

# Anti-self-dual conformal structures in neutral signature

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## 1 Introduction

We begin with some well-known facts from Riemannian geometry. Given an oriented Riemannian 4-manifold  $(\mathcal{M}, g)$ , the Hodge- $*$  operator is an involution on 2-forms. This induces a decomposition

$$\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2 \tag{1}$$

of 2-forms into self-dual and anti-self-dual components, which only depends on the conformal class  $[g]$ . Now choose  $g \in [g]$ . The Riemann tensor has the index symmetry  $R_{abcd} = R_{[ab][cd]}$  so can be thought of as a map  $\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$ . This map decomposes

under (1) as follows:

$$\mathcal{R} = \left( \begin{array}{c|c} C_+ + \frac{s}{12} & \phi \\ \hline \phi & C_- + \frac{s}{12} \end{array} \right). \quad (2)$$

The  $C_{\pm}$  terms are the self-dual and anti-self-dual parts of the Weyl tensor, the  $\phi$  terms are the tracefree Ricci curvature, and  $s$  is the scalar curvature which acts by scalar multiplication. The Weyl tensor is conformally invariant, so can be thought of as being defined by the conformal structure  $[g]$ . An *anti-self-dual conformal structure* is one with  $C_+ = 0$ . Such structures have a global twistor correspondence [3] which has been studied intensively; they have also been studied from a purely analytic point of view using elliptic techniques [51].

What happens in other signatures? In Lorentzian signature  $(+ + - -)$ , the Hodge- $*$  is not an involution (it squares to  $-1$  instead of  $1$ ) and there is no decomposition of 2-forms. In neutral  $(+ + - -)$  signature, the Hodge- $*$  is an involution, and there is a decomposition exactly as in the Riemannian case, depending on  $[g]$ . Thus anti-self-dual conformal structures exist in neutral signature. This article is devoted to their properties.

At the level of PDEs, the difference between neutral and Riemannian is that in the neutral case the gauge-fixed anti-self-duality equations are *ultrahyperbolic*, whereas in the Riemannian case they are elliptic. This results in profound differences, both locally and globally. Roughly speaking, the neutral case is far less rigid than the Riemannian case. For instance, any Riemannian anti-self-dual conformal structure must be analytic by the twistor construction. In the neutral case there is no general twistor construction, and in fact neutral conformal structures are not necessarily analytic. This lack of analyticity provides scope for rich local behaviour, as wave like solutions exist.

Assuming symmetries in the form of Killing vectors, one often finds that the equations reduce to *integrable systems*. Different integrable systems can be obtained by combining symmetries with geometric conditions for a metric in a conformal class. The story here in some sense parallels the case of the self-dual Yang–Mills equations in neutral signature, where imposing symmetries leads to many well-known integrable systems [37].

The subject of this review is the interplay between the ultrahyperbolic differential equations, and the anti-self-duality condition. We shall make a historical digression, and note that both concepts arose separately in mid 1930s.

Indeed, the ultrahyperbolic wave equation appears naturally in integral geometry, where the X-ray transform introduced in 1938 by John [25] can be used to construct all its smooth solutions. This takes a smooth function on  $\mathbb{R}P^3$  (a compactification of  $\mathbb{R}^3$ ) and integrates it over an oriented geodesic. The resulting function is defined on the Grassmannian  $Gr_2(\mathbb{R}^4)$  of two-planes in  $\mathbb{R}^4$  and satisfies the wave equation for a flat metric in  $(+ + - -)$  signature. To see it explicitly consider a smooth function

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$  which satisfies suitable decay conditions at infinity. For any oriented line  $L \subset \mathbb{R}^3$  define  $\psi(L) = \int_L f$ , or

$$\psi(x, y, w, z) = \int_{-\infty}^{\infty} f(xs + z, ys - w, s) ds, \quad (3)$$

where we have chosen an explicit parametrisation of all lines which are not perpendicular to the  $x_3$  axis. The dimension of the space of oriented lines is 4. This is greater than the dimension of  $\mathbb{R}^3$ , and one does not expect  $\psi$  to be arbitrary. Differentiating under the integral sign shows that  $\psi$  must satisfy the wave equation in neutral signature

$$\frac{\partial^2 \psi}{\partial x \partial w} + \frac{\partial^2 \psi}{\partial y \partial z} = 0. \quad (4)$$

John has demonstrated that equation (4) is the only condition constraining the range of the integral transform in this case, and that all smooth solutions to (4) arise by (3) from some  $f$ . One can regard the  $X$ -ray transform as the predecessor of twistor theory. In this context  $\mathbb{R}\mathbb{P}^3$  should be regarded as a totally real submanifold of a twistor space  $\mathbb{C}\mathbb{P}^3$ . In fact Woodhouse [56] showed that any local solution of (4) can be generated from a function on the real twistor space of  $\mathbb{R}^{2,2}$ . The twistor space is the set of totally null self-dual 2-planes and is three-dimensional, so we are again dealing with a function of three variables. To obtain the value  $\psi$  at a point  $p$ , one integrates  $f$  over all the planes through  $p$ . This was motivated by the Penrose transform with neutral reality conditions.

It is less well known that the ASD equation on Riemann curvature dates back to the same period as the work of John (at least 40 years before the seminal work of Penrose [45] and Atiyah–Hitchin–Singer [3]). It arose in the context of Wave Geometry – a subject developed in Hiroshima during the 1930s. Wave Geometry postulates the existence of a privileged spinor field which in the modern super-symmetric context would be called a Killing spinor. The integrability conditions come down to the ASD condition on a Riemannian curvature of the underlying complex space time. This condition implies vacuum Einstein equations. The Institute at Hiroshima where Wave Geometry had been developed was completely destroyed by the atomic bomb in 1945. Two of the survivors wrote up the results in a book [40]. In particular in [50] it was shown that local coordinates can be found such that the metric takes the form

$$g = \frac{\partial^2 \Omega}{\partial x \partial w} dx dw + \frac{\partial^2 \Omega}{\partial y \partial z} dy dz + \frac{\partial^2 \Omega}{\partial y \partial w} dy dw + \frac{\partial^2 \Omega}{\partial x \partial z} dx dz \quad (5)$$

and ASD vacuum condition reduces to a single PDE for one function  $\Omega$ :

$$\frac{\partial^2 \Omega}{\partial x \partial w} \frac{\partial^2 \Omega}{\partial y \partial z} - \frac{\partial^2 \Omega}{\partial x \partial z} \frac{\partial^2 \Omega}{\partial y \partial w} = 1. \quad (6)$$

This is nowadays known as the first heavenly equation after Plebanski who rediscovered it in 1975 [47]. If  $(\Omega, x, y, w, z)$  are all real, the resulting metric has neutral signature. The flat metric corresponds to  $\Omega = wx + zy$ . Setting

$$\Omega = wx + zy + \psi(x, y, w, z)$$

we see that up to the linear terms in  $\psi$  the heavenly equation reduces to the ultrahyperbolic wave equation (4). Later we shall see that the twistor method of solving (6) is a non-linear version of John's X-Ray transform. This concludes our historical digression.

The article is structured as follows. In Section 2 we introduce the local theory of neutral anti-self-dual conformal structures. It is convenient to use spinors, which for us will be a local tool to make the geometric structures more transparent. In Section 3 we explain how neutral ASD conformal structures are related to Lax pairs and hence integrable systems. We review various curvature restrictions on a metric in a conformal class (Ricci-flat, scalar flat Kähler etc), and show how these can be characterised in terms of their Lax pair. Section 4 is devoted to symmetries; in this section we make contact with many well known integrable systems. We discuss twistor theory in Section 5, explaining the differences between the Riemannian and neutral case, and describing various twistor methods of generating neutral ASD conformal structures. Despite the ultrahyperbolic nature of the equations, some strong global results have been obtained in recent years using a variety of techniques. We discuss these in Section 6.

The subject of neutral anti-self-dual conformal structures is rather diverse. We hope to present a coherent overview, but the different strands will not all be woven together. Despite this, we hope the article serves a useful purpose as a path through the literature.

## 2 Local geometry in neutral signature

**2.1 Conformal compactification.** We shall start off by describing a conformal compactification of the flat neutral metric. Let  $\mathbb{R}^{2,2}$  denote  $\mathbb{R}^4$  with a flat  $(+ + - -)$  metric. Its natural compactification is a projective quadric in  $\mathbb{R}\mathbb{P}^5$ . To describe it explicitly consider  $[\mathbf{x}, \mathbf{y}]$  as homogeneous coordinates on  $\mathbb{R}\mathbb{P}^5$ , and set  $\mathcal{Q} = |\mathbf{x}|^2 - |\mathbf{y}|^2$ . Here  $(\mathbf{x}, \mathbf{y})$  are vectors on  $\mathbb{R}^3$  with its natural inner product. The cone  $\mathcal{Q} = 0$  is projectively invariant, and the freedom  $(\mathbf{x}, \mathbf{y}) \sim (c\mathbf{x}, c\mathbf{y})$ , where  $c \neq 0$  is fixed to set  $|\mathbf{x}| = |\mathbf{y}| = 1$  which is  $S^2 \times S^2$ . We need to quotient this by the antipodal map  $(\mathbf{x}, \mathbf{y}) \rightarrow (-\mathbf{x}, -\mathbf{y})$  to obtain the conformal compactification<sup>1</sup>

$$\overline{\mathbb{R}^{2,2}} = (S^2 \times S^2)/\mathbb{Z}_2.$$

Parametrising the double cover of this compactification by stereographic coordinates we find that the flat metric  $|d\mathbf{x}|^2 - |d\mathbf{y}|^2$  on  $\mathbb{R}^{3,3}$  yields the metric

$$g_0 = 4 \frac{d\zeta d\bar{\zeta}}{(1 + \zeta\bar{\zeta})^2} - 4 \frac{d\chi d\bar{\chi}}{(1 + \chi\bar{\chi})^2} \quad (7)$$

on  $S^2 \times S^2$ . To obtain the flat metric on  $\mathbb{R}^{2,2}$  we would instead consider the intersection of the zero locus of  $\mathcal{Q}$  in  $\mathbb{R}^{3,3}$ , with a null hypersurface  $x_0 - y_0 = 1$ .

The metric  $g_0$  is conformally flat and scalar flat, as the scalar curvature is the difference between curvatures on both factors. It is also Kähler with respect to the natural complex structures on  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  with holomorphic coordinates  $(\zeta, \chi)$ . In

<sup>1</sup>This compactification can be identified with the Grassmannian  $\text{Gr}_2(\mathbb{R}^4)$  arising in the John transform (3).

Section 6.2 we shall see that  $g_0$  admits nontrivial scalar-flat Kähler deformations [53] globally defined on  $S^2 \times S^2$ .

**2.2 Spinors.** It is often convenient in four dimensions to use spinors, and the neutral signature case is no exception. The relevant Lie group isomorphism in neutral signature is

$$\mathrm{SO}(2, 2) \cong \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) / \mathbb{Z}_2. \quad (8)$$

We shall assume that the neutral four manifold  $(\mathcal{M}, g)$  has a spin structure. Therefore there exist real two-dimensional vector bundles  $S, S'$  (spin-bundles) over  $\mathcal{M}$  equipped with parallel symplectic structures  $\epsilon, \epsilon'$  such that  $T\mathcal{M} \cong S \otimes S'$  is a canonical bundle isomorphism, and

$$g(v_1 \otimes w_1, v_2 \otimes w_2) = \epsilon(v_1, v_2)\epsilon'(w_1, w_2)$$

for  $v_1, v_2 \in \Gamma(S)$  and  $w_1, w_2 \in \Gamma(S')$ . The two-component spinor notation [46] will be used in the paper. The spin bundles  $S$  and  $S'$  inherit connections from the Levi-Civita connection such that  $\epsilon, \epsilon'$  are covariant constant. We use the standard convention in which spinor indices are capital letters, unprimed for sections of  $S$  and primed for sections of  $S'$ . For example  $\mu_A$  denotes a section of  $S^*$ , the dual of  $S$ , and  $\nu^{A'}$  a section of  $S'$ .

The symplectic structures on spin spaces  $\epsilon_{AB}$  and  $\epsilon_{A'B'}$  (such that  $\epsilon_{01} = \epsilon_{0'1'} = 1$ ) are used to raise and lower indices. For example given a section  $\mu^A$  of  $S$  we define a section of  $S^*$  by  $\mu_A := \mu^B \epsilon_{BA}$ .

Spin dyads  $(o^A, \iota^A)$  and  $(o^{A'}, \iota^{A'})$  span  $S$  and  $S'$  respectively. We denote a normalised null tetrad of vector fields on  $\mathcal{M}$  by

$$e_{AA'} = \begin{pmatrix} e_{00'} & e_{01'} \\ e_{10'} & e_{11'} \end{pmatrix}.$$

This tetrad is determined by the choice of spin dyads in the sense that

$$o^A o^{A'} e_{AA'} = e_{00'}, \quad \iota^A o^{A'} e_{AA'} = e_{10'}, \quad o^A \iota^{A'} e_{AA'} = e_{01'}, \quad \iota^A \iota^{A'} e_{AA'} = e_{11'}.$$

The dual tetrad of one-forms by  $e^{AA'}$  determine the metric by

$$g = \epsilon_{AB} \epsilon_{A'B'} e^{AA'} \otimes e^{BB'} = 2(e^{00'} \odot e^{11'} - e^{10'} \odot e^{01'}) \quad (9)$$

where  $\odot$  is the symmetric tensor product. With indices, the above formula<sup>2</sup> for  $g$  becomes  $g_{ab} = \epsilon_{AB} \epsilon_{A'B'}$ .

The local basis  $\Sigma^{AB}$  and  $\Sigma^{A'B'}$  of spaces of ASD and SD two-forms are defined by

$$e^{AA'} \wedge e^{BB'} = \epsilon^{AB} \Sigma^{A'B'} + \epsilon^{A'B'} \Sigma^{AB}. \quad (10)$$

<sup>2</sup>Note that we drop the prime on  $\epsilon'$  when using indices, since it is already distinguished from  $\epsilon$  by the primed indices.

A vector  $V$  can be decomposed as  $V^{AA'} e_{AA'}$ , where  $V^{AA'}$  are the components of  $V$  in the basis. Its norm is given by  $\det(V^{AA'})$ , which is unchanged under multiplication of the matrix  $V^{AA'}$  by elements of  $\mathrm{SL}(2, \mathbb{R})$  on the left and right

$$V^{AA'} \longrightarrow \Lambda_B^A V^{BB'} \Lambda_{B'}^{A'}, \quad \Lambda \in \mathrm{SL}(2, \mathbb{R}), \quad \Lambda' \in \mathrm{SL}(2, \mathbb{R})'$$

giving (8). The quotient by  $\mathbb{Z}_2$  comes from the fact that multiplication on the left and right by  $-1$  leaves  $V^{AA'}$  unchanged.

Spinor notation is particularly useful for describing null structures. A vector  $V$  is null when  $\det(V^{AA'}) = 0$ , so  $V^{AA'} = \mu^A \nu^{A'}$  by linear algebra. In invariant language, this says that a vector  $V$  is null iff  $V = \mu \otimes \nu$  where  $\mu, \nu$  are sections of  $S, S'$ .

The decomposition of a 2-form into self-dual and anti-self-dual parts is straightforward in spinor notation. Let  $F_{AA'BB'}$  be a 2-form in indices. Now

$$\begin{aligned} F_{AA'BB'} &= F_{(AB)(A'B')} + F_{[AB][A'B']} + F_{(AB)[A'B']} + F_{[AB](A'B')} \\ &= F_{(AB)(A'B')} + c_{AB} \epsilon_{A'B'} + \phi_{AB} \epsilon_{A'B'} + \psi_{A'B'} \epsilon_{AB}. \end{aligned}$$

Here we have used the fact that in two dimensions there is a unique anti-symmetric matrix up to scale, so whenever an anti-symmetrized pair of spinor indices occurs we can substitute a multiple of  $\epsilon_{AB}$  or  $\epsilon_{A'B'}$  in their place. Now observe that the first two terms are incompatible with  $F$  being a 2-form, i.e.  $F_{AA'BB'} = -F_{BB'AA'}$ . So we obtain

$$F_{AA'BB'} = \phi_{AB} \epsilon_{A'B'} + \psi_{A'B'} \epsilon_{AB}, \quad (11)$$

where  $\phi_{AB}$  and  $\psi_{A'B'}$  are symmetric. This is precisely the decomposition of  $F$  into self-dual and anti-self dual parts. Which is which depends on the choice of volume form; we choose  $\psi_{A'B'} \epsilon_{AB}$  to be the self-dual part. Invariantly, we have

$$\Lambda_+^2 \cong S'^* \odot S'^*, \quad \Lambda_-^2 \cong S^* \odot S^*. \quad (12)$$

**2.3  $\alpha$ - and  $\beta$ -planes.** Suppose at a point  $x \in \mathcal{M}$  we are given a spinor  $\nu^{A'} \in S'_x$ . A two-plane  $\Pi_x$  is defined by all vectors of the form  $V^{AA'} = \mu^A \nu^{A'}$ , with varying  $\mu^A \in S$ . Now suppose  $V, W \in \Pi_x$ . Then  $g(V, W) = \nu^{A'} \nu^{B'} \epsilon_{A'B'} \mu^A \mu^B \epsilon_{AB} = 0$  since  $\epsilon_{A'B'}$  is antisymmetric. Therefore we say the two-plane is *totally null*. Furthermore, the 2-form  $V_{[a} W_{b]}$  is proportional to  $\mu_{A'} \mu_{B'} \epsilon_{AB}$ ; i.e. the two-plane is self-dual. In summary, a spinor in  $S$  defines a totally null self-dual two-plane, which is called an  $\alpha$ -plane. Similarly a spinor in  $S'$  defines a totally null anti-self-dual two-plane, called a  $\beta$ -plane.

**2.4 Anti-self-dual conformal structures in spinors.** A neutral conformal structure  $[g]$  is an equivalence class of neutral signature metrics, with the equivalence relation  $g \sim e^f g$  for any function  $f$ . Another way of viewing such a structure is as a line-bundle valued neutral metric; we will not need this description because in most cases we will be working with particular metrics within a conformal class.

Choose a  $g \in [g]$ . Then there is a Riemann tensor, which possesses certain symmetries under permutation of indices. In the same way that we deduced (11) for the decomposition of a 2-form in spinors, the Riemann tensor decomposes as [46]

$$\begin{aligned} R_{AA'BB'CC'DD'} &= C_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + C_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\ &\quad + \phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \phi_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \\ &\quad + \frac{s}{12}(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{A'D'}\epsilon_{B'C'}). \end{aligned}$$

This is the spinor version of (2). Here  $C_{A'B'C'D'}$ ,  $C_{ABCD}$  are totally symmetric, and correspond to  $C_+$ ,  $C_-$  in (2). The spinor  $\phi_{A'B'CD}$  is symmetric in its pairs of indices, and corresponds to  $\phi$  in (2). An anti-self-dual conformal structure is one for which  $C_{A'B'C'D'} = 0$ . In the next section we explain the geometric significance of this condition in more detail. It is appropriate here to recall the Petrov–Penrose classification [46] of the algebraic type of a Weyl tensor. In split signature this applies separately to  $C_{ABCD}$  and  $C_{A'B'C'D'}$ . In our case  $C_{A'B'C'D'} = 0$  and we are concerned with the algebraic type of  $C_{ABCD}$ . One can form a real polynomial of fourth order  $P(x)$  by defining  $\mu^A = (1, x)$  and setting  $P(x) = \mu^A \mu^B \mu^C \mu^D C_{ABCD}$ . The Petrov–Penrose classification refers to the position of roots of this polynomial, for example if there are four repeated roots then we say  $C_{ABCD}$  is type N. If there is a repeated root the metric is called *algebraically special* indexalgebraically special. There are additional complications in the split signature case [32] arising from the fact that real polynomials may not have real roots.

### 3 Integrable systems and Lax pairs

In this section we show how anti-self-dual conformal structures are related to integrable systems and Lax pairs. Let  $g \in [g]$  and let  $\nabla$  denote the Levi-Civita connection on  $\mathcal{M}$ . This connection induces spin connections on spin bundles which we also denote  $\nabla$ . Let us consider  $S'$ . The connection coefficients  $\Gamma_{AA'B'}^{C'}$  of  $\nabla$  are defined by

$$\nabla_{AA'}\mu^{C'} = e_{AA'}(\mu^{C'}) + \Gamma_{AA'B'}^{C'}\mu^{B'},$$

where  $\mu^{A'}$  is a section of  $S'$  in coordinates determined by the basis  $e_{AA'}$ . The  $\Gamma_{AA'B'}^{C'}$  symbols can be calculated in terms of the Levi-Civita connection symbols. They can also be read off directly from the Cartan equations  $de^{AA'} = e^{BA'} \wedge \Gamma_B^A + e^{AB'} \wedge \Gamma_{B'}^{A'}$ , where  $\Gamma_{B'}^{C'} = \Gamma_{AA'B'}^{C'} e^{AA'}$ . See [46] for details. Now given a connection on a vector bundle, one can lift a vector field on the base to a horizontal vector field on the total space. We follow standard notation and denote the local coordinates of  $S'$  by  $\pi^{A'}$ . Then the horizontal lifts  $\tilde{e}_{AA'}$  of  $e_{AA'}$  are given explicitly by

$$\tilde{e}_{AA'} := e_{AA'} + \Gamma_{AA'B'}^{C'} \pi^{B'} \frac{\partial}{\partial \pi^{C'}}.$$

Now we can state a seminal result of Penrose:

**Theorem 1** ([45]). *Given a neutral metric  $g$ , define a two-dimensional distribution on  $S'$  by  $\mathcal{D} = \text{span}\{L_0, L_1\}$ , where*

$$L_A := \pi^{A'} \tilde{e}_{AA'}. \quad (13)$$

*Then  $\mathcal{D}$  is integrable iff  $g$  is anti-self-dual.*

So when  $g$  is ASD,  $S'$  is foliated by surfaces. Since the  $L_A$  are homogeneous in the  $\pi^{A'}$  coordinates,  $\mathcal{D}$  defines a distribution on  $PS'$ , the projective version of  $S'$ .

The push down of  $\mathcal{D}$  from a point  $\pi^{A'} = v^{A'}$  in a fibre of  $S'$  to the base is the  $\alpha$ -plane defined by  $v^{A'}$ , as explained in Section 2.3. So the content of Theorem 1 is that  $g$  is ASD iff any  $\alpha$ -plane is tangent to an  $\alpha$ -surface, i.e. a surface that is totally null and self-dual at every point. Any such  $\alpha$ -surface lifts to a unique surface in  $PS'$ , or a one parameter family of surfaces in  $S'$ .

**3.1 Curvature restrictions and their Lax pairs.** A more recent interpretation of Theorem 1 is to regard  $L_A$  as a Lax pair for the ASD conformal structure. Working on  $PS'$ , with inhomogeneous fibre coordinate  $\lambda = \pi^{1'}/\pi^{0'}$ , the condition that  $\mathcal{D}$  commutes is the compatibility condition for the pair of linear equations

$$\begin{aligned} L_0 f &= (\tilde{e}_{00'} + \lambda \tilde{e}_{01'}) f = 0 \\ L_1 f &= (\tilde{e}_{10'} + \lambda \tilde{e}_{11'}) f = 0 \end{aligned}$$

to have a solution  $f$  for all  $\lambda \in \mathbb{R}$ , where  $f$  is a function on  $PS'$ . In integrable systems language,  $\lambda$  is the spectral parameter.

Here we describe various conditions that one can place on a metric  $g \in [g]$  on top of anti-self-duality. This provides a more direct link with integrable systems as in each case described below one can choose a spin frame, and local coordinates to reduce the special ASD condition to an integrable scalar PDE with corresponding Lax pair.

**3.1.1 Pseudo-hyperhermitian structures.** This is the neutral analogue of Riemannian hyperhermitian geometry. The significant point for us is that in four dimensions, pseudo-hyperhermitian metrics (defined below) are necessarily anti-self-dual.

Consider a structure  $(\mathcal{M}, I, S, T)$ , where  $\mathcal{M}$  is a four-dimensional manifold and  $I, S, T$  are anti-commuting endomorphisms of the tangent bundle satisfying

$$S^2 = T^2 = \mathbf{1}, \quad I^2 = -\mathbf{1}, \quad ST = -TS = \mathbf{1}. \quad (14)$$

This is called the algebra of para-quaternions [24] or split quaternions [12]. Consider the hyperboloid of almost complex structures on  $\mathcal{M}$  given by  $aI + bS + cT$ , for  $(a, b, c)$  satisfying  $a^2 - b^2 - c^2 = 1$ . If each of these almost complex structures is integrable, we call  $(\mathcal{M}, I, S, T)$  a *pseudo-hypercomplex* manifold.

So far we have not introduced a metric. A natural restriction on a metric given a pseudo-hypercomplex structure is to require it to be hermitian with respect to each of the complex structures. This is equivalent to the requirement:

$$g(X, Y) = g(IX, IY) = -g(SX, SY) = -g(TX, TY), \quad (15)$$



for all vectors  $X, Y$ . A metric satisfying (15) must be neutral. To see this consider the endomorphism  $S$ , which squares to the identity. Its eigenspaces decompose into  $+1$  and  $-1$  parts. Any eigenvector must be null from (15). So choosing an eigenbasis one can find 4 null vectors, from which it follows that the metric is neutral. Given a pseudo-hypercomplex manifold, we call a metric satisfying (15) a pseudo-hyperhermitian metric.

Given a local pseudo-hypercomplex structure in four dimensions one can construct many pseudo-hyperhermitian metrics for it as follows. Take a vector field  $V$  and let  $(V, IV, SV, TV)$  be an orthonormal basis in which the metric has diagonal components  $(1, 1, -1, -1)$ . The fact that these vectors are linearly independent follows from (14). It is easy to check that (15) holds for any two vectors in the above basis, and hence by linearity for any  $(X, Y)$ . By varying the length of  $V$  one obtains a different conformal class. However, even the conformal class is not uniquely specified. To see this take a vector  $W$  that is null for the metric specified by  $V$ , and form a new metric by the same procedure using  $W$ . Then  $W$  is not null in this new metric, so this metric must be in a different conformal class.

As mentioned above, it turns out that pseudo-hyperhermitian metrics are necessarily anti-self-dual. One way to formulate this is via the Lax pair formalism as follows:

**Theorem 2** ([14]). *Let  $e_{AA'}$  be four independent vector fields on a four-dimensional real manifold  $\mathcal{M}$ . Put*

$$L_0 = e_{00'} + \lambda e_{01'}, \quad L_1 = e_{10'} + \lambda e_{11'}.$$

If

$$[L_0, L_1] = 0 \tag{16}$$

for every value of a parameter  $\lambda$ , then  $g$  given by (9) a pseudo-hyperhermitian metric on  $\mathcal{M}$ . Given any four-dimensional pseudo-hyperhermitian metric there exists a null tetrad such that (16) holds.

Interpreting  $\lambda$  as the projective primed spin coordinate as in Section 3, we see that a pseudo-hyperhermitian metric must be ASD from Theorem 1. Theorem 2 characterises pseudo-hyperhermitian metrics as those which possess a Lax pair containing no  $\partial_\lambda$  terms.

We shall now discuss the local formulation of the pseudo-hyperhermitian condition as a PDE. Expanding equation (16) in powers of  $\lambda$  gives

$$[e_{A0'}, e_{B0'}] = 0, \quad [e_{A0'}, e_{B1'}] + [e_{A1'}, e_{B0'}] = 0, \quad [e_{A1'}, e_{B1'}] = 0. \tag{17}$$

It follows from (17), using the Frobenius theorem and the Poincaré lemma that one can choose coordinates  $(p^A, w^A)$  ( $A = 0, 1$ ) in which  $e_{AA'}$  take the form

$$e_{A0'} = \frac{\partial}{\partial p^A}, \quad e_{A1'} = \frac{\partial}{\partial w^A} - \frac{\partial \Theta^B}{\partial p^A} \frac{\partial}{\partial p^B},$$

where  $\Theta^B$  are a pair of functions satisfying a system of coupled non-linear ultra-hyperbolic PDEs.

$$\frac{\partial^2 \Theta_C}{\partial p_A \partial w^A} + \frac{\partial \Theta_B}{\partial p^A} \frac{\partial^2 \Theta_C}{\partial p_A \partial p_B} = 0. \quad (18)$$

Note the indices here are not spinor indices, they are simply a convenient way of labelling coordinates and the functions  $\Theta^A$ . We raise and lower them using the standard antisymmetric matrix  $\epsilon_{AB}$ , for example  $p_A := p^B \epsilon_{BA}$ , and the summation convention is used.

**3.1.2 Scalar-flat Kähler structures.** Let  $(\mathcal{M}, g)$  be an ASD four manifold and let  $J$  be a (pseudo-)complex structure such that the corresponding fundamental two-form is closed. This ASD Kähler condition implies that  $g$  is scalar flat, and conversely all scalar flat Kähler four manifolds are ASD [13].

In this section we shall show that in the scalar-flat Kähler case the spin frames can be chosen so that the Lax pair (13) consists of volume-preserving vector fields on  $\mathcal{M}$  together with two functions on  $\mathcal{M}$ . The following theorem has been obtained in a joint work of Maciej Przanowski and the first author. We shall formulate and prove it in the holomorphic category which will allow both neutral and Riemannian real slices.

**Theorem 3.** *Let  $e_{AA'} = (e_{00'}, e_{01'}, e_{10'}, e_{11'})$  be four independent holomorphic vector fields on a four-dimensional complex manifold  $\mathcal{M}$  and let  $f_1, f_2: \mathcal{M} \rightarrow \mathbb{C}$  be two holomorphic function. Finally, let  $v$  be a nonzero holomorphic four-form. Put*

$$L_0 = e_{00'} + \lambda e_{01'} - f_0 \lambda^2 \frac{\partial}{\partial \lambda}, \quad L_1 = e_{10'} + \lambda e_{11'} - f_1 \lambda^2 \frac{\partial}{\partial \lambda}. \quad (19)$$

Suppose that for every  $\lambda \in \mathbb{C}\mathbb{P}^1$

$$[L_0, L_1] = 0, \quad \mathcal{L}_{L_A} v = 0, \quad (20)$$

where  $\mathcal{L}_V$  denotes the Lie derivative. Then

$$\hat{e}_{AA'} = c^{-1} e_{AA'}, \quad \text{where } c^2 := v(e_{00'}, e_{01'}, e_{10'}, e_{11'}),$$

is a null-tetrad for an ASD Kähler metric. Every such metric locally arises in this way.

*Proof.* First assume that there exists a tetrad  $e_{AA'}$  and two functions  $f_A = (f_0, f_1)$  such that equations (20) are satisfied. For convenience write down equations  $[L_0, L_1] = 0$  in full

$$[e_{A0'}, e_{B0'}] = 0, \quad (21)$$

$$[e_{A0'}, e_{B1'}] + [e_{A1'}, e_{B0'}] = 0, \quad (22)$$

$$[e_{A1'}, e_{B1'}] = \epsilon_{AB} f^C e_{C1'}, \quad (23)$$

$$e^A{}_{0'} f_A = 0, \quad (24)$$

$$e^A{}_{1'} f_A = 0. \quad (25)$$

Define the almost complex structure  $J$  by

$$J(e_{A1'}) = -ie_{A1'}, \quad J(e_{A0'}) = ie_{A0'}.$$

Equations (21), and (23) imply that this complex structure is integrable. Let  $g$  be a metric corresponding to  $\hat{e}_{AA'}$  by (9). To complete this part of the proof we need to show that a fundamental two-form  $\omega$  defined by  $\omega(X, Y) = g(X, JY)$  is closed. First observe that

$$\omega = \frac{\partial}{\partial \lambda} (v(L_0, L_1, \cdot, \cdot))|_{\lambda=0}.$$

It is therefore enough to prove that  $\Sigma = v(L_0, L_1, \cdot, \cdot)$  is closed for each fixed  $\lambda$ . We shall establish this fact using equations (20), and  $dv = 0$ . Let us calculate

$$\begin{aligned} d\Sigma &= d(v(L_0, L_1, \cdot, \cdot)) = d(L_0 \lrcorner (v(L_1, \cdot, \cdot, \cdot))) \\ &= \mathcal{L}_{L_0}(v(L_1, \cdot, \cdot, \cdot)) - L_0 \lrcorner (dv(L_1, \cdot, \cdot, \cdot)) \\ &= [L_0, L_1] \lrcorner v + L_1 \lrcorner \mathcal{L}_{L_0}(v) - L_0 \lrcorner (L_1 \lrcorner dv) \\ &= -L_0 \lrcorner (\mathcal{L}_{L_1} v - L_0 \lrcorner (L_1 \lrcorner d(v))) = 0. \end{aligned}$$

Therefore  $\omega$  is closed which in the case of integrable  $J$  also implies  $\nabla\omega = 0$  [31].

*Converse.* The metric  $g$  is Kähler, therefore there exist local coordinates  $(w^A, \tilde{w}^A)$  and a complex valued function  $\Omega = \Omega(w^A, \tilde{w}^A)$  such that  $g$  is given by

$$g = \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} dw^A d\tilde{w}^B. \quad (26)$$

Choose a spin frame  $(o_{A'}, \iota_{A'})$  such that the tetrad of vector fields  $e_{AA'}$  is

$$e_{A0'} = o^{A'} e_{AA'} = \frac{\partial}{\partial w^A}, \quad e_{A1'} = \iota^{A'} e_{AA'} = \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial}{\partial \tilde{w}^B}.$$

The null tetrad for the metric (26) is  $\hat{e}_{AA'} = G^{-1} e_{AA'}$ , where

$$G = \det(g) = \frac{1}{2} \frac{\partial^2 \Omega}{\partial w_A \partial \tilde{w}^B} \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B}. \quad (27)$$

The Lax pair (13) is

$$L_A = \frac{\partial}{\partial w^A} - \lambda \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial}{\partial \tilde{w}^B} + l_A \frac{\partial}{\partial \lambda}.$$

Consider the Lie bracket

$$\begin{aligned} [L_0, L_1] &= l^2 \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial^3 \Omega}{\partial w_A \partial \tilde{w}^B \partial \tilde{w}^C} \frac{\partial}{\partial \tilde{w}^C} + l^A \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial}{\partial \tilde{w}^B} \\ &\quad + \left( \frac{\partial l^A}{\partial w^A} - \lambda \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial l^A}{\partial \tilde{w}^B} + l_A \frac{\partial l^A}{\partial \lambda} \right) \frac{\partial}{\partial \lambda}. \end{aligned}$$

The ASD condition is equivalent to integrability of the distribution  $L_A$ , therefore

$$[L_A, L_B] = \epsilon_{AB} \alpha^C L_C$$

for some  $\alpha^C$ . The lack of  $\partial/\partial w^A$  term in the Lie bracket above implies  $\alpha^C = 0$ . Analysing other terms we deduce the existence of  $f = f(w^A, \tilde{w}^A) \in \ker \square$  such that  $l_A = \lambda^2 \partial f / \partial w^A$ , and

$$\frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial}{\partial \tilde{w}^C} \left( \frac{\partial^2 \Omega}{\partial w_A \partial \tilde{w}_B} \right) = \frac{\partial f}{\partial w_A} \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^C}. \quad (28)$$

□

The real-analytic (+ + --) slices are obtained if  $e_{AA'}, \nu, f_1, f_2$  are all real. In this case we alter our definition of  $J$  by

$$J(e_{A1'}) = -e_{A1'}, \quad J(e_{A0'}) = e_{A0'}.$$

Therefore  $J^2 = \mathbf{1}$ , and  $g$  is pseudo-Kähler.

In the Euclidean case the quadratic-form  $g$  and the complex structure

$$J = i(e^{A0'} \otimes e_{A0'} - e^{A1'} \otimes e_{A1'})$$

are real but the vector fields  $e_{AA'}$  are complex.

As a corollary from the last theorem we can deduce a formulation of the scalar-flat Kähler condition [44]. Scalar-flat Kähler metric are locally given by (26) where  $\Omega(w^A, \tilde{w}^A)$  is a solution to a 4th order PDE (which we write as a system of two second order PDEs):

$$\frac{\partial f}{\partial w^A} = \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial \ln G}{\partial \tilde{w}_B}, \quad (29)$$

$$\square f = \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial^2 f}{\partial w_A \partial \tilde{w}_B} = 0. \quad (30)$$

Moreover (29,30) arise as an integrability condition for the linear system  $L_0 \Psi = L_1 \Psi = 0$ , where  $\Psi = \Psi(w^A, \tilde{w}^A, \lambda)$  and

$$L_A = \frac{\partial}{\partial w^A} - \lambda \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial}{\partial \tilde{w}_B} + \lambda^2 \frac{\partial f}{\partial w^A} \frac{\partial}{\partial \lambda}. \quad (31)$$

To see this note that in the proof of Theorem 3 we have demonstrated that  $f \in \ker \square$ . In the adopted coordinate system

$$\square = \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial^2}{\partial w_A \partial \tilde{w}_B},$$

which gives (33). Solving the algebraic system (28) for  $\partial f / \partial w^A$  yields (30).

**3.1.3 Null-Kähler structures.** A null-Kähler structure on a real four-manifold  $\mathcal{M}$  consists of an inner product  $g$  of signature  $(+ + - -)$  and a real rank-two endomorphism  $N : T\mathcal{M} \rightarrow T\mathcal{M}$  parallel with respect to this inner product such that

$$N^2 = 0 \quad \text{and} \quad g(NX, Y) + g(X, NY) = 0$$

for all  $X, Y \in T\mathcal{M}$ . The isomorphism  $\Lambda^2_+(\mathcal{M}) \cong \text{Sym}^2(S')$  between the bundle of self-dual two-forms and the symmetric tensor product of two spin bundles implies that the existence of a null-Kähler structure is in four dimensions equivalent to the existence of a parallel real spinor. The Bianchi identity implies the vanishing of the curvature scalar.

In [8] and [15] it was shown that null-Kähler structures are locally given by one arbitrary function of four variables, and admit a canonical form<sup>3</sup>

$$g = dw dx + dz dy - \Theta_{xx} dz^2 - \Theta_{yy} dw^2 + 2\Theta_{xy} dw dz, \quad (32)$$

with  $N = dw \otimes \partial/\partial y - dz \otimes \partial/\partial x$ .

Further conditions can be imposed on the curvature of  $g$  to obtain non-linear PDEs for the potential function  $\Theta$ . Define

$$f := \Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2. \quad (33)$$

- The Einstein condition implies that

$$f = xP(w, z) + yQ(w, z) + R(w, z),$$

where  $P, Q$  and  $R$  are arbitrary functions of  $(w, z)$ . In fact the number of the arbitrary functions can be reduced down to one by redefinition of  $\Theta$  and the coordinates. This is the hyper-heavenly equation of Plebański and Robinson [48] for non-expanding metrics of type  $[N] \times [\text{Any}]$ . (Recall that  $(\mathcal{M}, g)$  is called hyper-heavenly if the self-dual Weyl spinor is algebraically special).

- The conformal anti-self-duality (ASD) condition implies a 4th order PDE for  $\Theta$

$$\square f = 0, \quad (34)$$

where  $\square$  is the Laplace–Beltrami operator defined by the metric  $g$ . This equation is integrable: It admits a Lax pair

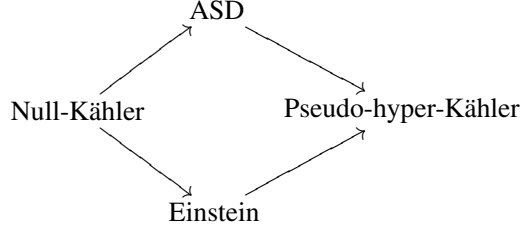
$$\begin{aligned} L_0 &= (\partial_w - \Theta_{xy}\partial_y + \Theta_{yy}\partial_x) - \lambda\partial_y + f_y\partial_\lambda, \\ L_1 &= (\partial_z + \Theta_{xx}\partial_y - \Theta_{xy}\partial_x) + \lambda\partial_x + f_x\partial_\lambda. \end{aligned}$$

and its solutions can in principle be found by twistor methods [15], or the dressing approach [7].

<sup>3</sup>The local form (32) is a special case of Walker's canonical form of a neutral metric which admits a two-dimensional distribution which is parallel and null [54]. Imposing more restrictions on Walker's metric leads to examples of conformally Osserman structures, i.e. metrics for which the eigenvalues of the operator  $Y^a \rightarrow C_{bcd}^a X^b Y^c X^d$  are constant on the unit pseudo-sphere  $\{X \in T\mathcal{M}, g(X, X) = \pm 1\}$ . These metrics are all SD or ASD according to [6].

- Imposing both conformal ASD and Einstein condition implies (possibly after a redefinition of  $\Theta$ ) that  $f = 0$ , which yields the celebrated second heavenly equation of Plebański [47]:

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0. \quad (35)$$



**3.1.4 Pseudo-hyper-Kähler structures.** Suppose we are given a pseudo-hypercomplex structure as defined in the previous section, i.e. a two-dimensional hyperboloid of integrable complex structures. In the previous section we defined a pseudo-hyperhermitian metric to be a metric that is hermitian with respect to each complex structure in the family. If we further require that the 2-forms

$$\omega_I(\cdot, \cdot) = g(\cdot, I\cdot), \quad \omega_S(\cdot, \cdot) = g(\cdot, S\cdot), \quad \omega_T(\cdot, \cdot) = g(\cdot, T\cdot), \quad (36)$$

be closed, we call say  $g$  is pseudo-hyper-Kähler. These define three symplectic forms, and Hitchin has termed such structures hypersymplectic<sup>4</sup> [23].

It follows from similar arguments to those in standard Riemannian Kähler geometry that  $(I, S, T)$  are covariant constant, and hence so are  $\omega_I, \omega_S, \omega_T$ .

As in the Riemannian case, pseudo-hyper-Kähler metrics are equivalent to Ricci-flat anti-self-dual metrics. One can deduce this by showing that the 2-forms (36) are self-dual, and since they are also covariant constant there exists a basis of covariant constant primed spinors. Then using the spinor Ricci identities one can deduce anti-self-duality and Ricci-flatness. See for details.

The Lax pair formulation for a pseudo-hyper-Kähler metric is as follows:

**Theorem 4** ([1], [36]). *Let  $e_{AA'}$  be four independent vector fields on a four-dimensional real manifold  $\mathcal{M}$ , and  $v$  be a 4-form. Put*

$$L_0 = e_{00'} + \lambda e_{01'}, \quad L_1 = e_{10'} + \lambda e_{11'}.$$

If

$$[L_0, L_1] = 0 \quad (37)$$

for every  $\lambda \in \mathbb{RP}^1$ , and

$$\mathcal{L}_{L_A} v = 0, \quad (38)$$

then  $f^{-1}e_{AA'}$  is a null tetrad for a pseudo-hyper-Kähler metric on  $\mathcal{M}$ , where  $f^2 = v(e_{00'}, e_{0'}, e_{10'}, e_{11'})$ . Given any four-dimensional pseudo-hyper-Kähler metric such a null tetrad and 4-form exists.

<sup>4</sup>Other terminology includes neutral hyper-Kähler [28] and hyper-para-Kähler [24].

The extra volume preserving condition (38) distinguishes this from Theorem 2. Alternatively Theorem 4 arises as a special case of Theorem 3 with  $f_A = 0$ .

**The Heavenly equations.** It was shown by Plebański [47] that one can always put a pseudo-hyper-Kähler metric into the form (5), where  $\Omega$  satisfies the *first Heavenly equation* (6). The function  $\Omega$  can be interpreted as the Kähler potential for one of the complex structures. Plebański also gave the alternative local form. The metric is given by (32), and the potential  $\Theta$  satisfies the *second Heavenly equation* (35). The Heavenly equations are non-linear ultrahyperbolic equations. These formulations are convenient for understanding local properties of pseudo-hyper-Kähler metrics, as they only depend on a single function satisfying a single PDE.

## 4 Symmetries

By a symmetry of a metric, we mean a conformal Killing vector, i.e. a vector field  $K$  satisfying

$$\mathcal{L}_K g = c g, \quad (39)$$

where  $c$  is a function. If  $c$  vanishes,  $K$  is called a *pure* Killing vector, otherwise it is called a *conformal* Killing vector. If  $c$  is a nonzero constant  $K$  is called a *homothety*. If we are dealing with a conformal structure  $[g]$ , a symmetry is a vector field  $K$  satisfying (39) for some  $g \in [g]$ . Then (39) will be satisfied for any  $g \in [g]$ , where the function  $c$  will depend on the choice of  $g \in [g]$ . Such a  $K$  is referred to as a conformal Killing vector for the conformal structure.

In neutral signature there are two types of Killing vectors: non-null and null. Unlike in the Lorentzian case where non-null vectors can be timelike or spacelike, there is essentially only one type of non-null vector in neutral signature. Note that a null vector for  $g \in [g]$  is null for all  $g \in [g]$ , so nullness of a vector with respect to a conformal structure makes sense.

**4.1 Non-null case.** Given a neutral four-dimensional ASD conformal structure  $(\mathcal{M}, [g])$  with a non-null conformal Killing vector  $K$ , the three-dimensional space  $\mathcal{W}$  of trajectories of  $K$  inherits a conformal structure  $[h]$  of signature  $(++-)$ , due to (39). The ASD condition on  $[g]$  results in extra geometrical structure on  $(\mathcal{W}, h)$ ; it becomes a Lorentzian *Einstein–Weyl* space. This is called the Jones–Tod construction, and is described in Section 4.1.2. The next section is an summary of Einstein–Weyl geometry.

**4.1.1 Einstein–Weyl geometry.** Let  $\mathcal{W}$  be a three-dimensional manifold. Given a conformal structure  $[h]$  of signature<sup>5</sup>  $(2, 1)$ , a connection  $D$  is said to preserve  $[h]$  if

$$Dh = \omega \otimes h, \quad (40)$$

---

<sup>5</sup>The formalism in this section works in general dimension and signature but we specialize to the case we encounter later.

for some  $h \in [h]$ , and a 1-form  $\omega$ . It is clear that if (40) holds for a single  $h \in [h]$  it holds for all, where  $\omega$  will depend on the particular  $h \in [h]$ . (40) is a natural condition; it is the requirement that null geodesics of any  $h \in [h]$  are also geodesics of  $D$ .

Given  $D$  we can define its Riemann and Ricci curvature tensors  $W^i_{jkl}$ ,  $W_{ij}$  in the usual way. The notion of a curvature scalar must be modified, because there is no distinguished metric in the conformal class to contract  $W_{ij}$  with. Given some  $h \in [h]$  we can form  $W = h^{ij} W_{ij}$ . Under a conformal transformation  $h \rightarrow \phi^2 h$ ,  $W$  transforms as  $W \rightarrow \phi^{-2} W$ . This is because  $W_{ij}$  unaffected by any conformal rescaling, being formed entirely out of the connection  $D$ .  $W$  is an example of a conformally weighted function, with weight  $-2$ .

One can now define a conformally invariant analogue of the Einstein equation as follows:

$$W_{(ij)} - \frac{1}{3} W h_{ij} = 0. \quad (41)$$

These are the *Einstein–Weyl equations*. Notice that the left-hand side is well defined tensor (i.e. weight 0), since the weights of  $W$  and  $h_{ij}$  cancel. Equation (41) is the Einstein–Weyl equation for  $(D, [h])$ . It says that given any  $h \in [h]$ , the Ricci tensor of  $W$  is tracefree when one defines the trace using  $h$ . Notice also that  $W_{ij}$  is not necessarily symmetric, unlike the Ricci-tensor for a Levi-Civita connection.

In the special case that  $D$  is the Levi-Civita connection of some metric  $h \in [h]$ , (41) reduces to the Einstein equation. This happens when  $\omega$  is exact, because under  $h \rightarrow \phi^2 h$ , we get  $\omega \rightarrow \omega + 2d(\ln\phi)$ , so if  $\omega$  is exact a suitable choice of  $\phi$  will transform it to 0, giving  $Dh = 0$  in (40). All Einstein metrics in 2+1 or 3 dimensions are spaces of constant curvature. The Einstein–Weyl condition allows non-trivial degrees of freedom. The general solution to (41) depends on four arbitrary functions of two variables.

In what follows, we refer to an Einstein–Weyl structure by  $(h, \omega)$ . The connection  $D$  is fully determined by this data using (40).

**4.1.2 Reduction by a non-null Killing vector; the Jones–Tod construction.** The Jones–Tod construction relates ASD conformal structures in four dimensions to Einstein–Weyl structures in three dimensions. In neutral signature it can be formulated as follows:

**Theorem 5** ([26]). *Let  $(\mathcal{M}, [g])$  be a neutral ASD four manifold with a non-null conformal Killing vector  $K$ . An Einstein–Weyl structure on the space  $\mathcal{W}$  of trajectories of  $K$  is defined by*

$$h := |K|^{-2} g - |K|^{-4} \mathbb{K} \odot \mathbb{K}, \quad \omega = 2|K|^{-2} *_g (\mathbb{K} \wedge d\mathbb{K}), \quad (42)$$

where  $|K|^2 := g(K, K)$ ,  $\mathbb{K} := g(K, \cdot)$ , and  $*_g$  is the Hodge-\* of  $g$ . All EW structures arise in this way. Conversely, let  $(h, \omega)$  be a three-dimensional Lorentzian EW structure on  $\mathcal{W}$ , and let  $(V, \eta)$  be a function and a 1-form on  $\mathcal{W}$  satisfying the generalised monopole equation

$$*_h(dV + \frac{1}{2}\omega V) = d\eta, \quad (43)$$

where  $*_h$  is the Hodge-\* of  $h$ . Then

$$g = V^2 h - (d\phi + \eta)^2$$



is a neutral ASD metric with non-null Killing vector  $\partial_\phi$ .

This is a local theorem, so we may assume  $\mathcal{W}$  is a manifold. A vector in  $\mathcal{W}$  is a vector field Lie-derived along the corresponding trajectory in  $\mathcal{M}$ , and one applies the formulae (42) to this vector field to obtain  $([h], \omega)$  on  $\mathcal{W}$ . In the Riemannian case it has been successfully applied globally in certain nice cases [33]. When one performs a conformal transformation of  $g$ , one obtains a conformal transformation of  $h$  and the required transformation of  $\omega$ , so this is a theorem about conformal structures, though we have phrased it in terms of particular metrics.

The Jones–Tod construction was originally discovered using twistor theory in [26]; since then other purely differential-geometric proofs have appeared [27], [11]; although these are in Riemannian signature the arguments carry over to the neutral case. In Section 5 we explain the twistorial argument that originally motivated the theorem.

**4.1.3 Integrable systems and the Calderbank–Pedersen construction.** Applying the Jones–Tod correspondence to the special ASD conditions discussed in Section 2 will yield special integrable systems in  $2 + 1$  dimensions. In each case of interest we shall assume that the symmetry preserves the special geometric structure in four dimensions. This will give rise to special Einstein–Weyl backgrounds, together with general solutions of the generalised monopole equation (43) on these backgrounds. We can then seek special monopoles such that the resulting ASD structure is conformal to pseudo-hyper-Kähler.

An elegant framework for this is provided by the Calderbank–Pedersen construction [11]. In this construction self-dual complex (or null) structures on  $\mathcal{M}$  correspond to shear-free geodesic congruences (SFGC) on  $\mathcal{W}$ . This gives rise to a classification of three-dimensional EW spaces according to the properties of associated congruences. Below we shall list the resulting reductions and integrable systems. In each case we shall specify the properties of the associated congruence without going into the details of the Calderbank–Pedersen correspondence.

**Scalar-flat Kähler with symmetry. The  $SU(\infty)$ -Toda equation.** Let  $(\mathcal{M}, g)$  be a scalar-flat Kähler metric in neutral signature, with a symmetry  $K$  Lie deriving the Kähler form  $\omega$ . One can follow the steps of LeBrun [33] to reduce the problem to a pair of coupled PDEs: the  $SU(\infty)$ -Toda equation and its linearisation. The key step in the construction is to use the moment map for  $K$  as one of the coordinates, i.e. define a function  $t: \mathcal{M} \rightarrow \mathbb{R}$  by  $dt = K \lrcorner \omega$ . Then  $x, y$  arise as isothermal coordinates on two-dimensional surfaces orthogonal to  $K$  and  $dt$ . The metric takes the form

$$g = V(e^u(dx^2 + dy^2) - dt^2) - \frac{1}{V}(d\phi + \eta)^2, \quad (44)$$

where the function  $u$  satisfies the  $SU(\infty)$ -Toda equation

$$(e^u)_{tt} - u_{xx} - u_{yy} = 0, \quad (45)$$

and  $V$  is a solution to its linearization – the generalised monopole equation (43). The corresponding EW space from the Jones–Tod construction is

$$h = e^u(dx^2 + dy^2) - dt^2, \quad \omega = 2u_t dt. \quad (46)$$

It was shown in [52] that the EW spaces that can be put in the form (45) are precisely those possessing a shear-free twist-free geodesic congruence. Given the Toda EW space, any solution to the monopole equation will yield a  $(+ + - -)$  scalar flat Kähler metric. The special solution  $V = cu_t$ , where  $c$  is a constant, will lead to a pseudo-hyper-Kähler metric with symmetry.

In [35] solutions to (45) were used to construct neutral ASD Ricci flat metrics without symmetries.

**ASD null-Kähler with symmetry. The dKP equation.** Let  $(\mathcal{M}, g, N)$  be an ASD null Kähler structure with a Killing vector  $K$  such that  $\mathcal{L}_K N = 0$ . In [15] it was demonstrated that there exist smooth real valued functions  $H = H(x, y, t)$  and  $W = W(x, y, t)$  such that

$$g = W_x(dy^2 - 4dxdt - 4H_x dt^2) - W_x^{-1}(d\phi - W_x dy - 2W_y dt)^2 \quad (47)$$

is an ASD null-Kähler metric on a circle bundle  $\mathcal{M} \rightarrow \mathcal{W}$  if

$$H_{yy} - H_{xt} + H_x H_{xx} = 0, \quad (48)$$

$$W_{yy} - W_{xt} + (H_x W_x)_x = 0. \quad (49)$$

All real analytic ASD null-Kähler metrics with symmetry arise from this construction.

With definition  $u = H_x$  the  $x$  derivative of equation (48) becomes

$$(u_t - uu_x)_x = u_{yy},$$

which is the dispersionless Kadomtsev–Petviashvili equation originally used in [17]. The corresponding Einstein–Weyl structure is

$$h = dy^2 - 4dxdt - 4udt^2, \quad \omega = -4u_x dt.$$

This EW structure possesses a covariant constant null vector with weight  $-\frac{1}{2}$ , and in fact every such EW structure with this property can be put into the above form. The covariant constancy is with respect to a derivative on weighted vectors that preserves their weight. Details can be found in [17].

The linear equation (49) is a (derivative of) the generalised monopole equation from the Jones–Tod construction. Given a dKP Einstein–Weyl structure, any solution to this monopole equation will yield an ASD Null Kähler structure in four dimensions. The special monopole  $V = H_x/2$  will yield a pseudo-hyper-Kähler structure with symmetry whose self-dual derivative is null.

**Pseudo-hypercomplex with symmetry. The hyper-CR equation.** Let us assume that a pseudo-hypercomplex four manifold admits a symmetry which Lie derives all (pseudo) complex structures. This implies [16] that the EW structure is locally given by

$$h = (dy + udt)^2 - 4(dx + wdt)dt, \quad \omega = u_x dy + (uu_x + 2u_y)dt,$$

where  $u(x, y, t)$  and  $w(x, y, t)$  satisfy a system of quasi-linear PDEs

$$u_t + w_y + uw_x - wu_x = 0, \quad u_y + w_x = 0. \quad (50)$$

The corresponding pseudo-hypercomplex metric will arise from any solution to this coupled system, and its linearisation (the generalised monopole (43)). The special monopole  $V = u_x/2$  leads to pseudo-hyper-Kähler metric with triholomorphic homothety.

**4.2 Null case.** Given a neutral four-dimensional ASD conformal structure  $(\mathcal{M}, [g])$  with a null conformal Killing vector  $K$ , the three-dimensional space of trajectories of  $K$  inherits a degenerate conformal structure of signature  $(+ - 0)$ , and the Jones–Tod construction does not hold. The situation was investigated in detail in [18] and [10]. It was shown that  $K$  defines a pair of totally null foliations of  $\mathcal{M}$ , one by  $\alpha$ -surface and one by  $\beta$ -surfaces; these foliations intersect along integral curves of  $K$  which are null geodesics. In spinors, if  $K^a = \iota^A o^{A'}$  then an  $\alpha$ -plane distribution is defined by  $o^{A'}$ , and a  $\beta$ -plane distribution by  $\iota^A$ , and it follows from the Killing equation that these distributions are integrable.

The main result from [18] is that there is a canonically defined *projective structure* on the two-dimensional space of  $\beta$ -surfaces  $U$  which arises as a quotient of  $\mathcal{M}$  by a distribution  $\iota^A e_{AA'}$ . A more general framework where the distribution  $\iota^A e_{AA'}$  is still integrable, but  $\iota^A o^{A'}$  is not a symmetry for any  $o^{A'} \in \Gamma(S')$  was recently developed by Calderbank [10] and extended by Nakata [42].

A projective structure is an equivalence class of connections, where two connections are equivalent if they have the same unparameterized geodesics. In Section 5 we will explain the twistor theory that led to the observation that projective structures are involved, and give a new example of a twistor construction.

It turns out that one can explicitly write down all ASD conformal structures with null conformal Killing vectors in terms of their underlying projective structures as follows:

**Theorem 6** ([18]). *Let  $(\mathcal{M}, [g], K)$  be a smooth neutral signature ASD conformal structure with null conformal Killing vector. Then there exist local coordinates  $(\phi, x, y, z)$  and  $g \in [g]$  such that  $K = \partial_\phi$  and  $g$  has one of the following two forms, according to whether the twist  $\mathbb{K} \wedge d\mathbb{K}$  vanishes or not ( $\mathbb{K} := g(K, \cdot)$ ):*

1.  $\mathbb{K} \wedge d\mathbb{K} = 0$ .

$$\begin{aligned} g = & (d\phi + (zA_3 - Q)dy)(dy - \beta dx) \\ & - (dz - (z(-\beta_y + A_1 + \beta A_2 + \beta^2 A_3))dx \\ & - (z(A_2 + 2\beta A_3) + P)dy)dx, \end{aligned} \quad (51)$$

where  $A_1, A_2, A_3, \beta, Q, P$  are arbitrary functions of  $(x, y)$ .

2.  $\mathbb{K} \wedge d\mathbb{K} \neq 0$ .

$$g = (d\phi + A_3\partial_z G dy + (A_2\partial_z G + 2A_3(z\partial_z G - G) - \partial_z\partial_y G)dx) \cdot (dy - zdx) - \partial_z^2 G dx(dz - (A_0 + zA_1 + z^2A_2 + z^3A_3)dx), \quad (52)$$

where  $A_0, A_1, A_2, A_3$  are arbitrary functions of  $(x, y)$ , and  $G$  is a function of  $(x, y, z)$  satisfying the following PDE:

$$(\partial_x + z\partial_y + (A_0 + zA_1 + z^2A_2 + z^3A_3)\partial_z)\partial_z^2 G = 0. \quad (53)$$

The functions  $A_i(x, y)$  in the metrics (51) and (52) determine projective structures on the two-dimensional space  $U$  in the following way. A two projective structure in two dimensions is equivalent to a second-order ODE

$$\frac{d^2 y}{dx^2} = A_3(x, y)\left(\frac{dy}{dx}\right)^3 + A_2(x, y)\left(\frac{dy}{dx}\right)^2 + A_1(x, y)\left(\frac{dy}{dx}\right) + A_0(x, y), \quad (54)$$

obtained by choosing local coordinates  $(x, y)$  and eliminating the affine parameter from the geodesic equation. The  $A_i$  functions can be expressed in terms of combinations of connection coefficients that are invariant under projective transformation. In (52) all the  $A_i, i = 0, 1, 2, 3$  functions occur explicitly in the metric. In (51) the function  $A_0$  does not explicitly occur. It is determined by the following equation:

$$A_0 = \beta_x + \beta\beta_y - \beta A_1 - \beta^2 A_2 - \beta^3 A_3. \quad (55)$$

If the projective structure is flat, i.e.  $A_i = 0$  and  $\beta = P = 0$  then (51) is Ricci flat [47], and in fact this is the most general ASD Ricci flat metric with a null Killing vector which preserves the pseudo-hyper-Kähler structure [4]. More generally, if the projective structure comes from a Riemannian metric on  $U$  then there will always exist a (pseudo-)Kähler structure in the conformal class  $[g]$  if  $G = z^2/2 + \gamma(x, y)z + \delta(x, y)$  for certain  $\gamma, \delta$  [9].

It is interesting that integrable systems are not involved in the null case, given their ubiquity in the non-null case.

## 5 Twistor theory

In Riemannian signature, given an ASD conformal structure  $(\mathcal{M}, [g])$  in four dimensions one can form a 2-sphere bundle over it, and endow this with an integrable complex structure by virtue of anti-self-duality [3]. The resulting complex manifold  $\mathcal{PT}$  is called the twistor space. The original manifold is the moduli space of rational curves in  $\mathcal{PT}$  preserved under a certain anti-holomorphic involution, and one can recover the conformal structure by looking at how the rational curves intersect one another. Hence the  $(\mathcal{M}, [g])$  is completely encoded in  $\mathcal{PT}$  and its anti-holomorphic involution. The

important feature of a successful twistor construction is that the original geometry becomes encoded in the holomorphic geometry of the twistor space, and can be recovered from this.

Neutral signature ASD conformal structures cannot be encoded purely in holomorphic geometry as in the Riemannian case. This is not surprising as generically they are not analytic. However, there is a recent twistor construction due to LeBrun–Mason [34] in the neutral case that uses a mixture of holomorphic and smooth ingredients; we review this in Section 5.2. Let us now review the differences in Riemannian and neutral signature.

In the Riemannian case, if one expresses the metric in terms of a null tetrad as in (9) then the basis vectors  $e_{AA'}$  must be complex, as there are no real null vectors. The spin bundles are complex two-dimensional vector bundles  $\mathbb{S}, \mathbb{S}'$ , with an isomorphism  $T_{\mathbb{C}}\mathcal{M} \cong \mathbb{S} \otimes \mathbb{S}'$ , at least locally. One then takes the projective bundle  $\mathbb{P}\mathbb{S}'$ , which has  $\mathbb{C}\mathbb{P}^1$  fibres. Even if  $\mathbb{S}'$  does not exist globally, the bundle  $\mathbb{P}\mathbb{S}'$  does exist globally, since the  $\mathbb{Z}_2$  obstruction to existence of a spin bundle is eliminated on projectivizing. Concretely,  $\mathbb{P}\mathbb{S}'$  is the bundle of complex self-dual totally null 2-planes; from this description it clearly exists globally.

Now one can form the  $L_A$  vectors as in Theorem 1, where now  $\pi^{A'}$  are complex (the homogeneous fibre coordinates of  $\mathbb{P}\mathbb{S}'$ ). The connection coefficients in the expression for  $\tilde{e}_{AA'}$  will now be complex, and satisfy certain Hermiticity properties that we need not go into. The  $L_A$  span a complex two-dimensional distribution on the complexified tangent space of  $\mathbb{P}\mathbb{S}'$ , and the Riemannian version of Theorem 1 is that this distribution is complex integrable iff the metric is ASD. Together with  $\partial_{\bar{\lambda}}$ , where  $\lambda$  is the inhomogeneous fibre coordinate on  $\mathbb{P}\mathbb{S}'$ , we obtain a complex three-dimensional distribution  $\Pi$ , satisfying  $\Pi \cap \bar{\Pi} = 0$ . If the metric is ASD,  $\Pi$  is complex integrable and defines a complex structure on  $\mathbb{P}\mathbb{S}'$ . This construction works globally. It was discovered by Atiyah, Hitchin and Singer [3].

In the neutral case one can complexify the real spin bundles  $S, S'$  and obtain  $T_{\mathbb{C}}\mathcal{M} \cong S_{\mathbb{C}} \otimes S'_{\mathbb{C}}$  as in the Riemannian case. One can define a complex distribution  $\Pi$  distribution on  $PS'_{\mathbb{C}}$ , by allowing  $\pi^{A'}$  in Theorem 1 to be complex. The key point is that the vectors  $L_A$  become totally real when  $\pi^{A'}$  is real. So on the hypersurface  $PS' \subset PS'_{\mathbb{C}}$ , the distribution  $\text{span}\{\Pi, \partial_{\bar{\lambda}}\}$  no longer satisfies  $\Pi \cap \bar{\Pi} = 0$ , so does not define an almost complex structure. When  $\pi^{A'}$  is not real, the distribution spanned by  $L_A$  and  $\partial_{\bar{\lambda}}$  does define an almost complex structure, which is integrable when  $g$  is ASD. We obtain two non-compact regions in  $PS'_{\mathbb{C}}$ , each of which possesses an integrable complex structure, separated by a hypersurface  $PS'$ . This is more complicated than the Riemannian case, where the end result is simply a complex manifold. Nevertheless, the construction is reversible in a precise sense given by Theorem of LeBrun–Mason which we review in Section 5.2 (Theorem 7).

Before describing the work of LeBrun–Mason we review the analytic case, where one can complexify and work in the holomorphic category.

**5.1 The analytic case.** In this section we work locally. Standard references for this material are [55], [22].

Suppose a neutral four-dimensional ASD conformal structure  $(\mathcal{M}, [g])$  is analytic in some coordinate system. Then we can complexify by letting the coordinates become complex variables, and we obtain a holomorphic conformal structure  $(\mathcal{M}_{\mathbb{C}}, [g_{\mathbb{C}}])$ . If each coordinate is defined in some connected open set on  $\mathbb{R}$ , then one thickens this slightly on both sides of the axis to obtain a region in  $\mathbb{C}$  on which the complex coordinate is defined. The holomorphic conformal structure is obtained by picking a real metric  $g$  and allowing the coordinates to be complex to obtain  $g_{\mathbb{C}}$ . Then  $[g_{\mathbb{C}}]$  is the equivalent class of  $g_{\mathbb{C}}$  up to multiplication by nonzero holomorphic functions.

From Theorem 1, which is valid equally for holomorphic metrics, we deduce that given any holomorphic  $\alpha$ -plane at a point, there is a holomorphic  $\alpha$ -surface through that point. Assuming we are working in a suitably convex neighbourhood so that the space of such  $\alpha$ -surfaces is Hausdorff, we define  $\mathcal{PT}$  to be this space.  $\mathcal{PT}$  is a three-dimensional complex manifold, since the space of  $\alpha$ -planes at a point is complex one-dimensional and each surface is of complex codimension two in  $\mathcal{M}_{\mathbb{C}}$ . This is summarised in the double fibration picture

$$\mathcal{M}_{\mathbb{C}} \xleftarrow{p} PS'_{\mathbb{C}} \xrightarrow{q} \mathcal{PT}, \quad (56)$$

where  $q$  is the quotient by the twistor distribution  $L_A$ .

If we had started with a Riemannian metric this would lead to the same twistor space, locally, as the Atiyah–Hitchin–Singer construction described above, though we shall not demonstrate this here. A point  $x \in \mathcal{M}$ , corresponds to an embedded  $\mathbb{CP}^1 \subset \mathcal{PT}$ , since there is a  $\mathbb{CP}^1$  of  $\alpha$ -surfaces through  $x$ . By varying the point  $x \in \mathcal{M}$  we obtain a four complex parameter family of  $\mathbb{CP}^1$ 's.

$\mathcal{PT}$  inherits an anti-holomorphic involution  $\sigma$ . To describe  $\sigma$ , note that there is an anti-holomorphic involution  $\tau$  of  $\mathcal{M}_{\mathbb{C}}$  that fixes real points, i.e. points of  $\mathcal{M} \subset \mathcal{M}_{\mathbb{C}}$ . This is just the map from a coordinate to its complex conjugate, so we can arrange our complexification regions in which the coordinates are defined so that  $\tau$  maps the regions to themselves. Now  $\tau$  will map holomorphic  $\alpha$ -surfaces to holomorphic  $\alpha$ -surfaces, so gives an anti-holomorphic involution  $\sigma$  on  $\mathcal{PT}$ . One way to see this is to note that  $\alpha$ -surfaces are totally geodesic as the geodesic shear free condition

$$\pi^{A'} \pi^{B'} \nabla_{AA'} \pi_{B'} = 0$$

is equivalent to  $C_{A'B'C'D'}$ , and consider the holomorphic geodesic equation. Using the fact that the connection coefficients are real, one can show that the involution  $\tau$  will map the null geodesics in an  $\alpha$ -surface to other null geodesics in another  $\alpha$ -surface. The  $\alpha$ -surfaces fixed by this are the real  $\alpha$ -surfaces in  $\mathcal{M}$ .

In terms of  $\mathcal{PT}$ , this last fact means that  $\sigma$  fixes an equator of each of the four complex parameter family of embedded  $\mathbb{CP}^1$ 's. Moreover, an  $\alpha$ -surface through a real point gets mapped to one through that same point since the point is fixed by  $\tau$ . So the  $\mathbb{CP}^1$ 's that are fixed by  $\sigma$  are a four real parameter family corresponding to  $\mathcal{M}$ , we call these *real*  $\mathbb{CP}^1$ 's.

How does one recover the neutral conformal structure from the data  $(\mathcal{PT}, \sigma)$ ? As described above,  $\mathcal{M}$  is the moduli space of  $\mathbb{CP}^1$ 's fixed by  $\sigma$ . Now a vector at a point in  $\mathcal{M}$  corresponds to a holomorphic section of the normal bundle  $\mathcal{O}(1) \oplus \mathcal{O}(1)$  of the corresponding real  $\mathbb{CP}^1$  in  $\mathcal{PT}$ , such that the section ‘points’ to another real  $\mathbb{CP}^1$ . We define a vector to be null if this holomorphic section has a zero. Since vanishing of a section of  $\mathcal{O}(1) \oplus \mathcal{O}(1)$  is a quadratic condition, this gives a conformal structure. One can prove that this conformal structure is ASD fairly easily, by showing that the required  $\alpha$ -surfaces must exist in terms of the holomorphic geometry.

Moreover, special conditions on a  $g_{\mathbb{C}} \in [g_{\mathbb{C}}]$  can be encoded into the holomorphic geometry of the twistor space:

- Holomorphic fibration  $\theta: \mathcal{PT} \rightarrow \mathbb{CP}^1$  corresponds to hyper-hermitian conformal structures [5], [14].
- Preferred section of  $\kappa^{-1/2}$  which vanishes at exactly two points on each twistor line corresponds to scalar-flat Kähler  $g_{\mathbb{C}}$  [49].
- Preferred section of  $\kappa^{-1/4}$  corresponds to ASD null-Kähler  $g_{\mathbb{C}}$  [15].
- Holomorphic fibration  $\theta: \mathcal{PT} \rightarrow \mathbb{CP}^1$  and holomorphic isomorphism

$$\theta^* \mathcal{O}(-4) \cong \kappa$$

correspond to hyper-Kähler  $g_{\mathbb{C}}$  [45], [3], [22].

Here  $\kappa$  is a holomorphic canonical bundle of  $\mathcal{PT}$ , and  $\mathcal{O}(-4)$  is a power of the tautological bundle on the base of  $\theta$ . To obtain a real metrics the structures above must be preserved by an anti-holomorphic involutions fixing a real equator of each rational curve in  $\mathcal{PT}$ .

It is worth saying a few words about the construction of solutions of integrable systems using the twistor correspondence. It is shown in Section 4 that a number of well-known integrable systems 2 + 1 dimensions are special cases of ASD conformal structures. Analytic solutions to these integrable systems therefore correspond<sup>6</sup> to twistor spaces  $\mathcal{PT}$ . There will be extra conditions on  $\mathcal{PT}$ , depending on the special case in question. However, solutions to the integrable systems are not always analytic.

**5.1.1 Symmetries and twistor spaces.** In Section 4 we discussed the appearance of Einstein–Weyl structures and projective structures in the cases of a non-null and null Killing vector respectively. In both cases twistor theory was the key factor in revealing these correspondences. We shall now explain this briefly. In [22], Hitchin gave three twistor correspondences. He considered complex manifolds containing embedded  $\mathbb{CP}^1$ 's with normal bundles  $\mathcal{O}(1)$ ,  $\mathcal{O}(2)$  and  $\mathcal{O}(1) \oplus \mathcal{O}(1)$  respectively. Kodaira deformation theory guarantees a local moduli space of embedded  $\mathbb{CP}^1$ 's, whose complex dimension is the dimension of the space of holomorphic sections of the corresponding normal bundle, i.e. 2, 3, 4 respectively. By examining how nearby curves intersect, he

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<sup>6</sup>This correspondence is not one-one due to coordinate freedom.

deduced that the moduli space inherits a holomorphic projective structure, Einstein–Weyl structure, or ASD conformal structure respectively. He also showed that the construction is reversible in each case.

Now given a four-dimensional holomorphic ASD conformal structure, its twistor space is the space of  $\alpha$ -surfaces, as described in Section 5.1. A conformal Killing vector preserves the conformal structure, so preserves  $\alpha$ -surfaces, giving a holomorphic vector field on the twistor space.

If the Killing vector is non-null then the vector field on twistor space  $\mathcal{PT}$  is non-vanishing. This is because the Killing vector is transverse to any  $\alpha$ -surface, as it is non-null. In this case one can quotient the three-dimensional twistor space by the induced vector field, and it can be shown [26] that the resulting two-dimensional complex manifold contains  $\mathbb{CP}^1$ 's with normal bundle  $\mathcal{O}(2)$ . Using Hitchin's results, this corresponds to a three-dimensional Einstein–Weyl structure. This is the twistorial version of the Jones–Tod construction, Theorem 5.

If the Killing vector is null then the induced vector field on the twistor space  $\mathcal{PT}$  vanishes on a hypersurface. This is because at each point, the Killing vector is tangent to a single  $\alpha$  surface. Hence it preserves a foliation by  $\alpha$ -surfaces, and vanishes at the hypersurface in twistor space corresponding to this foliation. However, one can show [18] that it is possible to continue the vector field on twistor space to a one-dimensional distribution  $\hat{K}$  that is nowhere vanishing. Quotienting  $\mathcal{PT}$  by this distribution gives a two-dimensional complex manifold  $Z$  containing  $\mathbb{CP}^1$ 's with normal bundle  $\mathcal{O}(1)$ . Using Hitchin's results, this corresponds to a two-dimensional projective structure. This is the twistorial version of the correspondence described in Section 4.2. The situation is illustrated by the following diagram.

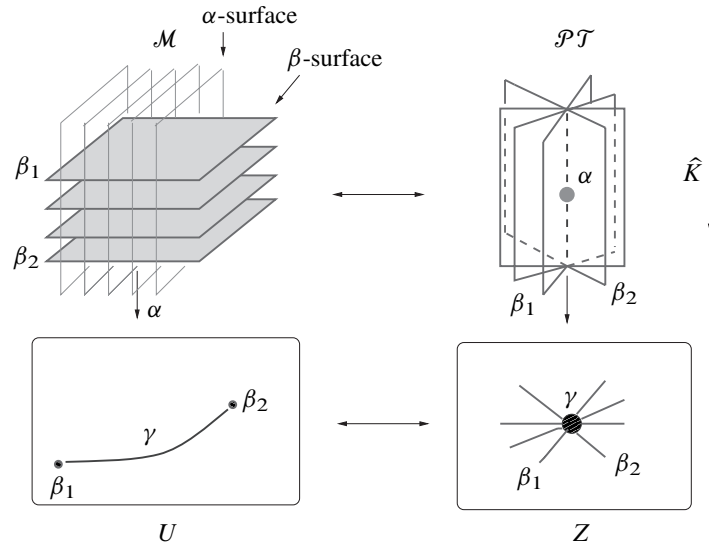


Figure 1. Relationship between  $\mathcal{M}$ ,  $U$ ,  $\mathcal{PT}$  and  $Z$ .



In  $\mathcal{M}$ , a one parameter family of  $\beta$ -surface is shown, each of which intersects a one parameter family of  $\alpha$ -surfaces, also shown. The  $\beta$ -surfaces correspond to a projective structure geodesic in  $U$ , shown at the bottom left.

The  $\beta$ -surfaces in  $\mathcal{M}$  correspond to surfaces in  $\mathcal{PT}$ , as discussed above. These surfaces intersect at the dotted line, which corresponds to the one parameter family of  $\alpha$ -surfaces in  $\mathcal{M}$ . When we quotient  $\mathcal{PT}$  by  $\hat{K}$  to get  $Z$ , the surfaces become twistor lines in  $Z$ , and the dotted line becomes a point at which the twistor lines intersect; this is shown on the bottom right. This family of twistor lines intersecting at a point corresponds to the geodesic of the projective structure.

**Example<sup>7</sup>.** Here we give an explicit construction of the twistor space of an analytic neutral ASD conformal structure with a null Killing vector, from the reduced projective structure twistor space. We take  $Z$  to be the total space of  $\mathcal{O}(1)$ . This is the twistor space of the flat projective structure. Now suppose we are given a 1-form  $\omega$  on  $U$ . We shall complexify the setup and regard  $\omega$  as holomorphic a holomorphic connection on a holomorphic line bundle  $B \rightarrow U$ . This gives rise to a holomorphic line bundle  $E \rightarrow Z$ , where the vector space over  $z \in Z$  is the space of parallel sections of  $B$  over the geodesic in  $U$  corresponding to  $z$ . The *twistor lines* in  $Z$  are the two-parameter family of embedded  $\mathbb{CP}^1$ 's, each corresponding to the set of geodesics through a single point in  $U$ . We denote the twistor line corresponding to a point  $x \in U$  by  $\hat{x}$ . Now  $E$  restricted to a twistor line  $\hat{x}$  is trivial, because to specify a parallel section of  $B$  through any geodesic through  $x$ , one need only know its value at  $x$ . This is a simple analogue of the Ward correspondence relating solutions of the anti-self-dual Yang–Mills equations on  $\mathbb{C}^4$  to vector bundles over the total space of  $\mathcal{O}(1) \oplus \mathcal{O}(1)$  that are trivial on twistor lines. The situation here is simpler since there are no PDEs involved; this is because there are no integrability conditions for a space of parallel sections to exist on a line. As with the Ward correspondence, the construction is reversible, i.e. given a holomorphic line bundle trivial on twistor lines one can find a connection on  $U$  to which it corresponds in the manner described above. We will not prove this here, it is simply a case of mimicking the argument for the Ward correspondence [55].

Now to create the twistor space  $\mathcal{PT}$ , we must tensor  $E$  with a line bundle  $L$  so that  $E \otimes L$  restricts to  $\mathcal{O}(1)$  on the twistor lines in  $Z$ . Then the total space of  $E \otimes L$  will have embedded  $\mathbb{CP}^1$ 's with normal bundle  $\mathcal{O}(1) \oplus \mathcal{O}(1)$ , so will be a twistor space for an ASD conformal structure. For  $L$  we choose the pull back of  $\mathcal{O}(1)$  to the total space of  $Z$ .

Let us now make the above explicit. Let  $\lambda, \tilde{\lambda}$  be the inhomogeneous coordinate on the two patches  $U_0, U_1$  of  $\mathbb{CP}^1$ . The total space of  $\mathcal{O}(1)$  can be coordinatized as follows. Let  $\mu$  be the fibre coordinate over  $U_0$ , and  $\tilde{\mu}$  the fibre coordinate over  $U_1$ . The line bundle transition relation on the overlap is  $\tilde{\mu} = \frac{1}{\lambda}\mu$ .

Now suppose we have a line bundle  $E \rightarrow Z = \mathcal{O}(1)$ , that is trivial on holomorphic sections of  $Z \rightarrow \mathbb{CP}^1$ . Let  $\tau, \tilde{\tau}$  be the fibre coordinates on the two patches, satisfying a transition relation  $\tilde{\tau} = F(\lambda, \mu)\tau$ , where  $F(\lambda, \mu)$  is holomorphic and nonvanishing on the overlap, i.e. for  $\lambda \in \mathbb{C} - \{0\}$ ,  $\mu \in \mathbb{C}$ . In sheaf terms,  $F$  is an element of

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<sup>7</sup>We thank Paul Tod for his help with this example.

$H^1(\mathcal{O}(1), \mathcal{O}^*)$ . Now the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0 \quad (57)$$

gives rise to a long exact sequence, part of which is:

$$\dots \rightarrow H^1(\mathcal{O}(1), \mathbb{Z}) \rightarrow H^1(\mathcal{O}(1), \mathcal{O}) \rightarrow H^1(\mathcal{O}(1), \mathcal{O}^*) \rightarrow H^2(\mathcal{O}(1), \mathbb{Z}) \rightarrow \dots \quad (58)$$

The first term in (58) vanishes and the final term is  $\mathbb{Z}$ , by topological considerations. The final term gives the Chern class of the line bundle determined by the element of  $H^1(\mathcal{O}(1), \mathcal{O}^*)$ . This vanishes for  $E$ , since it is trivial on twistor lines. The third arrow in (57) is the exponential map. Together these facts imply that  $F$  can be written  $F(\lambda, \mu) = e^{f(\lambda, \mu)}$ , where  $f(\lambda, \mu)$  is a holomorphic function on the overlap that may have zeros. After twisting by  $L$ , we obtain the following transition function for  $E \otimes L$ , again using  $\tau, \tilde{\tau}$  as fibre coordinates:

$$\tilde{\tau} = \frac{1}{\lambda} e^{f(\lambda, \mu)} \tau. \quad (59)$$

To find the conformal structure we must find the four parameter family of twistor lines in  $E \otimes L$ . The two parameter family in  $\mathcal{O}(1)$  is given in one patch by  $\mu(\lambda) = X\lambda + Y$ , and in the other by  $\tilde{\mu}(\tilde{\lambda}) = X + \tilde{\lambda}Y$ . Restricting to one of these we can split  $f$ :

$$f(\lambda, X\lambda + Y) = h(X, Y, \lambda) - \tilde{h}(X, Y, 1/\lambda), \quad (60)$$

where  $h$  and  $\tilde{h}$  are functions on  $U \times \mathbb{C}\mathbb{P}^1$  holomorphic in  $\lambda$  and  $1/\lambda$  respectively. For fixed  $(X, Y)$  there is then a further two parameter family of twistor lines, given by

$$\tau(\lambda) = e^{-h(X, Y, \lambda)} (W - \lambda Z) \quad (61)$$

in one patch, and

$$\tilde{\tau}(\tilde{\lambda}) = e^{-\tilde{h}(X, Y, \tilde{\lambda})} (\tilde{\lambda}W - Z). \quad (62)$$

It is easy to check that (59) is satisfied by (61) and (62).

One must now calculate the conformal structure on the moduli space of lines parametrised by  $X^a = (X, Y, W, Z)$  by determining the quadratic condition for a section of the normal bundle to a twistor line to vanish. The sections of the normal bundle to  $\hat{x} \subset \mathcal{PT}$  correspond to tangent vectors in  $T_x\mathcal{M}$ , and sections with one zero will determine null vectors and therefore the conformal structure.

Using the identity  $(\partial_X - \lambda\partial_Y)f = 0$  together with (60) we deduce (by Liouville theorem or using power series) that

$$\lambda \frac{\partial h}{\partial Y} - \frac{\partial h}{\partial X} = \lambda B(X, Y) - A(X, Y), \quad (63)$$

for some analytic functions  $A, B$ .

Now take the variation of  $\mu(\lambda)$  and  $\tau(\lambda)$  for a small change  $\delta X^a$  to obtain

$$0 = \delta\mu = \delta Y + \lambda\delta X, \quad (64)$$

$$0 = \delta\tau = e^{-h} \left( -\frac{\partial h}{\partial X} \delta X - \frac{\partial h}{\partial Y} \delta Y \right) (W - \lambda Z) + e^{-h} (\delta W - \lambda\delta Z). \quad (65)$$

Substituting  $\lambda = -\delta Y/\delta X$  from the first expression to the second, using (63) and multiplying the resulting expression by  $\delta X$  we find that the conformal structure is represented by the following metric:

$$g = dXdW + dYdZ - (WdX + ZdY)(A(X, Y)dX + B(X, Y)dY). \quad (66)$$

This conformal structure possesses the null conformal Killing vector  $K = W\partial_W + Z\partial_Z$ , which is twisting. The global holomorphic vector field on  $\mathcal{PT}$  induced by  $K$  is  $\tau\partial_\tau = \tilde{\tau}\partial_{\tilde{\tau}}$  where the equality holds on the intersection of the two coordinate patches. This vanishes on the hypersurface defined by  $\tau = 0$  in one patch and  $\tilde{\tau} = 0$  in the other, which intersects each twistor line at a single point, as we expect from the argument in Section 5.1.1. The 1-form  $\omega = AdX + BdY$  in  $g$  is the inverse Ward transform of  $F \in H^1(\mathcal{O}(1), \mathcal{O}^*)$ .

To compare with (52) one must transform to coordinates  $(\phi, x, y, z)$  in which  $K = \partial_\phi$ . Dividing by a conformal factor  $W$ , transforming with  $(\phi, x, y, z) = (\log W, Y, -X, Z/W)$ , and then translating  $\phi$  to eliminate an arbitrary one function of  $(x, y)$  gives

$$g = (d\phi + f(x, y)dx)(dy - zdx) - dzdx, \quad (67)$$

which is a special case of (52) with flat projective structure, and  $G = z^2/2 - zC(x, y)$ , where  $f = \partial_y C$ .

If we take the coordinates to be real we obtain a neutral metric. The twistor space  $\mathcal{PT}$  fibres over  $Z = \mathcal{O}(1)$  and this fibres over  $\mathbb{CP}^1$ , so  $\mathcal{PT}$  fibers over  $\mathbb{CP}^1$  and (67) is pseudo-hyperhermitian.

To construct an example of a conformal structure with *non-twisting* null Killing vector one uses an affine line bundle over  $Z = \mathcal{O}(1)$ ; see [18] for details.

**5.2 LeBrun–Mason construction.** Here we describe recent work of LeBrun and Mason in which a general, global twistor construction is given for neutral metrics. We will only be able to give a crude paraphrase, and refer the reader to the original paper [34] for details. Note that their paper uses the opposite duality conventions to ours; they use self-dual conformal structures with integrable  $\beta$ -plane distributions.

We described above how a neutral ASD conformal structure  $(\mathcal{M}, [g])$  gives rise to a complex structure on  $\mathbb{CP}^1$  bundle over  $\mathcal{M}$ , which degenerates on a hypersurface. The following theorem of LeBrun–Mason is a converse to this, and is the closest one can come to a general twistor construction in the neutral case:

**Theorem 7** ([34]). *Let  $\mathcal{M}$  be a smooth connected 4-manifold, and let  $\varpi: \mathcal{X} \rightarrow \mathcal{M}$  be a smooth  $\mathbb{CP}^1$ -bundle. Let  $\rho: \mathcal{X} \rightarrow \mathcal{X}$  be an involution which commutes with  $\varpi$ , and has as fixed-point set  $\mathcal{X}_\rho$  an  $S^1$ -bundle over  $\mathcal{M}$  which disconnects  $\mathcal{X}$  into two*

closed 2-disk bundles  $\mathcal{X}_\pm$  with common boundary  $\mathcal{X}_\varrho$ . Suppose that  $\Pi \subset T_{\mathbb{C}}\mathcal{X}$  is a distribution of complex 3-planes on  $\mathcal{X}$  such that

1.  $\varrho_*\Pi = \bar{\Pi}$ ;
2. the restriction of  $\Pi$  to  $\mathcal{X}_+$  is smooth and involutive,
3.  $\Pi \cap \bar{\Pi} = 0$  on  $\mathcal{X} - \mathcal{X}_\varrho$ ,
4.  $\Pi \cap \ker \varpi_*$  is the  $(0, 1)$  tangent space of the  $\mathbb{C}\mathbb{P}^1$  fibers of  $\varpi$ ,
5. the restriction of  $\Pi$  to a fiber of  $\mathcal{X}$  has first Chern class  $-4$  with respect to the complex orientation.

Then  $\mathcal{E} = \Pi \cap T\mathcal{X}_\varrho$  is an integrable distribution of real 2-planes on  $\mathcal{X}_\varrho$ , and  $\mathcal{M}$  admits a unique smooth split-signature ASD conformal structure  $[g]$  for which the  $\alpha$ -surfaces are the projections via  $\varpi$  of the integral manifolds of  $\mathcal{E}$ .

This theorem provides a global twistor construction for neutral ASD four manifolds, whereas the analytic construction of the last section only works locally.

At first sight the theorem does not seem like a promising method of generating ASD conformal structures, since the conditions required on the  $\mathbb{C}\mathbb{P}^1$  bundle over  $\mathcal{M}$  are complicated, and it is not clear how one might construct examples. This obstacle is overcome in [34] by deforming a simple example (another example was given by Nakata [41]).

Consider the conformally flat neutral metric  $g_0$  given by (7) on  $\mathcal{M} = S^2 \times S^2$  that is just the difference of the standard sphere metrics on each factor. The underlying manifold  $\mathcal{M}$  can be realised as the space of  $\mathbb{C}\mathbb{P}^1$ 's embedded in  $\mathbb{C}\mathbb{P}^3$  that are invariant under the complex conjugate involution, which we call the *real*  $\mathbb{C}\mathbb{P}^1$ 's; these real  $\mathbb{C}\mathbb{P}^1$ 's are the fibres of the bundle  $\mathcal{X}$ . The involution  $\varrho$  of  $\mathcal{X}$  is induced by the complex conjugate involution of  $\mathbb{C}\mathbb{P}^3$ . The fixed point set  $\mathcal{X}_\varrho$  consists of the invariant equators of the real  $\mathbb{C}\mathbb{P}^1$ 's, and is therefore a circle bundle over  $\mathcal{M}$ . The closed disc bundles  $\mathcal{X}_\pm$  are obtained by slicing the real  $\mathbb{C}\mathbb{P}^1$ 's at their invariant equator, and throwing away one of the open halves. The fixed point set of the complex conjugate involution is the standard embedding of  $\mathbb{R}\mathbb{P}^3$ , and this is the space of  $\alpha$ -surfaces in  $\mathcal{M} = S^2 \times S^2$ . Taking all the real  $\mathbb{C}\mathbb{P}^1$ 's through a point  $p \in \mathbb{R}\mathbb{P}^3$  gives an  $\alpha$ -surface.

To obtain the  $\Pi$  from Theorem 7, take  $\mathcal{X}_+$  and construct a map  $f$  to  $\mathbb{C}\mathbb{P}^3$  as follows. On the interior of  $\mathcal{X}_+$ ,  $f$  is a diffeomorphism onto  $\mathbb{C}\mathbb{P}^3 - \mathbb{R}\mathbb{P}^3$ . The boundary  $\partial\mathcal{X}_+$  gets mapped by  $f$  to  $\mathbb{R}\mathbb{P}^3$ , by taking a point in  $\partial\mathcal{X}_+$ , i.e. a holomorphic disc and a point  $p$  on  $\mathbb{R}\mathbb{P}^3$  lying on the intersection of the disc with  $\mathbb{R}\mathbb{P}^3$ , to the point  $p$ . Let  $f_*^{1,0}: T_{\mathbb{C}}\mathcal{X}_+ \rightarrow T^{1,0}\mathbb{C}\mathbb{P}^3$  be the  $(1, 0)$  part of the derivative of  $f$ . Then the  $\Pi$  of Theorem 7 is defined on  $\mathcal{X}_+$  by

$$\Pi = \ker f_*^{1,0} \subset T_{\mathbb{C}}\mathcal{X}_+.$$

Note that  $f$  maps the five-dimensional boundary  $\partial\mathcal{X}_+$  to the three-dimensional space  $\mathbb{R}\mathbb{P}^3$ ; this means that on the boundary  $\Pi$  restricts to the complexification of a real two-plane distribution, direct summed with the complexification of the direction into the disc. The  $\Pi$  here agrees with the one described at the beginning of Section 5, defined in terms of the twistor distribution  $L_A$  on  $PS'_C$ .

The idea for creating new spaces satisfying the conditions of Theorem 7 is to deform the standard embedding of  $\mathbb{R}\mathbb{P}^3 \subset \mathbb{C}\mathbb{P}^3$  slightly. One forms the bundle  $\mathcal{X}_+$  by taking the same  $S^2 \times S^2$  family of real holomorphic discs, now with boundary on the deformed embedding of  $\mathbb{R}\mathbb{P}^3$ . The complex distribution  $\Pi$  is formed in the same way as in the conformally flat case described above. One then patches two copies of  $\mathcal{X}_+$  together to form the bundle  $\mathcal{X}$ , and it is shown that this satisfies the conditions of the Theorem. It is also shown that the resulting conformal structure on  $S^2 \times S^2$  has the property that all null geodesics are embedded circles; conformal structures with this property are termed *Zollfrei*. It turns out that all ASD conformal structures close enough in a suitable sense to the conformally flat one are *Zollfrei*, and in fact the twistor description gives a complete understanding of ASD conformal structures near the standard one. The embedded  $\mathbb{R}\mathbb{P}^3$  is the real twistor space, i.e. the space of  $\alpha$ -surfaces in  $\mathcal{M}$ , and a significant portion of [34] is devoted to showing that for a space-time oriented *Zollfrei* 4-manifold the real twistor space must be  $\mathbb{R}\mathbb{P}^3$ , making contact with the picture of a deformed  $\mathbb{R}\mathbb{P}^3 \subset \mathbb{C}\mathbb{P}^3$ .

We mention that there is another twistor-like construction of smooth ASD conformal structures which avoids the holomorphic methods altogether [20]. In this approach one views the real twistor curves in  $\mathbb{R}\mathbb{P}^3$  as solutions to a system of two second order nonlinear ODEs. The ODEs have to satisfy certain conditions (expressed in terms of point invariants) if their solution spaces are equipped with ASD conformal structures.

## 6 Global results

In the last section we outlined the global twistor construction for neutral ASD four manifolds due to LeBrun–Mason, which they used to construct *Zollfrei* metrics on  $S^2 \times S^2$ . In this section we review the known explicit constructions of globally defined neutral ASD conformal structures on various compact and non-compact manifolds.

**6.1 Topological restrictions.** Existence of a neutral metric on a four manifold  $\mathcal{M}$  imposes topological restrictions on  $\mathcal{M}$ . A neutral inner product on a four-dimensional vector  $V$  space splits  $V$  into a direct sum  $V = V_+ \oplus V_-$ , where the inner product is positive definite on  $V_+$  and negative definite on  $V_-$ . So a neutral metric  $g$  on a four manifold  $\mathcal{M}$  splits the tangent bundle

$$T\mathcal{M} = T_+\mathcal{M} \oplus T_-\mathcal{M}, \quad (68)$$

where  $T_{\pm}$  are two-dimensional subbundles of  $T\mathcal{M}$ . Conversely, given such a splitting one can construct neutral metrics on  $\mathcal{M}$  by taking a difference of positive definite metrics on the vector bundles  $T_+\mathcal{M}$  and  $T_-\mathcal{M}$ .

If  $\mathcal{M}$  admits a non-vanishing 2-plane field  $\mathcal{E}$  (a real two-dimensional distribution), then a splitting of the form (68) can be found by taking  $\mathcal{E}$  to define  $T_+\mathcal{M}$ , choosing a Riemannian metric, and letting  $T_-\mathcal{M}$  be the orthogonal complement. So a four manifold  $\mathcal{M}$  admits a neutral metric iff it admits a 2-plane field.

The topological conditions for existence of a 2-plane field were discovered by Hirzebruch and Hopf and are as follows:

**Theorem 8** ([21]). *A compact smooth four-manifold  $\mathcal{M}$  admits a field of 2-planes iff  $\tau[\mathcal{M}]$  and  $\chi[\mathcal{M}]$  satisfy a pair of conditions*

$$\begin{aligned} 3\tau[\mathcal{M}] + 2\chi[\mathcal{M}] &\in \Omega(\mathcal{M}), \\ 3\tau[\mathcal{M}] - 2\chi[\mathcal{M}] &\in \Omega(\mathcal{M}), \end{aligned}$$

where

$$\Omega[\mathcal{M}] = \{\mu_{\mathcal{M}}(w, w) \in \mathbb{Z} : w \text{ are arbitrary elements in } H^2(\mathcal{M}, \mathbb{Z})/\text{Tor}\}.$$

Here  $\tau[\mathcal{M}]$ ,  $\chi[\mathcal{M}]$  are the signature and Euler characteristic respectively, and  $\mu_{\mathcal{M}}$  is the intersection form on  $H^2(\mathcal{M}, \mathbb{Z})/\text{Tor}$ .

A neutral metric implies that the structure group of the tangent bundle can be reduced to  $O(2, 2)$ , by choosing orthonormal bases in each patch. In fact  $O(2, 2)$  has four connected components, so there are various different orientability requirements one can impose. The simplest is to require the structure group to reduce to the identity component  $SO_+(2, 2)$ . It is shown in [39] that this is equivalent to the existence of a field of *oriented* 2-planes, i.e. an orientable two-dimensional subbundle of the tangent bundle. The topological restrictions imposed by this were discovered by Atiyah:

**Theorem 9** ([2]). *Let  $\mathcal{M}$  be a compact oriented smooth manifold of dimension 4, such that there exists a field of oriented 2-planes on  $\mathcal{M}$ . Then*

$$\chi[\mathcal{M}] \equiv 0 \pmod{2}, \quad \chi[\mathcal{M}] \equiv \tau[\mathcal{M}] \pmod{4}. \quad (69)$$

In fact Matsushita showed [38] that for a simply-connected 4-manifold, (69) are actually sufficient for the existence of an oriented field of 2-planes. A more subtle problem is to determine topological obstructions arising from existence of an ASD neutral metric. This deserves further study.

**6.2 Tod's scalar-flat Kähler metrics on  $S^2 \times S^2$ .** Consider  $S^2 \times S^2$  with the conformally flat metric described in Sections 2.1 and 5.2, i.e. the difference of the standard sphere metrics on each factor. Thinking of each sphere as  $\mathbb{C}\mathbb{P}^1$  and letting  $\zeta$  and  $\chi$  be non-homogeneous coordinates for the spheres, this metric is given by (7). As we have already said,  $g_0$  is scalar flat, indefinite Kähler. The obvious complex structure  $J$  gives a closed two form and  $\Omega := g_0(J, \cdot)$ . Moreover  $g_0$  clearly has a high degree of symmetry, since the 2-sphere metrics have rotational symmetry. In [53], Tod found deformations of  $g_0$  preserving the scalar-flat Kähler property, by using the explicit expression (44) for neutral scalar-flat Kähler metrics with symmetry. Take the explicit solution

$$e^u = 4 \frac{1 - t^2}{(1 + x^2 + y^2)^2}$$

to (45), which can be obtained by demanding  $u = f_1(x, y) + f_2(t)$ . There remains a linear equation for  $V$ . Setting  $W = V(1 - t^2)$  and performing the coordinate transformation  $t = \cos \theta$ ,  $\zeta = x + iy$  gives

$$g = 4W \frac{d\zeta d\bar{\zeta}}{(1 + \zeta\bar{\zeta})^2} - W d\theta^2 - \frac{\sin^2 \theta}{W} (d\phi + \eta)^2, \quad (70)$$

and  $W$  must solve a linear equation. This metric reduces to (7) for  $W = 1$ ,  $\eta = 0$ , with  $\theta, \phi$  standard coordinates for the second sphere. Tod shows that on differentiating the linear equation for  $W$  and setting  $Q = \frac{\partial W}{\partial r}$ , one obtains the ultrahyperbolic wave equation

$$\nabla_1^2 Q = \nabla_2^2 Q, \quad (71)$$

where  $\nabla_{1,2}$  are the Laplacians on the 2-spheres, and  $Q$  is independent of  $\phi$ , i.e. is axisymmetric for one of the sphere angles. Equation (71) can be solved using Legendre polynomials, and one obtains non-conformally flat deformations of (7) in this way. In the process one must check that  $W$  behaves in such a way that (70) extends over  $S^2 \times S^2$ .

The problem of relating these explicit metrics to the Zollfrei metrics on  $S^2 \times S^2$  known to exist by results described in Section 5.2 appears to be open.

In a recent paper, Kamada [29] rediscovered the above metrics, and showed that a compact neutral scalar-flat Kähler manifold with a Hamiltonian  $S^1$  symmetry must in fact be  $S^2 \times S^2$ . Here a Hamiltonian  $S^1$  symmetry is an  $S^1$  action preserving the Kähler form, and which possesses a moment map. In the case of  $S^2 \times S^2$  case, there is always a moment map since the manifold is simply connected. Without the symmetry, there are other neutral scalar-flat Kähler manifolds. For example, take a Riemann surface  $\Sigma$  with a constant curvature metric  $g$ . Then on  $\Sigma \times \Sigma$ , the metric  $\rho_1^* g - \rho_2^* g$  is neutral scalar-flat Kähler, where  $\rho_i$  are the projections onto the first and second factors.

**6.3 Compact neutral hyper-Kähler metrics.** The only compact four-dimensional Riemannian hyper-Kähler manifolds are the complex torus with the flat metric and  $K3$  with a Ricci-flat Calabi–Yau metric. In the neutral case, Kamada showed in [28] that a compact pseudo-hyper-Kähler four manifold must be either a complex torus or a primary Kodaira surface. In the complex torus case, the metric need not be flat, in contrast to the Riemannian case. Moreover in both cases one can write down explicit non-flat examples, in contrast to the Riemannian case where no explicit non-flat Calabi–Yau metric is known.

To write down explicit examples, consider the following hyper-Kähler metric

$$g = d\phi dy - dz dx - Q(x, y) dy^2, \quad (72)$$

for  $Q$  and arbitrary function. This is the neutral version of the pp-wave metric of general relativity [47], and is a special case of (51), where the underlying projective structure is flat. It is non-conformally flat for generic  $Q$ . Define complex coordinates  $z_1 = \phi + iz$ ,  $z_2 = x + iy$  on  $\mathbb{C}^2$ . By quotienting the  $z_1$ - and  $z_2$ -planes by lattices one obtains a product of elliptic curves, a special type of complex torus. If we require  $Q$  to

be periodic with respect to the  $z_2$  lattice, then (72) descends to a metric on this manifold. Likewise, a primary Kodaira surface can be obtained as a quotient of  $\mathbb{C}^2$  by a subgroup of the group of affine transformations, and again by assuming suitable periodicity in  $Q$  the metric (72) descends to the quotient. In our framework [18] we compactify the flat projective space  $\mathbb{R}^2$  to two-dimensional torus  $U = T^2$  with the projective structure coming from the flat metric. Both  $\phi$  and  $z$  in (72) are taken to be periodic, thus leading to  $\hat{\pi} : \mathcal{M} \rightarrow U$ , the holomorphic toric fibration over a torus. Assume the suitable periodicity on the function  $Q : U \rightarrow \mathbb{R}$ . This leads to a commutative diagram:

$$\begin{array}{ccc}
 \mathcal{M} & & \\
 \downarrow T^2 & \searrow \hat{\pi}^* Q & \\
 U & \xrightarrow{Q} & \mathbb{R}.
 \end{array}$$

In the framework of [28] and [19] the Kähler structure on  $\mathcal{M}$  is given by  $\omega_{\text{flat}} + i\partial\bar{\partial}(\hat{\pi}^* Q)$ , where  $(\partial, \omega_{\text{flat}})$  is the flat Kähler structure on the Kodaira surface induced from  $\mathbb{C}^2$ .

As remarked in [28], the existence of pseudo-hyper-Kähler metrics on complex tori other than a product of elliptic curves is an open problem.

**6.4 Ooguri–Vafa metrics.** In [43] Ooguri, Vafa and Yau constructed a class of non-compact neutral hyper-Kähler metrics on cotangent bundles of Riemann surfaces with genus  $\geq 1$ , using the Heavenly equation formalism. This is similar to (6), but one takes a different  $(++--)$  real section of  $\mathcal{M}_{\mathbb{C}}$ . Instead of using the real coordinates we set

$$w = \zeta, \quad y = \bar{\zeta}, \quad z = ip, \quad x = -i\bar{p}, \quad \zeta, p \in \mathbb{C}$$

with  $\Omega = i(p\bar{\zeta} - \bar{p}\zeta)$  corresponding to the flat metric. Let  $\Sigma$  be a Riemann surface with a local holomorphic coordinate  $\zeta$ , such that the Kähler metric on  $\Sigma$  is  $h_{\zeta\bar{\zeta}}d\zeta d\bar{\zeta}$ . Suppose that  $p$  is a local complex coordinate for fibres of the cotangent bundle  $T^*\Sigma$ . If  $\omega$  is the Kähler form for a neutral metric  $g$  then  $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Omega$  for a function  $\Omega$  on the cotangent bundle. Then the equation

$$\det g_{i\bar{j}} = -1$$

is equivalent to the first Heavenly equation (6), and gives a Ricci-flat ASD neutral metric.

The idea in [43] is to suppose that  $\Omega$  depends only on the globally defined function  $X = h^{\zeta\bar{\zeta}} p \bar{p}$ , which is the length of the cotangent vector corresponding to  $p$ . There is a globally defined holomorphic  $(2, 0)$ -form  $\omega = d\zeta \wedge dp$ , which is the holomorphic part of the standard symplectic form on the cotangent bundle, so  $(\zeta, p)$  are the holomorphic coordinates in the Plebański coordinate system. The heavenly equation reduces to an ODE for  $\Omega(X)$  and Ooguri–Vafa show that for solutions of this ODE to exist  $h$  must have constant negative curvature, so  $\Sigma$  has genus greater than one. In this case one can



solve the ODE to find

$$\Omega = 2\sqrt{A^2 + BX} + A \ln \frac{\sqrt{A^2 + BX} - A}{\sqrt{A^2 + BX} + A},$$

where  $A, B$  are arbitrary positive constants. The metric  $g$  is well behaved when  $X \rightarrow 0$  (or  $p \rightarrow 0$ ), as in this limit  $\Omega \rightarrow \ln(X)$  and  $g$  restricts to  $h_{\xi\bar{\xi}}d\xi d\bar{\xi}$  on  $\Sigma$  and  $-h^{\xi\bar{\xi}}dpd\bar{p}$  on the fibres. In the limit  $X \rightarrow \infty$  the metric is flat. To see it one needs to choose a uniformising coordinate  $\tau$  on  $\Sigma$  so that  $h$  is a metric on the upper half plane. Then make a coordinate transformation  $\zeta_1 = \tau\sqrt{p}, \zeta_2 = \sqrt{p}$ . The holomorphic two form is still  $d\zeta_1 \wedge d\zeta_2$ , and the Kähler potential  $\Omega = i(\zeta_2\bar{\zeta}_1 - \zeta_1\bar{\zeta}_2)\sqrt{B}$  yields the flat metric.

Ooguri–Vafa also observed that the pp-wave metric (72) can be put onto  $T^*\Sigma$ , by requiring  $Q(x, y)$  to satisfy certain symmetries. Globally defined neutral metrics on non-compact manifolds were also studied by Kamada and Machida in [30], where they obtained many neutral analogues of well known ASD Riemannian metrics.

## References

- [1] Ashtekar, A., Jacobson, T., and Smolin, L., A new characterization of half-flat solutions to einstein’s equation. *Comm. Math. Phys.* **115** (1988), 631–648.
- [2] Atiyah, M. F., Vector fields on manifolds. *Arbeitsgemeinschaft für Forschung des Landes Nordrhein Westfalen* **200** (1970), 7–24.
- [3] Atiyah, M. F., Hitchin, N. J., and Singer, I. M., Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A* **362** (1978), 425–461.
- [4] Barrett, J., Gibbons, G. W., Perry, M., and Pope, C., Kleinian geometry and the  $n = 2$  superstring. *Int. J. Mod. Phys. A* **9** (1994), 1457–1494.
- [5] Boyer, C. P., A note on hyperhermitian four-manifolds. *Proc. Amer. Math. Soc.* **102** (1988), 157–164.
- [6] Brozos-Vázquez, M., Garcia-Rio, E., and Vázquez-Lorenzo, R., Conformally osserman four-manifolds whose conformal Jacobi operators have complex eigenvalues. *Proc. Roy. Soc. London Ser. A* **462** (2006), 1425–1441.
- [7] Bogdanov, L. V., Dryuma, V. S., and Manakov S. V., On the dressing method for Dunajski anti-self-duality equation. Preprint 2006, arXiv:nlin/0612046.
- [8] Bryant, R. L., Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor. In *Global analysis and harmonic analysis* (Marseille–Luminy 1999), Semin. Congr. 4, Soc. Math. France, Paris 2000, 53–94.
- [9] Bryant, R. L., Dunajski, M., Eastwood, M., Metrisability of two-dimensional projective structures. Preprint 2008, arXiv:math.DG0801.0300v1.
- [10] Calderbank, D. M. J., Selfdual 4-manifolds, projective structures, and the Dunajski-West construction. Preprint 2006, arXiv:math.DG/0606754.
- [11] Calderbank, D. M. J., and Pedersen, H., Self-dual spaces with complex structures, Einstein-Weyl geometry and geodesics. *Ann. Inst. Fourier* **50** (2000), 921–963.

- [12] Dancer, A. S., Jorgensen, H. R., and Swann, A. F., Metric geometries over the split quaternions. *Rend. Sem. Mat. Univ. Politec. Torino* **63** (2005), 119–139.
- [13] Derdziński, A., Self-dual Kähler manifolds and Einstein manifolds of dimension four. *Compositio Math.* **49** (1983), 405–433.
- [14] Dunajski, M., The twisted photon associated to hyper-hermitian four-manifolds. *J. Geom. Phys.* **30** (1999), 266–281.
- [15] Dunajski, M., Anti-self-dual four-manifolds with a parallel real spinor. *Proc. Roy. Soc. London Ser. A* **458** (2002), 1205–1222.
- [16] Dunajski, M., A class of Einstein-Weyl spaces associated to an integrable system of hydrodynamic type. *J. Geom. Phys.* **51** (2004), 126–137.
- [17] Dunajski, M., Mason, L. J., and Tod, K. P., Einstein-Weyl geometry, the dKP equation and twistor theory. *J. Geom. Phys.* **37** (2001), 63–93.
- [18] Dunajski, M., and West, S., Anti-self-dual conformal structures from projective structures. *Comm. Math. Phys.* **272** (2007), 85–118.
- [19] Fino, A., Pedersen, H., Poon, Y.-S., and Sorensen, M. W., Neutral calabi-yau structures on kodaira manifolds. *Comm. Math. Phys.* **248** (2004), 255–268.
- [20] Grossman, D., Torsion-free path geometries and integrable second order ODE systems, *Selecta Math. (N.S.)* **6** (2000), 399–442.
- [21] Hirzebruch, F., and Hopf, H., Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten. *Math. Ann.* **136** (1958), 156–172.
- [22] Hitchin, N. J., Complex manifolds and Einstein’s equations. In *Twistor geometry and Non-Linear Systems*, Lecture Notes in Math. 970, Springer-Verlag, Berlin 1982, 73–99.
- [23] Hitchin, N. J., Hypersymplectic quotients. *Acta Acad. Sci. Tauriensis* **124** (1990), 169–180.
- [24] Ivanov, S., and Zamkovy, S., Parahermitian and paraquaternionic manifolds. *Differential Geom. Appl.* **23** (2005), 205–234.
- [25] John, F., The ultrahyperbolic wave equation with four independent variables. *Duke Math. J.* **4** (1938), 300–322.
- [26] Jones, P. E., and Tod, K. P., Minitwistor spaces and Einstein-Weyl spaces. *Classical Quantum Gravity* **2** (1985), 565–577.
- [27] Joyce, D. D., Explicit construction of self-dual 4-manifolds. *Duke Math. J.* **77** (1995), 519–552.
- [28] Kamada, H., Neutral hyperkähler structures on primary Kodaira surfaces. *Tsukuba J. Math.* **23** (1999), 321–332.
- [29] Kamada, H., Compact scalar-flat indefinite Kähler surfaces with Hamiltonian  $S^1$ -symmetry. *Comm. Math. Phys.* **254** (2005), 23–44.
- [30] Kamada, H., and Machida, Y., Self-duality of metrics of type  $(2, 2)$  on four-dimensional manifolds. *Tôhoku Math. J.* **49** (1997), 259–275.
- [31] Kobayashi, S., and Nomizu, K., *Foundations of differential geometry, Volume II*. Interscience Tracts in Pure Appl. Math. 15, John Wiley & Sons, New York, London, Sydney 1969.
- [32] Law, P., Classification of the Weyl curvature spinors of neutral metrics in four dimensions. *J. Geom. Phys.* (2006), 2093–2108.

- [33] LeBrun, C., Explicit self-dual metrics on  $\mathbb{C}\mathbb{P}^2 \# \cdots \# \mathbb{C}\mathbb{P}^2$ . *J. Differential Geom.* **34** (1991), 223–253.
- [34] LeBrun, C., and Mason, L. J., Nonlinear gravitons, null geodesics, and holomorphic disks. *Duke Math. J.* **136** (2007), 205–273.
- [35] Malykh, A. A., Nutku, Y., and Sheftel, M. B., Lift of noninvariant solutions of heavenly equations from three to four dimensions and new ultra-hyperbolic metrics. *J. Phys. A* **40** (2007), 9371–9386.
- [36] Mason, L. J., and Newman, E. T., A connection between the einstein and yang-mills equations. *Comm. Math. Phys.* **121** (1989), 659–668.
- [37] Mason, L. J., and Woodhouse, N. M. J., *Integrability, self-duality, and twistor theory*. London Math. Soc. Monogr. 15, Oxford University Press, New York 1996.
- [38] Matsushita, Y., Fields of 2-planes on compact simply-connected smooth 4-manifolds. *Math. Ann.* **280** (1988), 687–689.
- [39] Matsushita, Y., and Law, P., Hitchin-Thorpe type inequalities for pseudo-Riemannian 4-manifolds of metric signature  $(+ + - -)$ . *Geom. Dedicata* **87** (2001), 65–89.
- [40] Mimura, Y., and Takeno, H., *Wave geometry*. Sci. Rep. Res. Inst. Theoret. Phys. Hiroshima Univ. 2, 1962.
- [41] Nakata, F., Singular self-dual Zollfrei metrics and twistor correspondence. *J. Geom. Phys.* **57** (2007), 1477–1498.
- [42] Nakata, F., Self-dual Zollfrei conformal structures with alpha-surface foliation. *J. Geom. Phys.* **57** (2007), 2077–2097.
- [43] Ooguri, H., and Vafa, C., Geometry of  $N = 2$  strings. *Nuclear Phys. B* **361** (1991), 469–518.
- [44] Park, Q. H., 2D sigma model approach to 4D instantons. *Internat. J. Modern Phys. A* **7** (1992), 1415–1447.
- [45] Penrose, R., Nonlinear gravitons and curved twistor theory. *General Relativity and Gravitation* **7** (1976), 31–52.
- [46] Penrose, R., and Rindler, W., *Spinors and Space-time 1 & 2*. Cambridge Monogr. Math. Phys., Cambridge University Press, Cambridge 1988.
- [47] Plebański, J. F., Some solutions of complex Einstein equations. *J. Math. Phys.* **16** (1975), 2395–2402.
- [48] Plebański, J. F., and Robinson, I., Left-degenerate vacuum metrics. *Phys. Rev. Lett.* **37** (1976), 493–495.
- [49] Pontecorvo, M., On twistor spaces of anti-self-dual hermitian surfaces. *Trans. Amer. Math. Soc.* **331** (1992), 653–661.
- [50] Sibata, T., and Morinaga, K., A complete and simpler treatment of wave geom. *Proc. Amer. Math. Soc.* **102** (1988), 157–164.
- [51] Taubes, C. H., The existence of anti-self-dual conformal structures. *J. Differential Geom.* **36** (1992), 163–253.
- [52] Tod, K. P., Scalar-flat Kähler and hyper-Kähler metrics from Painlevé-III. *Classical Quantum Gravity* **12** (1995), 1535–1547.
- [53] Tod, K. P., Indefinite conformally-ASD metrics on  $S^2 \times S^2$ . In *Further Advances in Twistor Theory* (2001), vol. III, Chapman & Hall/CRC, 61–63.

- [54] Walker, A. G., Canonical form for a riemannian space with a parallel field of null planes. *Quart. J. Math. Oxford* **1** (1950), 69–79.
- [55] Ward, R. S., and Wells, R. O., *Twistor geometry and field theory*. Cambridge Monogr. Math. Phys., Cambridge University Press, Cambridge 1990.
- [56] Woodhouse, N. M. J., Contour integrals for the ultrahyperbolic wave equation. *Proc. Roy. Soc. London Ser. A* **438** (1992), 197–206.

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