - g(\mathcal{Q}_W V, BX)$.
(*c*) $g(\mathcal{T}_V BX + \mathcal{A}_{CX} V, JW) = g(\mathcal{P}_W V, CX) + g(\mathcal{Q}_W V, BX)$,

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V, W \in \Gamma(ker\pi_*)$.

Also in this section we define totally geodesic map and prove:

Theorem 4.4. Let π be a Langrangian Riemannian submersion from an almost Hermitian manifold (M, g, J) to a Riemannian manifold (B, g_B). Then π is totally Geodesic map if and only if

$$\mathcal{T}_V JW = \mathcal{Q}_V W$$

and

$$\mathcal{A}_X JW = \mathcal{Q}_X W,$$

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

In section 5, we define different type of product manifolds and obtain decomposition theorems on the total space of the submersion. We, in particular, prove one of the main results:

Theorem 5.2. Let π be Langrangian Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then M is a twisted product manifold of the form $M_{(ker\pi_*)^\perp} \times_f M_{(ker\pi_*)}$ if and only if

$$\mathcal{T}_V JX = -g(X, \mathcal{T}_V V)\|V\|^{-2} JV - \mathcal{P}_X V,$$

and

$$\mathcal{A}_X JY = \mathcal{P}_X Y$$

for any $X,\ Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$, where $M_{(ker\pi_*)^\perp}$ and $M_{(ker\pi_*)}$ are integral manifolds of the distributions $(ker\pi_*)^\perp$ and $(ker\pi_*)$ respectively.

2. Preliminaries

In this section, we recall the notions of almost complex structure, the notion of Riemannian submersions between Riemannian manifolds and give a brief review of basic facts of Riemannian submersions. For the notion of submersion we follow B. O′ Neill [7] while for anti-invariant Riemannian submersions we have taken extracts from B. Sahin [10].

An almost complex structure on a smooth manifold M is a smooth tensor field J of type(1,1) with the property that $J^2 = -I$. A smooth manifold equipped with such an almost complex structure is called an almost complex manifold. An almost complex manifold is necessarily orientable and is of even dimension. An almost complex manifold (M, J) endowed with a chosen Riemannian metric g and satisfying

$$g(JX, JY) = g(X, Y), \tag{2.1}$$

for all $X, Y \in \Gamma(TM)$, is called an almost Hermitian manifold. The Levi-Civita connection ∇ of the almost Hermitian manifold M can be extended to the whole tensor algebra on M, and in this way we obtain tensor fields like $(\nabla_X J)$ and that

$$(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y, \tag{2.2}$$

for all $X,\ Y \in \Gamma(TM)$.
An almost Hermitian manifold M is called Kaehler manifold if

$$(\nabla_X J)Y = 0, \tag{2.3}$$

and is called nearly Kaehler manifold if

$$(\nabla_X J)Y + (\nabla_Y J)X = 0, \tag{2.4}$$

for all $X, Y \in \Gamma(TM)$.

Let (M, g) and (B, g_B) be two Riemannian manifolds with $\dim(M) = m$, $\dim(B) = n$ and $m > n$. A Riemannian submersion $\pi : M \to B$ is a map of M onto B satisfying the following axioms:

(S_1) π has maximal rank,

that is, each derivative map π_* of π is onto and hence, for each $b \in B$, $\pi^{-1}(b)$ is a submanifold of M of dimension= $\dim M - \dim B$. The submanifolds $\pi^{-1}(b)$ are called fibers. A vector field on M is called vertical vector field if it is always tangent to fibers and it is called horizontal if it is always orthogonal to fibers.

The second axiom may now be stated in the following form:

(S_2) The differential π_* preserves the length of the horizontal vectors.

If the submersions are considered as the generalization to an isometry $M \to B$ to the case $\dim M \geq \dim B$, then the notion bears a comparison with the generalization to $\dim M \leq \dim B$, that is, with an isometric immersion. The behavior of immersion is described by a single tensor, the second fundamental form while for a submersion two such tensors are defined, one of which is the second fundamental form of all the fibers.

A vector field X on M is called basic if X is horizontal and is π related to a vector field X_* on B, that is, $\pi_* X_p = X_{*_{\pi(p)}}$ for all $p \in M$. We denote by $(ker\pi_*)$ and $(ker\pi_*)^\perp$ the vertical and horizontal distribution on

M. Also note that we denote the projection morphisms on the distributions $(ker\pi_*)$ and $(ker\pi_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$ respectively. The letters U, V, W will always denote vertical vector fields and X, Y, Z horizontal vector fields.

We recall the following lemma from O'Neill [7]:

Lemma 2.1. Let $\pi : M \to B$ be a Riemannian submersion between Riemannian manifolds and X, Y be basic vector fields on M, then

(a) $g(X, Y) = g_B(X_*, Y_*)o\pi$.
(b) $\mathcal{H}[X, Y]$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, *i.e.*, $([X, Y]^{\mathcal{H}}) = [X_*, Y_*]$.
(c) $[V, X]$ is vertical, for any vector $V \in \Gamma(ker\pi_*)$.

(d) $(\nabla_X Y)^{\mathcal{H}}$ is the basic vector field corresponding to $\nabla^*_{X_*} Y_*$, where ∇^* is a Levi-Civita connection on B.

The geometry of Riemannian submersions is characterized by O'Neill's tensor $\mathcal{T}$ and $\mathcal{A}$ defined for any arbitrary vector fields E and F on M by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F \tag{2.5}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.6}$$

where ∇ is the Levi-Civita connection on M. The tensors $\mathcal{T}$ serves as the second fundamental form of the fibers and hence, it is easy to see that a Riemannian submersion $\pi : M \to B$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(TM)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, i.e., $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, i.e., $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$\mathcal{T}_U W = \mathcal{T}_W U, \quad \forall\, U, W \in \Gamma(ker\pi_*) \tag{2.7}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y], \quad \forall\, X, Y \in \Gamma(ker\pi_*)^\perp. \tag{2.8}$$

(2.8) shows that $\mathcal{A}$ is necessarily the integrability tensor of the horizontal distribution $(ker\pi_*)^\perp$on M.

On the other hand, from (2.5) and (2.6) we have the following lemma:

Lemma 2.2 [7]. Let X, Y be horizontal vector fields and V, W vertical vector fields. Then

1. $\nabla_V W = T_V W + \hat{\nabla}_V W,$
2. $\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X,$
3. $\nabla_X V = A_X V + \mathcal{V}\nabla_X V,$
4. $\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y,$

where $\hat{\nabla}_V W = \mathcal{V}(\nabla_V W)$. Furthermore, if X is basic, $\mathcal{H}(\nabla_V X) = \mathcal{A}_X V$.

Finally, we recall the second fundamental form of a map between Riemannian manifolds [1]. Let (M, g) and (B, g_B) be Riemannian manifolds and suppose that $\phi : M \to B$ be a smooth map between them. Then the differential ϕ_* of ϕ can be viewed as a section of the bundle $\mathrm{Hom}(TM, \phi^{-1}(TB)) \to M$, where $\phi^{-1}(TB)$

is the pullback bundle which has fibers $(\phi^{-1}(TB))_p = T_{\phi(p)}B,\ p \in M$. Hom$(TM, \phi^{-1}(TB))$ has a connection ∇ induced from Levi-Civita connection ∇^M and the pullback connection. The second fundamental form ϕ is then given by

$$(\nabla\phi_*)(X,Y) = \nabla^{\phi}_X\phi_*(Y) - \phi_*(\nabla^M_X Y) \tag{2.9}$$

for $X, Y \in \Gamma(TM)$, where ∇^{ϕ} is the pullback connection. It is known that the second fundamental form is symmetric.

3. Integrability Conditions

In this section, we define anti-invariant Riemannian submersions and Langrangian Riemannian submersions and study the geometry of distributions $(ker\pi_*)$ and $(ker\pi_*)^{\perp}$ and obtain the integrability conditions for the distribution $(ker\pi_*)^{\perp}$ for such submersions.

Definition 3.1. *[10] A Riemannian submersion π from an almost Hermitian manifold (M, g, J) to a Riemannian manifold (B, g_B) such that $(ker\pi_*)$ is anti-invariant with respect to J i.e., $J(ker\pi_*) \subseteq (ker\pi_*)^{\perp}$, is called an anti-invariant Riemannian submersion .*

Let $\pi : (M, g, J) \to (B, g_B)$ be an anti-invariant Riemannian submersion from an almost Hermitian manifold M to a Riemannian manifold B. From the above definition, we see that $J(ker\pi_*)^{\perp} \cap (ker\pi_*) \neq \{0\}$ and hence we have,

$$(ker\pi_*)^{\perp} = J(ker\pi_*) \oplus \mu, \tag{3.1}$$

where μ denote the orthogonal complementary distribution to $J(ker\pi_*)$ in $(ker\pi_*)^{\perp}$ and it is invariant under J. Thus, for $X \in \Gamma(ker\pi_*)^{\perp}$ we have

$$JX = BX + CX, \tag{3.2}$$

where $BX \in \Gamma(ker\pi_*)$ and $CX \in \Gamma(\mu)$.

On the other hand, since $\pi_*((ker\pi_*)^{\perp}) = TB$ and π is a Riemannian submersion, using (3.2) it can be shown that $g_B(\pi_*JV, \pi_*CX) = 0$, for any $X \in \Gamma(ker\pi_*)^{\perp}$ and $V \in \Gamma(ker\pi_*)$ which implies that

$$TB = \pi_*(J(ker\pi_*)) \oplus \pi_*(\mu). \tag{3.3}$$

Now, we prove

Lemma 3.1. *Let π be an anti-invariant Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then*

$$(i) \qquad g(CY, JV) = 0, \tag{3.4}$$

$$(ii) \qquad g(\nabla_X CY, JV) = -2g(CY, JA_X V) + g(CY, T_V BX) + g(CY, A_{CX}V), \tag{3.5}$$

for any $X, Y \in \Gamma(ker\pi_*)^{\perp}$ and $V \in \Gamma(ker\pi_*)$.

Proof. (i) For $Y \in \Gamma(ker\pi_*)^{\perp}$ and $V \in \Gamma(ker\pi_*)$, using (3.2) we have

$$\begin{aligned} g(CY, JV) &= g(JY, JV) - g(BY, JV) \\ &= g(JY, JV), \end{aligned}$$

since, $BY \in \Gamma(ker\pi_*)$ and $JV \in \Gamma(ker\pi_*)^\perp$.

But from (2.1), $g(JY, JV) = g(Y, V) = 0$ and hence we get (3.4).

(*ii*) On using (2.2), (2.4), (3.4) and Lemma 2.2 for $X, Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$, we have

$$\begin{aligned} g(\nabla_X CY, JV) &= -g(CY, \nabla_X JV) \\ &= -g(CY, J\nabla_X V) + g(CY, (\nabla_V J)X) \\ &= -g(CY, JA_X V) + g(CY, \nabla_V(BX + CX)) - g(CY, JA_X V) \\ &= -2g(CY, J\mathcal{A}_X V) + g(CY, \nabla_V BX) - g(CY, [V, CX] + \nabla_{CX} V) \end{aligned}$$

Since $[V, CX] \in \Gamma(ker\pi_*)$, hence we get

$$g(\nabla_X CY, JV) = -2g(CY, JA_X V) + g(CY, T_V BX) + g(CY, A_{CX} V),$$

which completes the proof. □

Note: Whenever it is need we have supposed the horizontal vector field to be basic.

For any arbitrary tangent vector fields E and F on M, we set

$$(\nabla_E J)F = \mathcal{P}_E F + \mathcal{Q}_E F, \tag{3.6}$$

where $\mathcal{P}_E F$ (respectively $\mathcal{Q}_E F$) denote the horizontal (respectively vertical) part of $(\nabla_E J)F$.

For a Kaehler manifold M, we have

$$\mathcal{P} = \mathcal{Q} = 0, \qquad \forall\ E,\ F \in \Gamma(TM). \tag{3.7}$$

If M is a nearly Kaehler manifold, then it can be easily checked that both $\mathcal{P}$ and $\mathcal{Q}$ are anti-symmetric in E and F, i.e.

$$\mathcal{P}_E F = -\mathcal{P}_F E \text{ and } \mathcal{Q}_E F = -\mathcal{Q}_F E. \tag{3.8}$$

Using (3.6), we prove the following:

Lemma 3.2. *Let π be an anti-invariant Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then we have*

$$g(\nabla_X CY, JV) = -g(CY, J\mathcal{A}_X V) - g(CY, \mathcal{P}_X V) \tag{3.9}$$

for any $X, Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$.

Proof. For $X,\ Y \in \Gamma(ker\pi_*)^\perp$ *and* $V \in \Gamma(ker\pi_*)$, using (2.2) we have

$$\begin{aligned} g(\nabla_X CY, JV) &= -g(CY, \nabla_X JV) \\ &= -g(CY, (\nabla_X J)V) - g(CY, J\nabla_X V). \end{aligned}$$

The result then follows from (3.6) and Lemma 2.2.
□

From Lemma 3.1 and Lemma 3.2, it follows:

Proposition 3.1. *Let π be an anti-invariant Riemannian submersion from nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then*

$$g(CY, \mathcal{P}_X V) = g(CY, J\mathcal{A}_X V) - g(CY, \mathcal{A}_{CX} V) - g(CY, \mathcal{T}_V BX), \tag{3.10}$$

for any $X,\ Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$.

Now, for the integrability of the distribution $(ker\pi_*)^\perp$ we have

Theorem 3.1. *Let π be an anti-invariant Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then the following assertions are equivalent to each other*

(*a*) $(ker\pi_*)^\perp$ *is integrable.*

(*b*) $g_B((\nabla\pi_*)(Y, BX) - (\nabla\pi_*)(X, BY),\ \pi_* JV) = g(CY, J\mathcal{A}_X V) + g(CY, \mathcal{P}_X V)$
$- g(CX, J\mathcal{A}_Y V) - g(CX, \mathcal{P}_Y V) + 2g(\mathcal{P}_X Y, JV).$

(*c*) $g(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) = g(CY, J\mathcal{A}_X V) + g(CY, \mathcal{P}_X V)$
$- g(CX, J\mathcal{A}_Y V) - g(CX, \mathcal{P}_Y V) + 2g(\mathcal{P}_X Y, JV),$

for any $X,\ Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$.

Proof. For $Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$, we see from definition 3.1, $JV \in (ker\pi_*)^\perp$ and $JY \in ker\pi_* \oplus \mu$. For $X \in \Gamma(ker\pi_*)^\perp$, using (2.1) and (2.2) we get

$$\begin{aligned} g([X, Y], V) &= g(J[X, Y], JV) \\ &= g(\nabla_X JY, JV) - g((\nabla_X J)Y, JV) - g(\nabla_Y JX, JV) + g((\nabla_Y J)X, JV) \end{aligned}$$

Further, by using (2.9), (3.2), (3.8) and Lemma 3.2, we obtain

$$\begin{aligned} g([X, Y], V) &= g(\nabla_X BY, JV) - g(\nabla_Y BX, JV) \\ &+ g(\nabla_X CY, JV) - g(\nabla_Y CX, JV) - g(\mathcal{P}_X Y, JV) + g(\mathcal{P}_Y X, JV) \\ &= g_B(\pi_*\nabla_X BY, \pi_* JV) - g_B(\pi_*\nabla_Y BX, \pi_* JV) \\ &+ g(\nabla_X CY, JV) - g(\nabla_Y CX, JV) - 2g(\mathcal{P}_X Y, JV) \\ &= g_B((\nabla\pi_*)(Y, BX) - (\nabla\pi_*)(X, BY), \pi_* JV) - g(CY, J\mathcal{A}_X V) \\ &- g(CY, \mathcal{P}_X V) + g(CX, J\mathcal{A}_Y V) + g(CX, \mathcal{P}_Y V) - 2g(\mathcal{P}_X Y, JV). \end{aligned}$$

Hence, $(ker\pi_*)^\perp$ is integrable if and only if

$$\begin{aligned} g_B((\nabla\pi_*)(Y, BX) - (\nabla\pi_*)(X, BY),\ \pi_* JV) &= g(CY, J\mathcal{A}_X V) + g(CY, \mathcal{P}_X V) \\ &- g(CX, J\mathcal{A}_Y V) - g(CX, \mathcal{P}_Y V) + 2g(\mathcal{P}_X Y, JV), \end{aligned} \tag{3.11}$$

which shows that $(a) \Leftrightarrow (b)$.

On the other hand, by using (2.9) and Lemma 2.2, we obtain

$$\begin{aligned} (\nabla\pi_*)(Y, BX) - (\nabla\pi_*)(X, BY) &= -\pi_*(\nabla_Y BX - \nabla_X BY) \\ &= \pi_*(\mathcal{A}_X BY - \mathcal{A}_Y BX). \end{aligned} \tag{3.12}$$

Hence, (3.11) and (3.12) shows that $(b) \Leftrightarrow (c)$.
□

Definition 3.2. *[10] An anti-invariant Riemannian submersion π is said to be Langrangian Riemannian submersion if $J(ker\pi_*) = (ker\pi_*)^\perp$. If $\mu \neq \{0\}$ in (3.1), then π is said to be a proper anti-invariant Riemannian submersion.*

For a Langrangian Riemannian submersion, we have the following corollary:

Corollary 3.1. *Let $\pi : (M, g, J) \to (B, g_B)$ be a Langrangian Riemannian submersion from a nearly Kaehler manifold (M, g, J) onto a Riemannian manifold (B, g_B). Then the following assertions are equivalent to each other*

(a) $(ker\pi_*)^\perp$ is integrable,
(b) $g_B((\nabla\pi_*)(Y, JX) - (\nabla\pi_*)(X, JY), \pi_* JV) = 2g(\mathcal{P}_X Y, JV)$,
(c) $\mathcal{A}_X JY - \mathcal{A}_Y JX = 2\mathcal{P}_X Y$,

for any $X, Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$.

Since $\mathcal{P} = 0$ if M is a Kaehler manifold, then Theorem 3.1 and Corollary 3.1 reduces to Theorem 3.1 and Corollary 3.1 of B Sahin [10].

Since, the vertical vector fields are π-related to zero vectors. It is noted that the vertical distribution is integrable.

We now have

Theorem 3.2. *Let π be an anti-invariant Riemannian submersion from nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B), then we have the following*

(a) $g_B((\nabla\pi_*)(W, JV) - (\nabla\pi_*)(V, JW), \pi_* CX) = 2g(\mathcal{P}_V W, CX) + 2g(\mathcal{Q}_V W, BX)$,

(b) $g(\mathcal{A}_{JW} V - \mathcal{A}_{JV} W, CX) = 2g(\mathcal{P}_V W, CX) + 2g(\mathcal{Q}_V W, BX)$,

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V, W \in \Gamma(ker\pi_*)$.

Proof. For any $X \in \Gamma(ker\pi_*)^\perp$ and $V, W \in \Gamma(ker\pi_*)$, using (2.1), (2.4) and (3.2) we have

$$\begin{aligned} g([V, W], X) &= g(J[V, W], JX) \\ &= g(\nabla_V JW, JX) - g(\nabla_W JV, JX) - 2g((\nabla_V J)W, JX) \end{aligned}$$

Now using (2.9), we get

$$\begin{aligned} g([V, W], X) &= -g_B((\nabla\pi_*)(V, JW), \pi_* CX) + g_B((\nabla\pi_*)(W, JV), \pi_* CX) \\ &- 2g(\mathcal{P}_V W, JX) - 2g(\mathcal{Q}_V W, JX) \end{aligned}$$

Since $(ker\pi_*)$ is integrable, we get

$$g_B((\nabla\pi_*)(W, JV) - (\nabla\pi_*)(V, JW), \pi_* CX) = 2g(\mathcal{P}_V W, JX) + 2g(\mathcal{Q}_V W, JX). \tag{3.13}$$

On the other hand, using (2.9) we have

$$\begin{aligned}(\nabla\pi_*)(W, JV) - (\nabla\pi_*)(V, JW) &= \pi_*(\nabla_V JW) - \pi_*(\nabla_W JV) \\ &= \pi_*(\mathcal{A}_{JW} V - \mathcal{A}_{JV} W). \qquad (3.14)\end{aligned}$$

From (3.13) and (3.14), we get (*b*).
□

Next, we have

Theorem 3.3. *Let π be an anti-invariant Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then we have*

(*a*) $g_B((\nabla\pi_*)(V, JX), \pi_* JW) - g_B((\nabla\pi_*)(W, JX), \pi_* JV) = 2g(\mathcal{P}_V W, CX) + 2g(\mathcal{Q}_V W, BX)$.

(*b*) $g(JV, \mathcal{T}_W BX) - g(JW, \mathcal{T}_V BX) + g(JV, \mathcal{A}_{CX} W) - g(JW, \mathcal{A}_{CX} V) = 2g(\mathcal{P}_V W, CX) + 2g(\mathcal{Q}_V W, BX)$,

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

Proof. Since $(ker\pi_*)$ is integrable then for any $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$ using (2.1), (3.2), (3.8) and (2.9), we have

$$\begin{aligned}0 &= -g(\nabla_V BX, JW) - g(\nabla_V CX, JW) + g(\nabla_W BX, JV) + g(\nabla_W CX, JV) - 2g(\mathcal{P}_V W, CX) - 2g(\mathcal{Q}_V W, BX) \\ &= g_B((\nabla\pi_*)(V, BX), \pi_* JW) + g_B((\nabla\pi_*)(V, CX), \pi_* JW) - g_B((\nabla\pi_*)(W, BX), \pi_* JV) \\ &\quad -g_B((\nabla\pi_*)(W, CX), \pi_* JV) - 2g(\mathcal{P}_V W, CX) - 2g(\mathcal{Q}_V W, BX),\end{aligned} \tag{3.15}$$

Since, $\nabla^\pi_V \pi_* X = 0$, we get (*b*).

On the other hand by using Lemma 2.2 and (3.2) in the above calculation, we have

$$\begin{aligned}0 &= g(\nabla_V JW, JX) - g(\nabla_W JV, JX) - g((\nabla_V J)W, JX) - g((\nabla_W J)V, JX) \\ &= -g(\nabla_V JX, JW) + g(\nabla_W JX, JV) - 2g(\mathcal{P}_V W, CX) - 2g(\mathcal{Q}_V W, BX) \\ &= -g(\nabla_V BX, JW) - g(\nabla_V CX, JW) + g(\nabla_W BX, JV) + g(\nabla_W CX, JV) - 2g(\mathcal{P}_V W, CX) - 2g(\mathcal{Q}_V W, BX) \\ &= -g(\mathcal{T}_V BX, JW) - g(\mathcal{A}_{CX} V, JW) + g(\mathcal{T}_W BX, JV) + g(\mathcal{A}_{CX} W, JV) - 2g(\mathcal{P}_V W, CX) - 2g(\mathcal{Q}_V W, BX)\end{aligned}$$

which gives (*b*).
□

For a Langrangian Riemannian submersion, from Theorem 3.3 we have following corollary:

Corollary 3.2. *Let π be Langrangian Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then we have*

(a) $g_B((\nabla\pi_*)(V, JX), \pi_*JW) - g_B((\nabla\pi_*)(W, JX), \pi_*JV) = 2g(\mathcal{Q}_V W, JX)$.

(b) $g(\mathcal{T}_W JX, JV) - g(\mathcal{T}_V JX, JW) = 2g(\mathcal{Q}_V W, JX)$.

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

For the case of Kaehler manifold, we have the following results:

Theorem 3.4. *Let π be an anti-invariant Riemannian submersion from a Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B), then we have*

(a) $g_B((\nabla\pi_*)(W, JV), \pi_*CX) = g_B((\nabla\pi_*)(V, JW), \pi_*CX)$,

(b) $\mathcal{A}_{JW}V - \mathcal{A}_{JV}W \in J(ker\pi_*)$,

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V, W \in \Gamma(ker\pi_*)$.

Proof. For any $X \in \Gamma(ker\pi_*)^\perp$ and $V, W \in \Gamma(ker\pi_*)$, using (2.1), (2.4) and (2.9) we have

$$\begin{aligned} g([V, W], X) &= g(\nabla_V JW, JX) - g(\nabla_W JV, JX) \\ &= -g_B((\nabla\pi_*)(V, JW), \pi_*CX) + g_B((\nabla\pi_*)(W, JV), \pi_*CX) \end{aligned}$$

Since $(ker\pi_*)$ is integrable, we get

$$g_B((\nabla\pi_*)(W, JV), \pi_*CX) = g_B((\nabla\pi_*)(V, JW), \pi_*CX) \tag{3.18}$$

On the other hand using (2.9), we have

$$\begin{aligned} (\nabla\pi_*)(W, JV) - (\nabla\pi_*)(V, JW) &= -\pi_*(\nabla_W JV) + \pi_*(\nabla_V JW) \\ &= \pi_*(\mathcal{A}_{JW}V - \mathcal{A}_{JV}W). \end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we get (*b*).

□

Next, we have

Theorem 3.5. *Let π be an anti-invariant Riemannian submersion from a Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then we have*

(a) $g_B((\nabla\pi_*)(V, JX), \pi_*JW) = g_B((\nabla\pi_*)(W, JX), \pi_*JV)$.

(b) $g(\mathcal{T}_V BX, JW) - g(\mathcal{T}_W BX, JV) = g(\mathcal{A}_{CX}W, JV) - g(\mathcal{A}_{CX}V, JW)$

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

Proof. For any $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$, we have

$$\begin{aligned} 0 &= g([V,W],X) \\ &= -g(\nabla_V JX, JW) + g(\nabla_W JX, JV) \\ &= g_B((\nabla\pi_*)(V,JX), \pi_* JW) - g_B((\nabla\pi_*)(W,JX), \pi_* JV), \end{aligned}$$

which gives (*a*).

On the other hand, using (2.3) and Lemma 2.2, we have

$$\begin{aligned} 0 &= g([V,W],X) \\ &= -g(\nabla_V JX, JW) + g(\nabla_W JX, JV) \\ &= -g(\mathcal{T}_V BX, JW) - g(\mathcal{A}_{CX} V, JW) + g(\mathcal{T}_W BX, JV) + g(\mathcal{A}_{CX} W, JV), \end{aligned}$$

which gives (*b*). □

For Langrangian Riemannian submersion, we have following corollary;

Corollary 3.3. *Let π be Langrangian Riemannian submersion from a Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then we have*

(*a*) $g_B((\nabla\pi_*)(V,JX), \pi_* JW) = g_B((\nabla\pi_*)(W,JX), \pi_* JV)$.

(*b*) $\mathcal{T}_V JW = \mathcal{T}_W JV$,

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

4. Totally Geodesic Foliations

In this section, we study the foliations of the two distributions and obtain the equivalent conditions for totally geodesicness. We have the following characterizations:

Theorem 4.1. *Let π be an anti-invariant Riemannian submersion from nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then the following assertions are equivalent to each other*

(*a*) $(ker\pi_*)^\perp$ *defines a totally geodesic foliation on M.*
(*b*) $g(\mathcal{A}_X BY, JV) = g(CY, J\mathcal{A}_X V) + g(CY, \mathcal{P}_X V) + g(\mathcal{P}_X Y, JV)$.
(*c*) $g_B((\nabla\pi_*)(X,JY), \pi_* JV) = -g(CY, J\mathcal{A}_X V) - g(CY, \mathcal{P}_X V) - g(\mathcal{P}_X Y, JV)$,

for any $X,\ Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$.

Proof. For any $X,\ Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$ using (2.1), (3.2), Lemma 2.2 and Lemma 3.2, we have

$$\begin{aligned} g(\nabla_X Y, V) &= g(\nabla_X BY, JV) + g(\nabla_X CY, JV) - g((\nabla_X J)Y, JV) \\ &= g(\mathcal{A}_X BY, JV) - g(CY, J\mathcal{A}_X V) - g(CY, \mathcal{P}_X V) - g(\mathcal{P}_X Y, JV), \end{aligned}$$

it follows (*a*) ⇔ (*b*).

Now, by using (2.9) we have

$$\begin{aligned} g(\mathcal{A}_X BY, JV) &= g(\nabla_X JY, JV) - g(\nabla_X CY, JV) \\ &= -g_B((\nabla\pi_*)(X, JY), \pi_* JV) + g_B(\nabla^\pi_X \pi_*(JY), \pi_* JV) - g(\nabla_X CY, JV) \\ &= -g_B((\nabla\pi_*)(X, JY), \pi_* JV) + g(\nabla_X CY, JV) - g(\nabla_X CY, JV) \\ &= -g((\nabla\pi_*)(X, JY), \pi_* JV), \end{aligned}$$

which shows $(b) \Leftrightarrow (c)$.
□

For Langrangian Riemannian submersion, we have

Corollary 4.1. *Let π be a Langrangian Riemannian submersion from nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then the following conditions are equivalent to each other*

- (a) $(ker\pi_*)^\perp$ *defines a totally geodesic foliation.*
- (b) $\mathcal{A}_X JY = \mathcal{P}_X Y.$
- (c) $g_B((\nabla\pi_*)(X, JY), \pi_* JV) = g(\mathcal{P}_Y X, JV),$

for any $X,\ Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$.

For the totally geodesicness of the foliations of $(ker\pi_*)$, we have

Theorem 4.2. *Let π be an anti-invariant Riemannian submersion from an almost Hermitian manifold (M, g, J) to a Riemannian manifold (B, g_B). Then, the following assertions are equivalent to each other*

- (a) $(ker\pi_*)$ *defines a totally geodesic foliation on M.*
- (b) $g_B((\nabla\pi_*)(V, JX), \pi_* JW) = g(\mathcal{P}_V W, CX) + g(\mathcal{Q}_V W, BX).$
- (c) $g(\mathcal{T}_V BX + \mathcal{A}_{CX} V, JW) = -g(\mathcal{P}_V W, CX) - g(\mathcal{Q}_V W, BX),$

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

Proof. For any $V, W \in \Gamma(ker\pi_*)$ and $X \in \Gamma(ker\pi_*)^\perp$, using (2.1) we have

$$g(\nabla_V W, X) = -g(JW, \nabla_V JX) - g((\nabla_V J)W, JX).$$

The Riemannian submersion π and (2.9) imply that

$$g(\nabla_V W, X) = g_B((\nabla\pi_*)(V, JX), \pi_* JW) - g(\nabla^\pi_V \pi_*(CX), \pi_* CX) - g(\mathcal{P}_V W, CX) - g(\mathcal{Q}_V W, BX). \tag{4.1}$$

Hence, $(ker\pi_*)$ defines a totally geodesic foliation on M if and only if

$$g_B((\nabla\pi_*)(V, JX), \pi_* JW) = g(\mathcal{P}_V W, CX) + g(\mathcal{Q}_V W, BX), \tag{4.2}$$

which shows that $(a) \Leftrightarrow (b)$.

Using (2.9) we derive

$$g_B(\pi_* JW, (\nabla\pi_*)(V, JX)) = -g_B(\pi_* JW, \pi_* \nabla_V JX).$$

Since, π is a Riemannian submersion by using (3.2) we have

$$\begin{aligned} &= -g(JW, \nabla_V JX) \\ &= -g(JW, \nabla_V BX + [V, CX] + \nabla_{CX} V) \end{aligned}$$

Since $[V, CX] \in \Gamma(ker\pi_*)$, by using Lemma 2.2 we get

$$g_B(\pi_* JW, (\nabla\pi_*)(V, JX) = -g(JW, \mathcal{T}_V BX + \mathcal{A}_{CX} V). \tag{4.3}$$

The result then follows from (4.2) and (4.3). □

For Langrangian Riemannian submersion we have following corollary:

Corollary 4.2. *Let π be a Langrangian Riemannian submersion from an almost Hermitian manifold (M, g, J) to a Riemannian manifold (B, g_B). Then the following conditions are equivalent*

(*a*) *$(ker\pi_*)$ defines a totally geodesic foliation on M.*
(*b*) *$g_B((\nabla\pi_*)(V, JX), \pi_* JW) = g(\mathcal{Q}_V W, JX)$.*
(*c*) *$\mathcal{T}_V JW = \mathcal{Q}_V W$,*

for $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

For a nearly Kaehler manifold M, the above two results can be stated as:

Theorem 4.3. *Let π be an anti-invariant Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then, the following assertions are equivalent to each other*

(*a*) *$(ker\pi_*)$ defines a totally geodesic foliation on M.*
(*b*) *$g_B((\nabla\pi_*)(V, JX), \pi_* JW) = -g(\mathcal{P}_W V, CX) - g(\mathcal{Q}_W V, BX)$.*
(*c*) *$g(\mathcal{T}_V BX + \mathcal{A}_{CX} V, JW) = g(\mathcal{P}_W V, CX) + g(\mathcal{Q}_W V, BX)$,*

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

For Langrangian Riemannian submersion we have

Corollary 4.3. *Let π be a Langrangian Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then the following conditions are equivalent*

(*a*) *$(ker\pi_*)$ defines a totally geodesic foliation on M.*
(*b*) *$g_B((\nabla\pi_*)(V, JX), \pi_* JW) = -g(\mathcal{Q}_W V, JX)$.*
(*c*) *$\mathcal{T}_V JW = -\mathcal{Q}_W V$,*

for $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

As we know that for the case of Kaehler manifold $\mathcal{P} = \mathcal{Q} = 0$, then from the above results we have the result of B. Sahin [10].

Definition 4.1. *[1] A differential map π between two Riemannian manifolds is called totally geodesic if $\nabla\pi_* = 0$.*

For a Langrangian Riemannian submersion, we have the following characterization.

Theorem 4.4. *Let π be a Langrangian Riemannian submersion from an almost Hermitian manifold (M, g, J) to a Riemannian manifold (B, g_B). Then π is totally Geodesic map if and only if*

$$\mathcal{T}_V JW = \mathcal{Q}_V W \qquad \text{and} \qquad \mathcal{A}_X JW = \mathcal{Q}_X W,$$

for any $X \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

Proof. We know that the second fundamental form of a Riemannian submersion satisfies

$$(\nabla\pi_*)(X, Y) = 0,\ \forall\ X, Y \in \Gamma(ker\pi_*)^\perp. \tag{4.4}$$

For $V,\ W \in \Gamma(ker\pi_*)$ using (2.9), we have

$$\begin{aligned} (\nabla\pi_*)(V, W) &= -\pi_*(\nabla_V W) \\ &= \pi_*(J(J\nabla_V W)) \\ &= \pi_*(J\mathcal{T}_V JW - J\mathcal{Q}_V W) \end{aligned} \tag{4.5}$$

On the other hand side, for $W \in \Gamma(ker\pi_*)$ and $X \in \Gamma(ker\pi_*)^\perp$ using (2.9) we obtain

$$\begin{aligned} (\nabla\pi_*)(X, W) &= -\pi_*(\nabla_X W) \\ &= \pi_*(J(J\nabla_X W)) \\ &= \pi_*(J(\nabla_X JW - (\nabla_X J)W)) \\ &= \pi_*(J\mathcal{A}_X JW - J\mathcal{Q}_X W) \end{aligned} \tag{4.6}$$

Since J is non-singular, then the result follows from (4.4), (4.5) and (4.6). □

5. Total Manifold as Product Manifold

In this section, we obtain some decomposition theorems for an anti-invariant Riemannian submersion and Langrangian Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B).

Definition 5.1. *[10] Let g_N be metric be a Riemannian metric tensor on the manifold $N = M \times B$ and assume that the canonical foliations $\mathcal{D}_M$ and $\mathcal{D}_B$ intersect perpendicularly everywhere. Then g is a metric tensor of*

(*i*) *a usual product of Riemannian manifolds if and only if $\mathcal{D}_M$ and $\mathcal{D}_B$ are totally geodesic foliations.*

(*ii*) *a twisted product if and only if $\mathcal{D}_M$ is a totally geodesic foliation and $\mathcal{D}_B$ is a totally umbilical foliation.*

We have the following decomposition theorem for an anti-invariant Riemannian submersion which follows from Theorem 4.1 and Theorem 4.3 in terms of second fundamental form of such submersions.

Theorem 5.1. *Let π be an anti-invariant Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then M is a locally product manifold if and only if*

$$g_B((\nabla\pi_*)(X, JY), \pi_* JV) = -g(CY, J\mathcal{A}_X V) - g(CY, \mathcal{P}_X V) - g(\mathcal{P}_X Y, JV),$$

and

$$g_B((\nabla\pi_*)(V, JX), \pi_* JW) = -g(\mathcal{P}_W V, CX) - g(\mathcal{Q}_W V, BX),$$

for any $X,\ Y \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

From Corollary 4.1 and Corollary 4.3 , we have the following decomposition theorem for Langrangian Riemannian submersion.

Corollary 5.1. *Let π be a Langrangian Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then M is a locally product if and only if*

$$g_B((\nabla\pi_*)(X, JY), \pi_* JV) = g(\mathcal{P}_Y X, JV),$$

and

$$g_B((\nabla\pi_*)(V, JX), \pi_* JW) = -g(\mathcal{Q}_W V, JX),$$

for any $X,\ Y \in \Gamma(ker\pi_*)^\perp$ and $V,\ W \in \Gamma(ker\pi_*)$.

Next, we obtain the decomposition theorem which is related to the notion of twisted product manifold.

Theorem 5.2. *Let π be Langrangian Riemannian submersion from a nearly Kaehler manifold (M, g, J) to a Riemannian manifold (B, g_B). Then M is a twisted product manifold of the form $M_{(ker\pi_*)^\perp} \times_f M_{(ker\pi_*)}$ if and only if*

$$\mathcal{T}_V JX = -g(X, \mathcal{T}_V V)\|V\|^{-2} JV - \mathcal{P}_X V,$$

and

$$\mathcal{A}_X JY = \mathcal{P}_X Y$$

for any $X,\ Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$, where $M_{(ker\pi_*)^\perp}$ and $M_{(ker\pi_*)}$ are integral manifolds of the distributions $(ker\pi_*)^\perp$ and $(ker\pi_*)$ respectively.

Proof. For any $X \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$ using (2.1), (2.2) and Lemma 2.2, we get

$$\begin{aligned} g(\nabla_V W, X) &= -g(\nabla_V X, W) \\ &= -g(J\nabla_V X, JW) \\ &= -g(\nabla_V JX - (\nabla_V J)X, JW) \\ &= -g(\mathcal{T}_V JX - \mathcal{P}_V X, JW). \end{aligned}$$

This implies that $(ker\pi_*)$ is totally umbilical if and only if

$$\mathcal{T}_V JX - \mathcal{P}_V X = -X(\lambda) JV, \tag{5.1}$$

where λ is some function on M.

Now, from (5.1) we have

$$g(-X(\lambda)JV, JV) = g(\mathcal{T}_V JX - \mathcal{P}_V X, JV).$$

$$\begin{aligned} -X(\lambda)\|V\|^2 &= g(\mathcal{T}_V JX, JV) - g(\mathcal{P}_V X, JV) \\ &= -g(JX, \nabla_V JV) - g(\mathcal{P}_V X, JV) \\ &= g(\nabla_V JX, JV) - g(\mathcal{P}_V X, JV) \\ &= g((\nabla_V J)X + J\nabla_V X, JV) - g(\mathcal{P}_V X, JV) \\ &= g(\mathcal{P}_V X, JV) + g(J\nabla_V X, JV) - g(\mathcal{P}_V X, JV) \\ &= g(\mathcal{T}_V X, V) \\ &= -g(X, \mathcal{T}_V V), \end{aligned}$$

which implies that

$$X(\lambda) = g(X, \mathcal{T}_V V)\|V\|^{-2} \tag{5.2}$$

and hence (5.1) and (5.2) gives

$$\mathcal{T}_V JX = -g(X, \mathcal{T}_V V)\|V\|^{-2} JV - \mathcal{P}_X V.$$

The result follows from Corollary 4.1. □

Now, we prove the non-existence of a twisted product manifold of the form $M_{(ker\pi_*)} \times_f M_{(ker\pi_*)^\perp}$ for Langrangian Riemannian submersion. We have

Theorem 5.3. *There do not exist Langragian Riemannian submersion from an Almost Hermitian manifold* (M, g, J) *to a Riemannian manifold* (B, g_B) *such that M is a locally proper twisted product manifold of the form* $M_{(ker\pi_*)} \times_f M_{(ker\pi_*)^\perp}$*.*

Proof. Let us suppose that π be a Langragian Riemannian submersion from an almost Hermitian manifold (M, g, J) to a Riemannian manifold (B, g_B) and M is a locally proper twisted product manifold of the form $M_{(ker\pi_*)} \times_f M_{(ker\pi_*)^\perp}$. Then by the Definition 5.1, $M_{(ker\pi_*)}$ is a totally geodesic foliation and $M_{(ker\pi_*)^\perp}$ is a totally umbilical foliation.

If h is the second fundamental form of $M_{(ker\pi_*)^\perp}$, then we get

$$\begin{aligned} g(\nabla_X Y, V) &= g(\mathcal{H}\nabla_X Y + \mathcal{V}\nabla_X Y, V) \\ &= g(\mathcal{V}\nabla_X Y, V) \\ &= g(h(X, Y), V) \end{aligned}$$

for $X, Y \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(ker\pi_*)$.

Since, $M_{(ker\pi_*)^\perp}$ is totally umbilical foliation then we have

$$g(\nabla_X Y, V) = g(H, V) g(X, Y), \tag{5.3}$$

where H is the mean curvature vector field of $V \in \Gamma(ker\pi_*)$.

On the other hand, from (2.1), (3.6) and Lemma 2.2 we have

$$g(\nabla_X Y, V) = -g(JY, \mathcal{A}_X JV - \mathcal{Q}_X V). \tag{5.4}$$

Thus from (5.3) and (5.4), we have

$$\mathcal{A}_X JV - \mathcal{Q}_X V = -g(H, V)JX$$

$$\begin{aligned} -g(H, V)\|X\|^2 &= g(\mathcal{A}_X JV - \mathcal{Q}_X V, JX) \\ &= g(\nabla_X JV - (\nabla_X J)V, JX) \\ &= -g(V, \nabla_X X) \\ &= -g(V, \mathcal{A}_X X). \end{aligned}$$

From (2.8) we know that $\mathcal{A}_X X = 0$, which implies that

$$g(H, V)\|X\|^2 = 0.$$

Since g is a Riemannian metric and $H \in \Gamma(ker\pi_*)$, we conclude that $H = 0$. It means that $(ker\pi_*)^\perp$ is totally geodesic, so M is usual product of Riemannian manifolds. Thus, it completes the proof. □

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