

ANTI-INVARIANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD WITH VANISHING CONTACT BOCHNER CURVATURE TENSOR

KENTARO YANO

0. Introduction

In 1949, by using a complex coordinate system Bochner [3] (see also Yano and Bochner [23]) introduced, as an analogue of the Weyl conformal curvature tensor in a Riemannian manifold, what we now call the Bochner curvature tensor in a Kaehlerian manifold. In 1967 Tachibana [13] gave a tensor expression of this curvature tensor in a real coordinate system. Since then the tensor has been studied by Chen [5], Ishihara [25], Liu [14], Matsumoto [10], Sato [17], Tachibana [14], Takagi [15], Watanabe [15], Yamaguchi [17], and the present author [5], [19], [20], [21], [22], [25].

Let M^{2m} be a real $2m$ -dimensional Kaehlerian manifold with the almost complex structure F , and M^n an n -dimensional Riemannian manifold isometrically immersed in M^{2m} . If $T_x(M^n) \perp FT_x(M^n)$, where $T_x(M^n)$ denotes the tangent space to M^n at a point x of M^n and is identified with its image under the differential of the immersion, then we call M^n a *totally real* or *anti-invariant* submanifold of M^{2m} . Since the rank of F is $2m$, we have $n \leq 2m - n$, that is, $n \leq m$.

The totally real submanifolds of a Kaehlerian manifold have been studied by Chen [4], Houh [6], Kon [7], [26], [27], Ludden [8], [9], Ogue [4], Okumura [8], [9] and the present author [8], [9], [21], [22], [26], [27].

As a theorem connecting the Weyl conformal curvature tensor and the Bochner curvature tensor, Blair [1] proved

Theorem A. *Let M^{2n} , $n \geq 4$, be a Kaehlerian manifold with vanishing Bochner curvature tensor, and M^n a totally geodesic, totally real submanifold of M^{2n} . Then M^n is conformally flat.*

Generalizing this theorem of Blair, the present author [21] established the following theorems.

Theorem B. *Let M^n , $n \geq 4$, be a totally umbilical, totally real submanifold of a Kaehlerian manifold M^{2m} with vanishing Bochner curvature tensor. Then M^n is conformally flat.*

Theorem C. *Let M^3 be a totally geodesic, totally real submanifold of a*

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Kaehlerian manifold M^{2m} with vanishing Bochner curvature tensor. Then M^3 is conformally flat.

Theorem D. *Let M^n , $n \geq 4$, be a totally real submanifold of a Kaehlerian manifold M^{2n} with vanishing Bochner curvature tensor. If the second fundamental tensors of M^n commute, then M^n is conformally flat.*

The main purpose of the present paper is to obtain theorems, analogous to the above theorems, for anti-invariant submanifolds of a Sasakian manifold with vanishing contact Bochner curvature tensor. For anti-invariant submanifolds of a Sasakian manifold, see Blair and Ogiue [2], Yamaguchi, Kon and Ikawa [16], Yano and Kon [28], [29], and for the contact Bochner curvature tensor see Matsumoto and Chūman [11].

First of all, in § 1 we recall the definition and the fundamental properties of a Sasakian manifold. In § 2 we define a curvature tensor in a Sasakian manifold which is called the contact Bochner curvature tensor and corresponds to the Bochner curvature tensor in a Kaehlerian manifold.

§ 3 is devoted to general discussions on anti-invariant submanifolds of a Sasakian manifold, and § 4 to the study of anti-invariant submanifolds of a Sasakian manifold with vanishing contact Bochner curvature tensor.

In the last two sections (§§ 5 and 6) we study Sasakian manifolds with vanishing contact Bochner curvature tensor regarded as fibred spaces with invariant Riemannian metric (see Yano and Ishihara [24]).

1. Sasakian manifolds

We first of all recall the definition and the fundamental properties of almost contact manifolds for the later use. Let M^{2m+1} be a $(2m+1)$ -dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^\alpha\}$ in which there are given a tensor field φ_λ^μ of type $(1,1)$, a vector field ξ^μ and a 1-form η_λ satisfying

$$(1.1) \quad \varphi_\lambda^\mu \varphi_\mu^\lambda = -\delta_\mu^\lambda + \eta_\mu \xi^\mu, \quad \varphi_\lambda^\mu \xi^\lambda = 0, \quad \eta_\lambda \varphi_\mu^\lambda = 0, \quad \eta_\lambda \xi^\lambda = 1,$$

where and in the sequel the indices $\alpha, \beta, \dots, \kappa, \lambda, \mu, \dots$ run over the range $\{1, 2, \dots, 2m+1\}$. Such a set (φ, ξ, η) consisting of a tensor field φ , a vector field ξ and a 1-form η is called an *almost contact structure*, and a manifold with an almost contact structure an *almost contact manifold* (see Sasaki [12]).

If the Nijenhuis tensor

$$(1.2) \quad N_{\mu\lambda}^\alpha = \varphi_\mu^\alpha \partial_\alpha \varphi_\lambda^\mu - \varphi_\lambda^\alpha \partial_\alpha \varphi_\mu^\mu - (\partial_\mu \varphi_\lambda^\alpha - \partial_\lambda \varphi_\mu^\alpha) \varphi_\alpha^\mu$$

formed with φ_λ^μ satisfies

$$(1.3) \quad N_{\mu\lambda}^\alpha + (\partial_\mu \eta_\lambda - \partial_\lambda \eta_\mu) \xi^\alpha = 0,$$

where $\partial_\mu = \partial/\partial x^\mu$, then the almost contact structure is said to be *normal* and

the manifold is called a *normal almost contact manifold*.

Suppose that in an almost contact manifold there is given a Riemannian metric $g_{\mu\lambda}$ such that

$$(1.4) \quad g_{\tau\beta}\varphi_\mu^\tau\varphi_\lambda^\beta = g_{\mu\lambda} - \eta_\mu\eta_\lambda, \quad \eta_\lambda = g_{\lambda\kappa}\xi^\kappa,$$

then the almost contact structure is said to be *metric*, and the manifold is called an *almost contact metric manifold*. In view of the second equation of (1.4) we shall write ξ_λ instead of η_λ in the sequel. In an almost contact metric manifold, the tensor field $\varphi_{\mu\lambda} = \varphi_\mu^\alpha g_{\alpha\lambda}$ is skew-symmetric.

If an almost contact metric structure satisfies

$$(1.5) \quad \varphi_{\mu\lambda} = \frac{1}{2}(\partial_\mu\xi_\lambda - \partial_\lambda\xi_\mu),$$

then the almost contact metric structure is called a *contact structure*. A manifold with a normal contact structure is called a *Sasakian manifold*.

It is well known that in a Sasakian manifold we have

$$(1.6) \quad \nabla_\lambda\xi^\kappa = \varphi_\lambda^\kappa,$$

$$(1.7) \quad \nabla_\mu\varphi_\lambda^\kappa = -g_{\mu\lambda}\xi^\kappa + \delta_\mu^\kappa\xi_\lambda,$$

where ∇_λ denotes the operator of covariant differentiation with respect to $g_{\mu\lambda}$. (1.6) written as $\nabla_\lambda\xi_\kappa = \varphi_{\lambda\kappa}$ shows that ξ^κ is a Killing vector field.

(1.6), (1.7) and the Ricci identity

$$\nabla_\nu\nabla_\mu\xi^\kappa - \nabla_\mu\nabla_\nu\xi^\kappa = K_{\nu\mu\lambda}^\kappa\xi^\lambda,$$

where $K_{\nu\mu\lambda}^\kappa$ is the curvature tensor, give

$$(1.8) \quad K_{\nu\mu\lambda}^\kappa\xi^\lambda = \delta_\nu^\kappa\xi_\mu - \delta_\mu^\kappa\xi_\nu,$$

or

$$(1.9) \quad K_{\nu\mu\lambda}^\kappa\xi_\kappa = \xi_\nu g_{\mu\lambda} - \xi_\mu g_{\nu\lambda}.$$

From (1.9) by contraction we have

$$(1.10) \quad K_{\mu\lambda}^\kappa\xi^\lambda = 2m\xi_\mu,$$

where $K_{\mu\lambda} = K_{\alpha\mu\lambda}^\alpha$ is the Ricci tensor.

(1.6), (1.7) and the Ricci identity

$$\nabla_\nu\nabla_\mu\varphi_\lambda^\kappa - \nabla_\mu\nabla_\nu\varphi_\lambda^\kappa = K_{\nu\mu\alpha}^\kappa\varphi_\lambda^\alpha - K_{\nu\mu\lambda}^\alpha\varphi_\alpha^\kappa$$

imply

$$(1.11) \quad K_{\nu\mu\alpha}^\kappa\varphi_\lambda^\alpha - K_{\nu\mu\lambda}^\alpha\varphi_\alpha^\kappa = -\varphi_\nu^\kappa g_{\mu\lambda} + \varphi_\mu^\kappa g_{\nu\lambda} - \delta_\nu^\kappa\varphi_{\mu\lambda} + \delta_\mu^\kappa\varphi_{\nu\lambda},$$

from which, by contraction, it follows that

$$(1.12) \quad K_{\mu\alpha}\varphi_{\lambda}^{\alpha} + K_{\beta\mu\lambda\alpha}\varphi^{\beta\alpha} = -(2m - 1)\varphi_{\mu\lambda},$$

where $\varphi^{\beta\alpha} = g^{\beta\lambda}\varphi_{\lambda}^{\alpha}$, $g^{\beta\lambda}$ being contravariant components of the metric tensor. Since $K_{\beta\mu\lambda\alpha}\varphi^{\beta\alpha}$ is skew-symmetric in μ and λ , we have from (1.12)

$$(1.13) \quad K_{\mu\alpha}\varphi_{\lambda}^{\alpha} + K_{\lambda\alpha}\varphi_{\mu}^{\alpha} = 0.$$

From (1.12) we also find

$$(1.14) \quad K_{\beta\alpha\mu\lambda}\varphi^{\beta\alpha} = 2K_{\mu\alpha}\varphi_{\lambda}^{\alpha} + 2(2m - 1)\varphi_{\mu\lambda}.$$

2. Contact Bochner curvature tensor

As an analogue of the Bochner curvature tensor in a Kaehlerian manifold, we define the contact Bochner curvature tensor in a Sasakian manifold by

$$(2.1) \quad \begin{aligned} B_{\nu\mu\lambda}^{\epsilon} &= K_{\nu\mu\lambda}^{\epsilon} + (\delta_{\nu}^{\epsilon} - \xi_{\nu}\xi^{\epsilon})L_{\mu\lambda} - (\delta_{\mu}^{\epsilon} - \xi_{\mu}\xi^{\epsilon})L_{\nu\lambda} + L_{\nu}^{\epsilon}(g_{\mu\lambda} - \xi_{\mu}\xi_{\lambda}) \\ &\quad - L_{\mu}^{\epsilon}(g_{\nu\lambda} - \xi_{\nu}\xi_{\lambda}) + \varphi_{\nu}^{\epsilon}M_{\mu\lambda} - \varphi_{\mu}^{\epsilon}M_{\nu\lambda} + M_{\nu}^{\epsilon}\varphi_{\mu\lambda} - M_{\mu}^{\epsilon}\varphi_{\nu\lambda} \\ &\quad - 2(\varphi_{\nu\mu}M_{\lambda}^{\epsilon} + M_{\nu\mu}\varphi_{\lambda}^{\epsilon}) + (\varphi_{\nu}^{\epsilon}\varphi_{\mu\lambda} - \varphi_{\mu}^{\epsilon}\varphi_{\nu\lambda} - 2\varphi_{\nu\mu}\varphi_{\lambda}^{\epsilon}), \end{aligned}$$

where

$$(2.2) \quad \begin{aligned} L_{\mu\lambda} &= \frac{1}{2(m + 2)}[-K_{\mu\lambda} - (L + 3)g_{\mu\lambda} + (L - 1)\xi_{\mu}\xi_{\lambda}], \\ L_{\mu}^{\epsilon} &= L_{\mu\alpha}g^{\alpha\epsilon}, \end{aligned}$$

$$(2.3) \quad L = g^{\mu\lambda}L_{\mu\lambda},$$

$$(2.4) \quad M_{\mu\lambda} = -L_{\mu\alpha}\varphi_{\lambda}^{\alpha}, \quad M_{\nu}^{\epsilon} = M_{\nu\alpha}g^{\alpha\epsilon}.$$

From (2.2) and (2.3) it follows that

$$(2.5) \quad L = -\frac{K + 2(3m + 2)}{4(m + 1)},$$

where K is the scalar curvature of the manifold.

Using (1.10) we have, from (2.2),

$$(2.6) \quad L_{\mu\lambda}\xi^{\lambda} = -\xi_{\mu},$$

which, together with the first equation of (2.4), yields

$$(2.7) \quad M_{\mu\alpha}\varphi_{\lambda}^{\alpha} = L_{\mu\lambda} + \xi_{\mu}\xi_{\lambda}.$$

We can easily verify that the contact Bochner curvature tensor satisfies the following identities :

$$(2.8) \quad B_{\nu\mu\lambda}{}^\epsilon = -B_{\mu\nu\lambda}{}^\epsilon, \quad B_{\nu\mu\lambda}{}^\epsilon + B_{\mu\lambda\nu}{}^\epsilon + B_{\lambda\nu\mu}{}^\epsilon = 0, \quad B_{\alpha\mu\lambda}{}^\alpha = 0,$$

$$(2.9) \quad B_{\nu\mu\lambda\kappa} = -B_{\nu\mu\kappa\lambda}, \quad B_{\nu\mu\lambda\kappa} = B_{\lambda\kappa\nu\mu},$$

where $B_{\nu\mu\lambda\kappa} = B_{\nu\mu\lambda}{}^\alpha g_{\alpha\kappa}$ and

$$(2.10) \quad B_{\nu\mu\lambda}{}^\kappa \xi_\kappa = 0, \quad B_{\nu\mu\alpha}{}^\kappa \varphi_\lambda{}^\alpha = B_{\nu\mu\lambda}{}^\alpha \varphi_\alpha{}^\kappa, \quad B_{\nu\mu\lambda}{}^\kappa \varphi^{\nu\mu} = 0.$$

3. Anti-invariant submanifolds of a Sasakian manifold

We consider an n -dimensional Riemannian manifold M^n , $n > 1$, covered by a system of coordinate neighborhoods $\{V; y^\mu\}$ and isometrically immersed in a Sasakian manifold M^{2m+1} , and denote the immersion by

$$(3.1) \quad x^\epsilon = x^\epsilon(y^\mu)$$

where and in the sequel the indices h, i, j, \dots run over the range $\{1', 2', \dots, n'\}$. We put

$$(3.2) \quad B_i{}^\epsilon = \partial_i x^\epsilon \quad (\partial_i = \partial/\partial y^i),$$

and denote $2m + 1 - n$ mutually orthogonal unit vectors normal to M^n by $C_y{}^\epsilon$, where and in the sequel the indices x, y, z run over the range $\{(n + 1)', \dots, (2m + 1)'\}$.

Then the metric tensor g_{ji} of M^n and that of the normal bundle are respectively given by

$$(3.3) \quad g_{ji} = g_{\mu i} B_{j\mu}{}^\lambda,$$

$$(3.4) \quad g_{zy} = g_{\mu z} C_{y\mu}{}^\lambda,$$

where $B_{ji}{}^\lambda = B_j{}^\mu B_i{}^\lambda$ and $C_{zy}{}^\lambda = C_z{}^\mu C_y{}^\lambda$.

If the transform by $\varphi_\lambda{}^\epsilon$ of any vector tangent to M^n is orthogonal to M^n , we say that the submanifold M^n is *anti-invariant* in M^{2m+1} . Since the rank of $\varphi_\lambda{}^\epsilon$ is $2m$, we have $n - 1 \leq 2m + 1 - n$, that is, $n \leq m + 1$.

For an anti-invariant submanifold M^n in M^{2m+1} , we have equations of the form

$$(3.5) \quad \varphi_\lambda{}^\epsilon B_i{}^\lambda = -f_i{}^x C_x{}^\epsilon,$$

$$(3.6) \quad \varphi_\lambda{}^\epsilon C_y{}^\lambda = f_y{}^i B_i{}^\epsilon + f_y{}^x C_x{}^\epsilon,$$

$$(3.7) \quad \xi^\epsilon = \xi^i B_i{}^\epsilon + \xi^x C_x{}^\epsilon.$$

Using $\varphi_{\mu\lambda} = -\varphi_{\lambda\mu}$ we have, from (3.5) and (3.6),

$$(3.8) \quad f_{iy} = f_{iy}^z g_{zy},$$

where $f_{iy} = f_i^z g_{zy}$ and $f_{yz} = f_y^z g_{zy}$ and

$$(3.9) \quad f_{yx} = -f_{xy},$$

where $f_{yx} = f_y^z g_{zx}$.

Applying φ to (3.5), (3.6) and (3.7) and using (1.1), (3.8), (3.9) we find

$$(3.10) \quad \begin{aligned} (i) \quad & f_i^y f_y^h = \delta_i^h - \xi_i \xi^h, \\ (ii) \quad & f_i^y f_y^x = -\xi_i \xi^x, \\ (iii) \quad & f_y^z f_z^h = \xi_y \xi^h, \\ (iv) \quad & f_y^z f_z^x = -\delta_y^x + \xi_y \xi^x + f_y^i f_i^x, \\ (v) \quad & f_x^i \xi^x = 0, \\ (vi) \quad & f_i^x \xi^i = f_y^x \xi^y, \\ (vii) \quad & \xi_i \xi^i + \xi_y \xi^y = 1, \end{aligned}$$

where $\xi_i = g_{ih} \xi^h$ and $\xi_y = g_{yh} \xi^h$, (vii) being a consequence of $\xi_i \xi^i = 1$.

Differentiating (3.5), (3.6) and (3.7) covariantly over M^n and using (1.6), (1.7), (3.10), equations of Gauss

$$(3.11) \quad \nabla_j B_i^x = h_{ji}^x C_x^x$$

and those of Weingarten

$$(3.12) \quad \nabla_j C_x^x = -h_{ji}^i B_i^x,$$

where ∇_j denotes the operator of covariant differentiation over M^n , and h_{ji}^x and $h_{ji}^i = h_{ji}^z g^{ti} g_{zy}$ are the second fundamental tensors of M^n with respect to the normals C_x^x , we find

$$(3.13) \quad \begin{aligned} (i) \quad & -g_{ji} \xi^h + \delta_j^h \xi_i = -h_{ji}^x f_x^h + h_{jx}^h f_i^x, \\ (ii) \quad & \nabla_j f_i^x = g_{ji} \xi^x - h_{ji}^y f_y^x, \\ (iii) \quad & \nabla_j f_y^h = \delta_j^h \xi_y + h_{jx}^h f_y^x, \\ (iv) \quad & \nabla_j f_y^x = -h_{ji}^x f_y^i + h_{jy}^i f_i^x, \\ (v) \quad & \nabla_j \xi^h = h_{jy}^h \xi^y, \\ (vi) \quad & \nabla_j \xi^x = -f_j^x - h_{ji}^x \xi^i. \end{aligned}$$

I. The case in which ξ^x is tangent to M^n . Suppose that $n = m + 1$. Then the codimension of M^n is $2m + 1 - n = n - 1$, and consequently $[f_y^h, \xi^h]$ and $\begin{bmatrix} f_i^y \\ \xi_i \end{bmatrix}$ are both square matrices and satisfy

$$[f_y^h, \xi^h] \begin{bmatrix} f_i^y \\ \xi_i \end{bmatrix} = \text{unit matrix}$$

because of (3.10) (i). Thus we have

$$\begin{bmatrix} f_i^x \\ \xi_i \end{bmatrix} [f_y^i, \xi^i] = \text{unit matrix},$$

from which it follows that

$$(3.14) \quad f_i^x f_y^i = \delta_y^i, \quad f_i^x \xi^i = 0, \quad \xi_i f_y^i = 0, \quad \xi_i \xi^i = 1.$$

By remembering that $\xi_i \xi^i + \xi_x \xi^x = 1$, we further find $\xi^x = 0$ and hence ξ^x is tangent to M^n .

In general suppose that ξ^x is tangent to M^n , that is, $\xi^x = 0$. Then (3.10) becomes

$$(3.15) \quad \begin{aligned} (\text{i}) \quad & f_i^y f_y^h = \delta_i^h - \xi_i \xi^h, \\ (\text{ii}) \quad & f_i^y f_y^x = 0, \\ (\text{iii}) \quad & f_y^z f_z^h = 0, \\ (\text{iv}) \quad & f_y^z f_z^x = -\delta_y^x + f_y^i f_i^x, \\ (\text{v}) \quad & f_i^x \xi^i = 0, \\ (\text{vi}) \quad & \xi_i \xi^i = 1. \end{aligned}$$

From (3.15)(iii) and (iv) we see that f_y^x defines a so-called *f-structure* in the normal bundle (see Yano [18]). In this case (3.13) becomes

$$(3.16) \quad \begin{aligned} (\text{i}) \quad & -g_{ji} \xi^h + \delta_j^h \xi_i = -h_{ji}^x f_x^h + h_j^h x f_i^x, \\ (\text{ii}) \quad & \nabla_j f_i^x = -h_{ji}^y f_y^x, \\ (\text{iii}) \quad & \nabla_j f_y^h = h_j^h x f_y^x, \\ (\text{iv}) \quad & \nabla_j f_y^x = -h_{ji}^x f_y^i + h_j^i y f_i^x, \\ (\text{v}) \quad & \nabla_j \xi^h = 0, \\ (\text{vi}) \quad & h_{ji}^x \xi^i + f_j^x = 0. \end{aligned}$$

(3.16)(v) shows that whenever the vector field ξ^x is tangent to an anti-invariant submanifold of a Sasakian manifold, it is parallel over the submanifold.

(3.16)(i) shows that an anti-invariant submanifold tangent to ξ^x cannot be totally umbilical or totally contact umbilical. For, if h_{ji}^x is of the form $(\alpha g_{ji} + \beta \xi_j \xi_i)h^x$, then from (3.16)(i) we have

$$-g_{ji} \xi^h + \delta_j^h \xi_i = -(\alpha g_{ji} + \beta \xi_j \xi_i)h^x f_x^h + (\alpha \delta_j^h + \beta \xi_j \xi^h)h_x f_i^x,$$

from which, by contracting with respect to h and j and using (3.15)(v) we obtain

$$(n-1)\xi_i = (n-1)\alpha h_x f_i^x + \beta h_x f_i^x ,$$

and consequently transvecting with ξ^i and using (3.15)(v) give $(n-1)\xi_i \xi^i = 0$, which is a contradiction for $n > 1$.

We now come back to the case $n = m + 1$. In this case, from the first equation of (3.14) and (3.15)(iv), we have $f_y^z f_z^x = 0$ or $f_{yx} f^{yx} = 0$ because $f_{yx} = f_y^z g_{zx}$ is skew-symmetric and $f_y^x = 0$. Thus (3.16)(ii) becomes

$$(3.17) \quad \nabla_j f_i^x = 0 ,$$

from which, using the Ricci identity we obtain

$$(3.18) \quad K_{kji}^h f_h^x - K_{kjy}^x f_i^y = 0 ,$$

where K_{kji}^h (respectively, K_{kjy}^x) is the curvature tensor of M^n (respectively, the normal bundle of M^n).

From (3.18) we have, taking account of the first equation of (3.14) and (3.15)(i),

$$(3.19) \quad K_{kjy}^x f_i^y f_x^h = K_{kji}^h ,$$

$$(3.20) \quad K_{kji}^h f_y^i f_h^x = K_{kjy}^x ,$$

because of $K_{kji}^h \xi^i = 0$ derived from (3.16)(v). (3.19) and (3.20) show that $K_{kji}^h = 0$ and $K_{kjy}^x = 0$ are equivalent.

II. The case in which ξ^x is normal to M^n . Now suppose that ξ^x is normal to M^n , that is, $\xi^h = 0$. Then (3.10) becomes

$$(3.21) \quad \begin{aligned} (\text{i}) \quad & f_i^y f_y^h = \delta_i^h , \\ (\text{ii}) \quad & f_i^y f_y^x = 0 , \\ (\text{iii}) \quad & f_y^z f_z^h = 0 , \\ (\text{iv}) \quad & f_y^z f_z^x = -\delta_y^x + \xi_y \xi^x + f_y^i f_i^x , \\ (\text{v}) \quad & f_x^i \xi^x = 0 , \\ (\text{vi}) \quad & f_y^x \xi^y = 0 , \\ (\text{vii}) \quad & \xi_y \xi^y = 1 . \end{aligned}$$

(3.21) (iii), (iv) and (vi) show that f_y^x defines an f -structure in the normal bundle. In this case, (3.13) becomes

$$(3.22) \quad \begin{aligned} (\text{i}) \quad & -h_{ji}^x f_x^h + h_j^h x f_i^x = 0 , \\ (\text{ii}) \quad & \nabla_j f_i^x = g_{ji} \xi^x - h_{ji}^y f_y^x , \\ (\text{iii}) \quad & \nabla_j f_y^h = \delta_j^h \xi_y + h_j^h x f_y^x , \\ (\text{iv}) \quad & \nabla_j f_y^x = -h_{ji}^x f_y^i + h_j^i y f_i^x , \end{aligned}$$

- (v) $h_j^h{}_y \xi^y = 0$,
(vi) $\nabla_j \xi^x = -f_j^x$.

From (3.21)(i) it follows that $f_{iy} f^{yi} = n$, and consequently by (3.21)(iv) and (vii) we find

$$-f_{zy} f^{zy} = -(2m + 1 - n) + 1 + n = -2(m - n).$$

Thus, if $n = m$, then we have $f_y^x = 0$, and (3.21) and (3.22) become respectively

- (3.23) (i) $f_i^y f_y^h = \delta_i^h$,
(ii) $f_i^x f_y^i = \delta_y^x - \xi_y \xi^x$,
(iii) $f_x^h \xi^x = 0$,
(iv) $\xi_x \xi^x = 1$;

(3.24) (i) $-h_{ji}^x f_x^h + h_j^h{}_x f_i^x = 0$,
(ii) $\nabla_j f_i^x = g_{ji} \xi^x$,
(iii) $\nabla_j f_y^h = \delta_j^h \xi_y$,
(iv) $-h_{ji}^x f_y^i + h_j^i{}_y f_i^x = 0$,
(v) $h_j^h{}_y \xi^y = 0$,
(vi) $\nabla_i \xi^x = -f_i^x$.

Suppose that M^n is totally umbilical, and put $h_{ji}^x = g_{ji} h^x$. Then from (3.24)(i) we have

$$-g_{ji} h^x f_x^h + \delta_j^h h_x f_i^x = 0,$$

which implies $h^x f_x^h = 0$ for $n > 1$. From (3.24)(iv) it follows that

$$-h^x f_{yj} + h_y f_j^x = 0,$$

from which, by transvecting with h^y and using $f_{yy} h^y = 0$ we have $h_y h^y f_j^x = 0$, and consequently $h_y h^y = 0$ and hence $h_y = 0$. Thus M^n must be totally geodesic.

By (3.24)(ii) and (vi), we find

$$\nabla_j \nabla_i \xi^x = -g_{ji} \xi^x,$$

from which, using the Ricci identity we obtain

$$K_{kjy}{}^x \xi^y = 0.$$

On the other hand, from (3.24)(ii) and (vi), we have, using the Ricci identity,

$$-K_{kji}^h f_h^x + K_{kjy}^x f_i^y = -f_k^x g_{ji} + f_j^x g_{ki},$$

which, together with (3.23)(i), implies that

$$(3.25) \quad K_{kji}^h = K_{kjy}^x f_i^y f_x^h + \delta_k^h g_{ji} - \delta_j^h g_{ki}$$

and that, in consequence of $K_{kjy}^x \xi^y = 0$ and (3.23)(ii),

$$(3.26) \quad K_{kjy}^x = K_{kji}^h f_j^i f_h^x + f_{yk} f_j^x - f_{yj} f_k^x.$$

(3.25) and (3.26) show that M^n is of constant curvature 1 if and only if the connection induced in the normal bundle is of zero curvature.

4. Anti-invariant submanifolds of a Sasakian manifold with vanishing contact Bochner curvature tensor

We first of all remember that the equations of Gauss, Codazzi and Ricci are respectively

$$(4.1) \quad K_{kji}^h = K_{\mu\lambda\varepsilon} B_{kji}^{\mu\lambda\varepsilon} + h_{khx} h_{ji}^x - h_{jhx} h_{ki}^x,$$

$$(4.2) \quad 0 = K_{\mu\lambda\varepsilon} B_{kji}^{\mu\lambda\varepsilon} C_y^\varepsilon - (\nabla_k h_{jiy} - \nabla_j h_{kit}),$$

$$(4.3) \quad K_{kjy}^x = K_{\mu\lambda\varepsilon} B_{kj}^{\mu\varepsilon} C_{yx}^\varepsilon - (h_k^t y h_{jtx} - h_j^t y h_{ktx}),$$

where $K_{\mu\lambda\varepsilon}$, K_{kji}^h and K_{kjy}^x are the covariant components of the curvature tensors of M^{2m+1} , M^n and the normal bundle respectively, $B_{kji}^{\mu\lambda\varepsilon} = B_k^\mu B_j^\lambda B_i^\varepsilon B_h^\varepsilon$ and $B_{kj}^{\mu\varepsilon} = B_k^\mu B_j^\varepsilon B_i^\lambda$.

We assume that the contact Bochner curvature tensor of M^{2m+1} vanishes identically. Then from (2.1) we have

$$(4.4) \quad \begin{aligned} & K_{\mu\lambda\varepsilon} + (g_{\nu\varepsilon} - \xi_\nu \xi_\varepsilon) L_{\mu\lambda} - (g_{\mu\varepsilon} - \xi_\mu \xi_\varepsilon) L_{\nu\lambda} + L_{\nu\varepsilon} (g_{\mu\lambda} - \xi_\mu \xi_\lambda) \\ & - L_{\mu\varepsilon} (g_{\nu\lambda} - \xi_\nu \xi_\lambda) + \varphi_{\nu\varepsilon} M_{\mu\lambda} - \varphi_{\mu\varepsilon} M_{\nu\lambda} + M_{\nu\mu} \varphi_{\lambda\varepsilon} - M_{\mu\varepsilon} \varphi_{\nu\lambda} \\ & - 2(\varphi_{\nu\mu} M_{\lambda\varepsilon} + M_{\nu\mu} \varphi_{\lambda\varepsilon}) + (\varphi_{\nu\mu} \varphi_{\lambda\varepsilon} - \varphi_{\mu\varepsilon} \varphi_{\nu\lambda} - 2\varphi_{\nu\mu} \varphi_{\lambda\varepsilon}) = 0, \end{aligned}$$

from which, by using $g_{\mu\varepsilon} B_{ji}^{\mu\varepsilon} = g_{ji}$, $\varphi_{\mu\varepsilon} B_{ji}^{\mu\varepsilon} = 0$, $\varphi_{\mu\varepsilon} B_j^\mu C_y^\lambda = -f_{jy}$, $\varphi_{\mu\varepsilon} C_{yx}^\varepsilon = f_{yx}$, $\xi_\nu B_k^\nu = \xi_k$ and $\xi_\nu C_y^\nu = \xi_y$, we find

$$(4.5) \quad \begin{aligned} & K_{\mu\lambda\varepsilon} B_{kji}^{\mu\lambda\varepsilon} + (g_{kh} - \xi_k \xi_h) L_{ji} - (g_{jh} - \xi_j \xi_h) L_{ki} \\ & + L_{kh} (g_{ji} - \xi_j \xi_i) - L_{jh} (g_{ki} - \xi_k \xi_i) = 0, \end{aligned}$$

$$(4.6) \quad \begin{aligned} & K_{\mu\lambda\varepsilon} B_{kji}^{\mu\lambda\varepsilon} C_y^\varepsilon - \xi_k \xi_y L_{ji} + \xi_j \xi_y L_{ki} + L_{ky} (g_{ji} - \xi_j \xi_i) \\ & - L_{jy} (g_{ki} - \xi_k \xi_i) - f_{ky} M_{ji} + f_{jy} M_{ki} + 2M_{kj} f_{iy} = 0, \end{aligned}$$

$$(4.7) \quad \begin{aligned} & K_{\mu\lambda\varepsilon} B_{kj}^{\mu\varepsilon} C_{yx}^\varepsilon - \xi_k \xi_x L_{jy} + \xi_j \xi_x L_{ky} - L_{kx} \xi_j \xi_y + L_{jx} \xi_k \xi_y - f_{kx} M_{jy} \\ & + f_{jx} M_{ky} - M_{kx} f_{jy} + M_{jx} f_{ky} - 2M_{kj} f_{yx} + (f_{kx} f_{jy} - f_{jx} f_{ky}) = 0, \end{aligned}$$

where

$$(4.8) \quad \begin{aligned} L_{ji} &= L_{\mu\lambda}B_{ji}^{\mu\lambda}, & L_{ky} &= L_{\mu\lambda}B_k{}^\mu C_y{}^\lambda, \\ M_{ji} &= M_{\mu\lambda}B_{ji}^{\mu\lambda}, & M_{ky} &= M_{\mu\lambda}B_k{}^\mu C_y{}^\lambda. \end{aligned}$$

Since $M_{\mu\lambda} = -L_{\mu\alpha}\varphi_\lambda{}^\alpha$, we have

$$M_{ji} = -L_{\mu\alpha}\varphi_\lambda{}^\alpha B_{ji}^{\mu\lambda} = L_{\mu\alpha}B_j{}^\mu f_i{}^x C_x{}^\alpha,$$

that is,

$$(4.9) \quad M_{ji} = L_{jx}f_i{}^x,$$

and also

$$M_{ky} = -L_{\mu\alpha}\varphi_\lambda{}^\alpha B_k{}^\mu C_y{}^\lambda = -L_{\mu\alpha}B_k{}^\mu (f_y{}^t B_t{}^\alpha + f_y{}^x C_x{}^\alpha),$$

that is,

$$(4.10) \quad M_{ky} = -L_{kt}f_y{}^t - L_{kx}f_y{}^x.$$

Thus (4.1), (4.2) and (4.3) can be written respectively as

$$(4.11) \quad \begin{aligned} K_{kjih} + (g_{kh} - \xi_k\xi_h)L_{ji} - (g_{jh} - \xi_j\xi_h)L_{ki} + L_{kh}(g_{ji} - \xi_j\xi_i) \\ - L_{jh}(g_{ki} - \xi_k\xi_i) - (h_{khx}h_{ji}{}^x - h_{jhx}h_{ki}{}^x) = 0, \end{aligned}$$

$$(4.12) \quad \begin{aligned} (\xi_k L_{ji} - \xi_j L_{ki})\xi_y - L_{ky}(g_{ji} - \xi_j\xi_i) + L_{jy}(g_{ki} - \xi_k\xi_i) \\ + f_{ky}M_{ji} - f_{jy}M_{ki} - 2M_{kj}f_{iy} - (\nabla_k h_{jty} - \nabla_j h_{kit}) = 0, \end{aligned}$$

$$(4.13) \quad \begin{aligned} K_{kjyx} - (\xi_k L_{jy} - \xi_j L_{ky})\xi_x - (L_{kx}\xi_j - L_{jx}\xi_k)\xi_y \\ + M_{ky}f_{jx} - M_{jy}f_{kx} + f_{ky}M_{jx} - f_{jy}M_{kx} - 2M_{kj}f_{yx} \\ + (f_{kx}f_{jy} - f_{jx}f_{ky}) + (h_k{}^t{}_y h_{jtx} - h_j{}^t{}_y h_{ktx}) = 0. \end{aligned}$$

I. The case in which the vector field ξ^* is tangent to M^n . We now assume that $n = m + 1$. Then the vector field ξ^* is tangent to M^n and $f_y{}^x = 0$. Thus (4.13) becomes

$$\begin{aligned} K_{kjyx} - f_{kx}M_{jy} + f_{jx}M_{ky} - M_{kx}f_{jy} + M_{jx}f_{ky} \\ + (f_{kx}f_{jy} - f_{jx}f_{ky}) + (h_k{}^t{}_y h_{jtx} - h_j{}^t{}_y h_{ktx}) = 0, \end{aligned}$$

from which, by transvecting with $f_i{}^y f_h{}^x$ and using $f_{jx}f_i{}^x = g_{ji} - \xi_j\xi_i$ derived from (3.15)(i), we find

$$(4.14) \quad \begin{aligned} K_{kjyx}f_i{}^y f_h{}^x - (g_{kh} - \xi_k\xi_h)M_{jy}f_i{}^y + (g_{jh} - \xi_j\xi_h)M_{ky}f_i{}^y \\ - M_{kx}f_h{}^x(g_{ji} - \xi_j\xi_i) + M_{jx}f_h{}^x(g_{ki} - \xi_k\xi_i) \\ + (g_{kh} - \xi_k\xi_h)(g_{ji} - \xi_j\xi_i) - (g_{jh} - \xi_j\xi_h)(g_{ki} - \xi_k\xi_i) \\ + (h_k{}^t{}_y h_{jtx} - h_j{}^t{}_y h_{ktx})f_i{}^y f_h{}^x = 0. \end{aligned}$$

We now assume that the second fundamental tensors are commutative. Then from (3.19) and (4.14) we have

$$(4.15) \quad \begin{aligned} & K_{kjih} + (g_{kh} - \xi_k \xi_h)N_{ji} - (g_{jh} - \xi_j \xi_h)N_{ki} \\ & + N_{kh}(g_{ji} - \xi_j \xi_i) - N_{jh}(g_{ki} - \xi_k \xi_i) \\ & + (g_{kh} - \xi_k \xi_h)(g_{ji} - \xi_j \xi_i) - (g_{jh} - \xi_j \xi_h)(g_{ki} - \xi_k \xi_i) = 0, \end{aligned}$$

where $N_{ji} = -M_{jy}f_i^y$.

Now since the vector field ξ^h is parallel, the Riemannian manifold M^n is locally a product of M^{n-1} and M^1 generated by ξ^h , and M^{n-1} is totally geodesic in M^n . We represent M^{n-1} in M^n by parametric equations $y^h = y^h(z^a)$ ($a, b, c, d, \dots = 1'', 2'', \dots, (n-1)''$), and put $B_b^h = \partial y^h / \partial z^b$. Then we have $\xi_t B_b^i = 0$, and the curvature tensor K_{dcba} of M^{n-1} is given by

$$(4.16) \quad K_{dcba} = K_{kjih} B_{dcba}^{kjih},$$

where $B_{dcba}^{kjih} = B_d^k B_c^j B_b^i B_a^h$. Thus transvecting (4.15) with B_{dcba}^{kjih} , we obtain

$$(4.17) \quad K_{dcba} + g_{da}C_{cb} - g_{ca}C_{ab} + C_{da}g_{cb} - C_{ca}g_{ab} = 0,$$

where $g_{cb} = g_{ji}B_c^j B_b^i$ is the metric tensor of M^{n-1} and

$$C_{cb} = N_{ji}B_c^j B_b^i + \frac{1}{2}g_{cb}.$$

(4.17) shows that the Weyl conformal curvature tensor of M^{n-1} vanishes, and M^{n-1} is conformally flat if $n-1 \geq 4$. Thus we have

Theorem 4.1. *Let M^n , $n \geq 5$, be an anti-invariant submanifold of a Sasakian manifold M^{2n-1} with vanishing contact Bochner curvature tensor. If the second fundamental tensors of M^n commute, then M^n is locally a product of a conformally flat Riemannian space and a 1-dimensional space.*

II. The case in which the vector field ξ^c is normal to M^n . We now consider the case in which the vector field ξ^c is normal to the anti-invariant submanifold M^n , so that $\xi^h = 0$. Then from (4.11) we obtain

$$(4.18) \quad \begin{aligned} & K_{kjih} + g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji} - L_{jh}g_{ki} \\ & - (h_{khx}h_{ji}^x - h_{jhx}h_{ki}^x) = 0. \end{aligned}$$

If M^n is umbilical, that is, if $h_{jix} = g_{ji}h_x$, then we can write (4.18) in the form

$$(4.19) \quad \begin{aligned} & K_{kjih} + g_{kh}(L_{ji} - \frac{1}{2}h_xh^xg_{ji}) - g_{jh}(L_{ki} - \frac{1}{2}h_xh^xg_{ki}) \\ & + (L_{kh} - \frac{1}{2}h_xh^xg_{kh})g_{ji} - (L_{jh} - \frac{1}{2}h_xh^xg_{jh})g_{ki} = 0, \end{aligned}$$

which shows that the Weyl conformal curvature tensor of M^n vanishes. Thus we have

Theorem 4.2. *Let M^n , $n \geq 4$, be a totally umbilical anti-invariant submanifold normal to the structure vector field ξ^* of a Sasakian manifold M^{2m+1} with vanishing contact Bochner curvature tensor. Then M^n is conformally flat.*

Next from (4.13) we obtain

$$(4.20) \quad \begin{aligned} K_{k j y x} + M_{k y} f_{j x} - M_{j y} f_{k x} + f_{k y} M_{j x} - f_{j y} M_{k x} + 2M_{k j} f_{y x} \\ + (f_{k x} f_{j y} - f_{j x} f_{k y}) + (h_k{}^t{}_y h_{j t x} - h_j{}^t{}_y h_{k t x}) = 0. \end{aligned}$$

If $n = m$, which implies that $f_y{}^x = 0$, and the second fundamental tensors of M^n commute, then from (4.20) we have

$$(4.21) \quad \begin{aligned} K_{k j y x} - f_{k x} M_{j y} + f_{j x} M_{k y} - M_{k x} f_{j y} + M_{j x} f_{k y} \\ + (f_{k x} f_{j y} - f_{j x} f_{k y}) = 0, \end{aligned}$$

from which, by transvecting with $f_t{}^y f_h{}^x$ and using (3.23)(i), we find

$$(4.22) \quad \begin{aligned} K_{k j y x} f_t{}^y f_h{}^x - g_{k h} M_{j y} f_t{}^y + g_{j h} M_{k y} f_t{}^y - M_{k y} f_h{}^y g_{j t} + M_{j y} f_h{}^y g_{k t} \\ + (g_{k h} g_{j t} - g_{j h} g_{k t}) = 0. \end{aligned}$$

Substituting (4.22) in (3.25) yields

$$(4.23) \quad K_{k j t h} - g_{k h} M_{j y} f_t{}^y + g_{j h} M_{k y} f_t{}^y - M_{k y} f_h{}^y g_{j t} + M_{j y} f_h{}^y g_{k t} = 0,$$

which shows that the Weyl conformal curvature tensor of M^n vanishes. Thus we have

Theorem 4.3. *Let M^n , $n \geq 4$, be an anti-invariant submanifold normal to the structure vector field ξ^* of a Sasakian manifold M^{2n+1} with vanishing contact Bochner curvature tensor. If the second fundamental tensors commute, then M^n is conformally flat.*

5. Sasakian manifolds as fibred spaces with invariant Riemannian metric

It is well known that in a Sasakian manifold we have

$$(5.1) \quad \mathcal{L}g_{\mu\nu} = 0, \quad \mathcal{L}\varphi_i{}^* = 0, \quad \mathcal{L}\xi_\lambda = 0,$$

where \mathcal{L} denotes the operator of Lie derivation with respect to the structure vector field ξ^* . Thus, assuming that ξ^* is regular, we can regard a Sasakian manifold M^{2m+1} as a fibred space with invariant Riemannian metric (see Yano and Ishihara [24]). Denoting $2m$ functionally independent solutions of

$$\xi^* \partial_\lambda u = 0$$

by $u^h(x)$, we see that u^h are local coordinates of the base space M^{2m} . We put

$$(5.2) \quad E_\lambda^h = \partial_\lambda u^h, \quad E_\lambda = \xi_\lambda, \quad E^* = \xi^*,$$

where and in the sequel the indices h, i, j, \dots run over the range $\{1', 2', \dots, (2m)'\}$. Then we have

$$E^i E_i^h = 0, \quad E^i E_i = 1.$$

Since E_i^h and E_i are linearly independent, we put

$$\begin{bmatrix} E_i^h \\ E_i \end{bmatrix}^{-1} = [E^i, E^i].$$

Then we have

$$(5.3) \quad E_i^h E^i_i = \delta_i^h, \quad E_i^h E^i = 0, \quad E_i E^i_i = 0, \quad E_i E^i = 1,$$

$$(5.4) \quad E_i^i E^i_i + E_i E^i = \delta_i^i.$$

For the Lie derivatives of E 's we have

$$(5.5) \quad \mathcal{L}E_i^h = 0, \quad \mathcal{L}E_i = 0, \quad \mathcal{L}E^i_i = 0, \quad \mathcal{L}E^i = 0.$$

Thus using $\mathcal{L}g_{\mu\lambda} = 0$ and (5.5) we see that

$$(5.6) \quad g_{ji} = g_{\mu\lambda} E_\mu^i E_\lambda^j$$

is the metric tensor of the base space M^{2m} . From (5.6) we have

$$(5.7) \quad g_{\mu\lambda} = g_{ji} E_\mu^j E_\lambda^i + E_\mu E_\lambda.$$

It will be easily verified that

$$(5.8) \quad E^i_i = E_\lambda^j g^{\lambda\kappa} g_{ji}, \quad E^i = E_\lambda g^{\lambda\kappa}, \quad E_i^h = E^\mu_i g_{\mu\lambda} g^{ih}, \quad E_i = E^\mu_i g_{\mu\lambda},$$

where g^{ih} are contravariant components of the metric tensor g_{ji} of the base space M^{2m} . Also using $\mathcal{L}\varphi_\lambda^i = 0$ and (5.5) we see that

$$(5.9) \quad F_i^h = \varphi_\lambda^i E_i^h E_\lambda^h$$

is a tensor field of type $(1, 1)$ of the base space M^{2m} and defines an almost complex structure of M^{2m} . From (5.6) and (5.9) we easily find

$$(5.10) \quad g_{ts} F_j^t F_i^s = g_{ji},$$

which shows that g_{ji} is a Hermitian metric with respect to this almost complex structure. Thus the base space M^{2m} is an almost Hermitian manifold.

From (5.9) it follows that

$$(5.11) \quad \varphi_\lambda^i E_i^h = F_i^h E_\lambda^h, \quad \varphi_\lambda^i E_\lambda^h = F_i^h E_\lambda^i, \quad \varphi_\lambda^i = F_i^h E_\lambda^i E_h^h.$$

For a function $f(u(x))$ on the base manifold M^{2m} we have

$$(5.12) \quad \partial_x f = E_i^i \partial_t f, \quad \partial_t f = E^i_i \partial_x f,$$

where $\partial_t = \partial/\partial u^i$.

Now using (5.7) we compute the Christoffel symbols $\{\gamma_{\mu}^{\nu}\}$ formed with $g_{\mu\nu}$ and find

$$(5.13) \quad \begin{aligned} \{\gamma_{\mu}^{\nu}\} &= \{\gamma_{\mu}^{\lambda}\} E_{\mu}^j E_{\lambda}^i E_{\nu}^{\epsilon} + (\partial_{\mu} E_{\lambda}^{\nu}) E_{\nu}^{\epsilon} + \frac{1}{2} (\partial_{\mu} E_{\lambda} + \partial_{\lambda} E_{\mu}) E_{\nu}^{\epsilon} \\ &\quad + E_{\mu} \varphi_{\lambda}^{\nu} + E_{\lambda} \varphi_{\mu}^{\nu}, \end{aligned}$$

where $\{\gamma_{\mu}^{\nu}\}$ are Christoffel symbols formed with $g_{\mu\nu}$. From (5.13) we have, in consequence of (5.11),

$$(5.14) \quad \partial_{\mu} E_{\lambda}^{\nu} - \{\gamma_{\mu}^{\nu}\} E_{\epsilon}^{\lambda} + \{\gamma_{\mu}^{\lambda}\} E_{\mu}^{\nu} E_{\lambda}^{\epsilon} = -(E_{\mu} E_{\lambda}^{\nu} + E_{\lambda} E_{\mu}^{\nu}) F_{\nu}^{\mu}.$$

Putting

$$(5.15) \quad \nabla_{\mu} E_{\lambda}^{\nu} = \partial_{\mu} E_{\lambda}^{\nu} - \{\gamma_{\mu}^{\nu}\} E_{\epsilon}^{\lambda} + \{\gamma_{\mu}^{\lambda}\} E_{\mu}^{\nu} E_{\lambda}^{\epsilon},$$

we have, from (5.14),

$$(5.16) \quad \nabla_{\mu} E_{\lambda}^{\nu} = -(E_{\mu} E_{\lambda}^{\nu} + E_{\lambda} E_{\mu}^{\nu}) F_{\nu}^{\mu}.$$

Thus putting $\nabla_j = E^{\mu} j \nabla_{\mu}$ we find

$$(5.17) \quad \nabla_j E_{\lambda}^{\nu} = -F_{j\lambda}^{\nu} E_{\nu},$$

from which it follows that

$$(5.18) \quad \nabla_j E_{\nu}^{\lambda} = -F_{j\lambda}^{\nu} E_{\nu},$$

where $F_{j\lambda} = F_{j\mu}^{\nu} g_{\mu\lambda}$. Thus by (5.9), (5.17) and (5.18) we obtain

$$(5.19) \quad \nabla_j F_{\lambda}^{\nu} = 0,$$

which shows that the base manifold M^{2m} is Kaehlerian.

From (5.16) and the Ricci identity

$$\nabla_{\nu} \nabla_{\mu} E_{\lambda}^{\nu} - \nabla_{\mu} \nabla_{\nu} E_{\lambda}^{\nu} = -K_{\nu\mu\lambda}^{\epsilon} E_{\epsilon}^{\nu} + K_{kji}^{\nu} E_{\nu}^k E_{\mu}^j E_{\lambda}^i,$$

we find

$$(5.20) \quad \begin{aligned} K_{kji}^{\nu} E_{\nu}^k E_{\mu}^j E_{\lambda}^i &= K_{\nu\mu\lambda}^{\epsilon} E_{\epsilon}^{\nu} - (E_{\nu} E_{\mu}^{\nu} - E_{\mu} E_{\nu}^{\nu}) E_{\lambda} \\ &\quad + (E_{\nu}^i \varphi_{\mu\lambda} - E_{\mu}^i \varphi_{\nu\lambda} - 2\varphi_{\nu\mu} E_{\lambda}^i) F_{\nu}^{\mu}, \end{aligned}$$

which implies that

$$(5.21) \quad K_{kji}^{\nu} = K_{\nu\mu\lambda}^{\epsilon} E_{\nu}^{\mu} E_{\mu}^{\lambda} + (F_{\nu\lambda} F_{ji} - F_{j\lambda} F_{ki} - 2F_{kj} F_{ih}),$$

where $E_{kji}^{\nu} = E_{\nu}^{\mu} E_{\mu}^{\lambda} E_{\lambda}^i E_{\lambda}^h$.

6. Sasakian manifolds with vanishing contact Bochner curvature tensor as a fibred space with invariant Riemannian metric

We now assume that the contact Bochner curvature tensor of the Sasakian manifold M^{2m+1} vanishes identically. Then transvecting (4.4) with $E_{kjh}^{\mu\lambda\kappa}$ we find

$$(6.1) \quad \begin{aligned} & K_{\nu\mu\lambda\kappa} E_{kjh}^{\nu\mu\lambda\kappa} + g_{kh} L_{ji} - g_{jh} L_{ki} + L_{kh} g_{ji} - L_{jh} g_{ki} \\ & + F_{kh} M_{ji} - F_{jh} M_{ki} + M_{kh} F_{ji} - M_{jh} F_{ki} \\ & - 2(F_{kj} M_{ih} + M_{kj} F_{ih}) + (F_{kh} F_{ji} - F_{jh} F_{ki} - 2F_{kj} F_{ih}) = 0, \end{aligned}$$

where

$$L_{ji} = L_{\mu\lambda} E^\mu{}_j E^\lambda{}_i, \quad M_{ji} = M_{\mu\lambda} E^\mu{}_j E^\lambda{}_i.$$

Thus we have

$$M_{ji} = -L_{\mu\alpha} \varphi_i{}^\alpha E^\mu{}_j E^\lambda{}_i = -L_{\mu\alpha} E^\mu{}_j F_i{}^\alpha E^\lambda{}_i,$$

that is,

$$(6.2) \quad M_{ji} = -L_{jt} F_i{}^\alpha E^\lambda{}_i,$$

which implies that

$$(6.3) \quad L_{ji} = M_{jt} F_i{}^\alpha E^\lambda{}_i.$$

Substituting (6.1) in (5.21) we find

$$(6.4) \quad \begin{aligned} & K_{kjh} + g_{kh} L_{ji} - g_{jh} L_{ki} + L_{kh} g_{ji} - L_{jh} g_{ki} + F_{kh} M_{ji} - F_{jh} M_{ki} \\ & + M_{kh} F_{ji} - M_{jh} F_{ki} - 2(F_{kj} M_{ih} + M_{kj} F_{ih}) = 0, \end{aligned}$$

from which, by transvecting with g^{kh} and using (6.2), we find

$$(6.5) \quad K_{jt} = -2(m+2)L_{ji} - Lg_{ji},$$

where $L = g^{ji} L_{ji}$, from which transvecting with g^{jt} gives

$$(6.6) \quad K = -4(m+1)L \quad \text{or} \quad L = -\frac{1}{4(m+1)}K.$$

Substituting (6.6) in (6.5) we find

$$(6.7) \quad L_{ji} = -\frac{1}{2(m+2)}K_{jt} + \frac{1}{8(m+1)(m+2)}Kg_{ji}.$$

Thus (6.4) shows that the Bochner curvature tensor of the base space M^{2m} vanishes. Hence we have

Theorem 6.1. *Let M^{2m+1} be a Sasakian manifold with vanishing contact Bochner curvature tensor regarded as a fibred space with invariant Riemannian metric. Then the Bochner curvature tensor of the Kählerian base space vanishes.*

Bibliography

- [1] D. E. Blair, *On the geometric meaning of the Bochner tensor*, Geometriae Dedicata, **4** (1975) 33–38.
- [2] D. E. Blair & K. Ogiue, *Geometry of integral submanifolds of a contact distribution*, Illinois J. Math. **19** (1975) 269–276.
- [3] S. Bochner, *Curvature and Betti numbers. II*, Ann. of Math. **50** (1949) 77–93.
- [4] B. Y. Chen & K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc. **193** (1974) 257–266.
- [5] B. Y. Chen & K. Yano, *Manifolds with vanishing Weyl or Bochner curvature tensor*, J. Math. Soc. Japan **27** (1975) 106–112.
- [6] C. S. Houh, *Some totally real minimal surfaces in $\mathbb{C}P^2$* , Proc. Amer. Math. Soc. **40** (1973) 240–244.
- [7] M. Kon, *Totally real submanifolds in a Kählerian manifold*, J. Differential Geometry **11** (1976) 251–257.
- [8] G. D. Ludden, M. Okumura & K. Yano, *Totally real submanifolds of complex manifolds*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fiz. Mat. Natur. **58** (1975) 346–353.
- [9] ——, *A totally real surface in $\mathbb{C}P^2$ that is not totally geodesic*, Proc. Amer. Math. Soc. **53** (1975) 186–190.
- [10] M. Matsumoto, *On Kählerian spaces with parallel or vanishing Bochner curvature tensor*, Tensor, N. S. **20** (1969) 25–28.
- [11] M. Matsumoto & G. Chūman, *On the C-Bochner curvature tensor*, TRU. Math. **5** (1969) 21–30.
- [12] S. Sasaki, *Almost contact manifolds*, Lecture notes. I, 1965, Tōhoku University.
- [13] S. Tachibana, *On the Bochner curvature tensor*, Natural Sci. Rep., Ochanomizu Univ. **18** (1967) 15–19.
- [14] S. Tachibana & R. C. Liu, *Notes on Kählerian metrics with vanishing Bochner curvature tensor*, Kōdai Math. Sem. Rep. **22** (1970) 313–321.
- [15] H. Takagi & Y. Watanabe, *On the holonomy groups of Kählerian manifold with vanishing Bochner curvature tensor*, Tōhoku Math. J. **25** (1973) 177–184.
- [16] S. Yamaguchi, M. Kon & T. Ikawa, *C-totally real submanifolds*, J. Differential Geometry **11** (1976) 59–64.
- [17] S. Yamaguchi & S. Sato, *On complex hypersurfaces with vanishing Bochner curvature tensor in Kählerian manifolds*, Tensor, N. S. **22** (1971) 77–81.
- [18] K. Yano, *On a structure defined by a tensor field f of type $(1, 1)$ satisfying $f^* + f = 0$* , Tensor, N. S. **14** (1963) 99–109.
- [19] ——, *Manifolds and submanifolds with vanishing Weyl or Bochner curvature tensor*, Proc. Symposia in Pure Math. **27** (1975) 253–262.
- [20] ——, *On complex conformal connections*, Kōdai Math. Sem. Rep. **26** (1975) 137–151.
- [21] ——, *Totally real submanifolds of a Kählerian manifold*, J. Differential Geometry **11** (1976) 351–359.
- [22] ——, *Differential geometry of totally real submanifolds*, Topics in differential geometry, Academic Press, New York, 1976, 173–184.
- [23] K. Yano & S. Bochner, *Curvature and Betti numbers*, Ann. of Math. Studies, No. 32, Princeton University Press, Princeton, 1953.
- [24] K. Yano & S. Ishihara, *Fibred spaces with invariant Riemannian metric*, Kōdai Math. Sem. Rep. **19** (1967) 317–360.
- [25] ——, *Kählerian manifolds with constant scalar curvature whose Bochner curvature tensor vanishes*, Hokkaido Math. J. **3** (1974) 297–304.

- [26] K. Yano & M. Kon, *Totally real submanifolds of complex space forms*. I, Tôhoku Math. J. **28** (1976) 215–225.
- [27] ———, *Totally real submanifolds of complex space forms*. II, Kôdai Math. Sem. Rep. **27** (1976) 385–399.
- [28] ———, *Anti-invariant submanifolds of Sasakian space forms*. I, Tôhoku Math. J. **29** (1977) 9–23.
- [29] ———, *Anti-invariant submanifolds of Sasakian space forms*. II, J. Korean Math. Soc. **13** (1976) 1–14.

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