Eur. Phys. J. B (2015) 88: 109
DOI: 10.1140/epjb/e2015-50798-9

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Anti-synchronization for stochastic memristor-based neural networks with non-modeled dynamics via adaptive control approach

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Received 14 November 2014 / Received in final form 4 January 2015
Published online 4 May 2015 –

Abstract. In this paper, exponential anti-synchronization in mean square of an uncertain memristor-based neural network is studied. The uncertain terms include non-modeled dynamics with boundary and stochastic perturbations. Based on the differential inclusions theory, linear matrix inequalities, Gronwall's inequality and adaptive control technique, an adaptive controller with update laws is developed to realize the exponential anti-synchronization. Adaptive controller can adjust itself behavior to get the best performance, according to the environment is changing or the environment has changed, which has the ability to adapt to environmental change. Furthermore, a numerical example is provided to validate the effectiveness of the proposed method.

1 Introduction

Memristor has received a great deal of attention because of its rich capabilities, especially in the aspect of store and access data [1] with the memory mode of human brain which is realized by adjusting the strength of synaptic connections between neurals [2]. The synapse is a crucial element in biological neural networks, and which is the bridge across which nerve cells (neurons) contact each other. Artificial neural network try to imitate the working mechanisms of their biological counterparts though the learning in the same way, and over the years the memristor has been considered to be the electronic equivalent of the synapse [3], we use the memristor to simulate the memory and learning function of synapse. Therefore, it is very interesting to investigate the memristor-based neural networks (MNN), in which the resistance of a memristive system depends on its past states and exactly this functionality can be used to mimic the synaptic connections in a (human) brain [4], and provide an in-depth understanding of key design implications of memristor-based memories, is a model more realistic for the description of real neural systems [5,6]. Additionally, memristor which works as synaptic weights demonstrates plentiful characteristics to some extent [7]. And its potential applications are in next generation computers and powerful brain-like neural computers.

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Meanwhile, as an important dynamic behaviour of nonlinear system [8,9], the stability and the synchronization of system, for their potential applications play an important role in many different fields, including image process, secure communication, information science, biological system, etc. Subsequently, the stability and the synchronization of memristor-based neural networks have been widely investigated (see Ref. [10]), including exponential synchronization [11–13], complete periodic synchronization [14], anti-synchronization [15,16] and so on. It is a common phenomenon that the synchronization of two state vectors have the same state trajectory but opposite signs control have important application significance, including anti-synchronization to lasers, which provides a new way to generate the special form of pulse, and anti-synchronization to communication systems, which enhances security and secrecy of communication by changing the form of synchronization and anti-synchronization in the process of digital signal transmission. Recently, the anti-synchronization control has been more widely applied to many fields, e.g., image processing, secure communication, information science, and harmonic oscillation generation. Moreover, the anti-synchronization analysis for memristive neural networks can obtain some amazing properties, richness of flexibility, and opportunities.

In the applications, there are typically some uncertain parameters and noise perturbations in real systems, which often affect their dynamics, so it has practical implications to investigate synchronization issues of uncertain complex dynamical networks by using a pinning control strategy [17] and adaptive control [18–20]. To our best knowledge, the memristor-based neural networks models proposed and studied in the literature are deterministic. However, few works have been done on the anti-synchronization of a general MNN with uncertain terms and stochastic perturbations [21–23], which takes full use of them to strengthen the security and secrecy.

Motivated by the above discussion, in this paper, we consider the exponential anti-synchronization problem for a general stochastic MNN with non-modeled dynamics by using a novel adaptive control approach. The contributions of our paper are as follows. (i) The considered MNNs include the effects from external noise and non-modeled dynamics; (ii) the non-modeled dynamics which are bounded need not satisfy other constrained conditions; (iii) the exponential anti-synchronization analysis for a general MNN which demonstrates plentiful characteristics can be extended to other specific network forms. Therefore, our goal is to do our best to shorten the gap to quickly complete convergence for anti-synchronization problem of memristive neural networks with non-modeled dynamics and stochastic perturbations.

The rest of this paper is organized as follows. In Section 2, the model description of the MNN is given. In Section 3, a stochastic MNN with non-modeled dynamics is introduced, and some preliminaries about sufficient conditions are given to ensure synchronization of the memristor-based neural network. In Section 4, an effective adaptive controller and the proposed adaptive law are given to ensure that the MNN can achieve the exponential anti-synchronization in the mean square sense. In Section 5, a numerical example is provided to illustrate the effectiveness of our proposed results. Finally, this paper ends with conclusions in Section 6.

2 Model description

In order to understand the MNN well, we describe the circuit of a general class of neural networks in Figure 1 as follows. Taking the ith subsystem as the form of analysis, and the KCL equation [10] is written as:

$$\dot{x}_i(t) = -\frac{1}{C_i}\left[\sum_{j=1}^{n}\left(\frac{1}{R_{ij}}+\frac{1}{F_{ij}}\right)\times \mathrm{sgn}_{ij}+\frac{1}{R_i}\right]x_i(t) + \frac{1}{C_i}\sum_{j=1}^{n}\frac{f_j(x_j(t))}{R_{ij}}\times \mathrm{sgn}_{ij} + \frac{1}{C_i}\sum_{j=1}^{n}\frac{g_j(x_j(t-\tau))}{F_{ij}}\times \mathrm{sgn}_{ij}+\frac{I_i}{C_i}$$

$$t\geq 0,\quad i=1,2,\ldots,N, \tag{1}$$

where $x_i(t)$ is the voltage of the capacitor C_i, R_{ij} denote the resistor between the feedback function $f_i(x_i(t))$

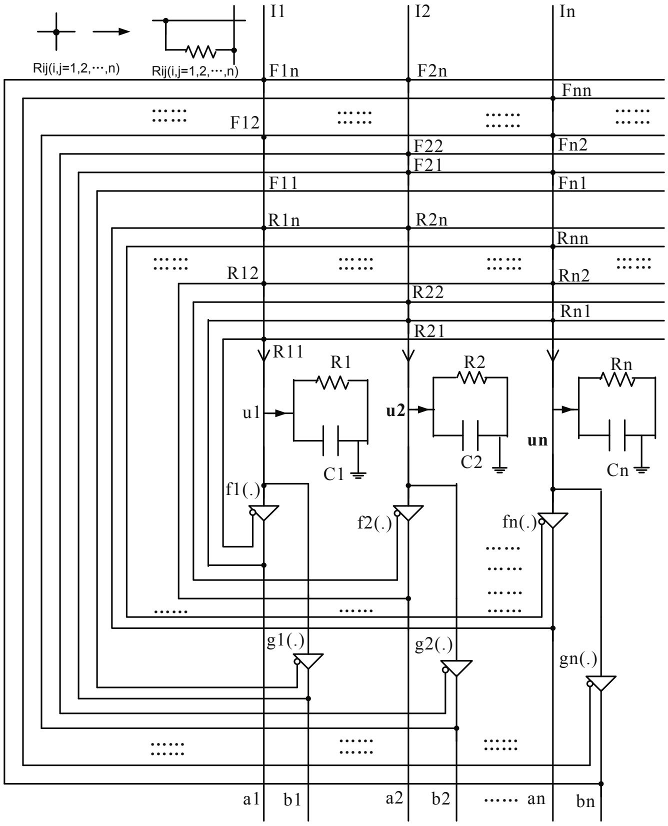

Fig. 1. Circuit of neural network, where $x_i(\cdot)$ is the state of the ith subsystem, $f_j(\cdot)$, $g_j(\cdot)$ are the amplifiers, $R_{ij}(\cdot)$ is the connection resistor between the amplifier $f_j(\cdot)$ and state $x_i(\cdot)$ and $F_{ij}(\cdot)$ is the connection resistor between the amplifier $g_j(\cdot)$ and state $x_i(\cdot)$, R_i and C_i are the resistor and capacitor, I_i is the external input, a_i, b_i are the outputs.

and $x_i(t)$, and F_{ij} denote the resistor between the feedback function $g_i(x_i(t-\tau))$ and $x_i(t)$. τ corresponds to the transmission delay and it can be considered as a constant or a time-varying function, R_i represents the parallel-resistor corresponding to the capacitor C_i; I_i is the external input or bias, and

$$\mathrm{sgn}_{ij}=\begin{cases}1, & i\neq j,\\ -1, & i=j.\end{cases}$$

In this paper, I_i is selected as null matrix, and therefore it is omitted in the following models.

From equation (1), which can be written as:

$$\dot{x}_i(t) = -\sigma_i x_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t)) + \sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau)),$$

$$t\geq 0,\quad i=1,2,\ldots,N, \tag{2}$$

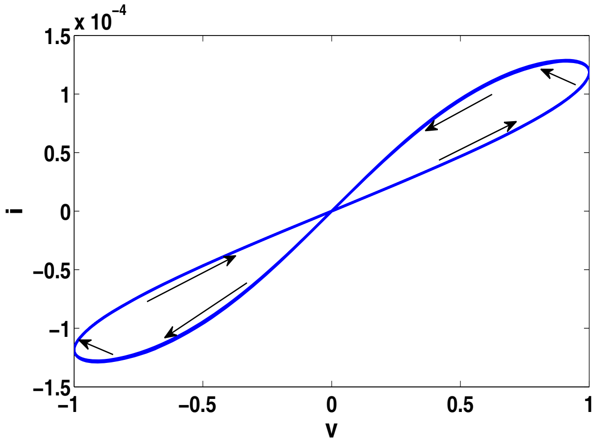

Fig. 2. Typical current-voltage (i-v) characteristics of a memristor, the pinched hysteresis loop is due to the nonlinear relationship between the memristance current and voltage.

where

$$\sigma_i = \frac{1}{C_i}\left[\sum_{j=1}^{n}\left(\frac{1}{R_{ij}} + \frac{1}{F_{ij}}\right) \times \mathrm{sgn}_{ij} + \frac{1}{R_i}\right],$$

$$a_{ij} = \frac{\mathrm{sgn}_{ij}}{C_i R_{ij}}, \quad b_{ij} = \frac{\mathrm{sgn}_{ij}}{C_i F_{ij}}.$$

For the model of memristor-based neural networks, the memductances of the memristors W_{ij}, M_{ij} and P_i, respectively take place of the resistors R_{ij}, F_{ij} and R_i of a general class of neural networks (1). With the pinched hysteresis loop in the current-voltage characteristic of memristors in Figure 2, we give a general type of MNN as follows:

$$\begin{aligned}\dot{x}_i(t) &= -\sigma_i(x_i(t))x_i(t) + \sum_{j=1}^{n} a_{ij}(x_i(t))f_j(x_j(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij}(x_i(t))g_j(x_j(t-\tau)), \\ &\quad t \geq 0, i = 1, 2, \ldots, N, \end{aligned} \tag{3}$$

where

$$\sigma_i(x_i(t)) = \frac{1}{C_i}\left[\sum_{j=1}^{n}(W_{ij} + M_{ij}) \times \mathrm{sgn}_{ij} + \frac{1}{R_i}\right],$$

$$a_{ij}(x_i(t)) = \frac{W_{ij}}{C_i} \times \mathrm{sgn}_{ij}, \quad b_{ij}(x_i(t)) = \frac{M_{ij}}{C_i} \times \mathrm{sgn}_{ij}.$$

Then

$$\sigma_i(x_i(t)) = \begin{cases} \hat{\sigma}_i, & |x_i| < T_i, \\ \check{\sigma}_i, & |x_i| > T_i, \end{cases}$$

$$a_{ij}(x_i(t)) = \begin{cases} \hat{a}_{ij}, & |x_i| < T_i, \\ \check{a}_{ij}, & |x_i| > T_i, \end{cases}$$

$$b_{ij}(x_i(t)) = \begin{cases} \hat{b}_{ij}, & |x_i| < T_i, \\ \check{b}_{ij}, & |x_i| > T_i, \end{cases}$$

where the switching jumps $T_i > 0$, $\hat{\sigma}_i > 0$, $\check{\sigma}_i > 0$, $\hat{a}_{ij}$, $\check{a}_{ij}$, $\hat{b}_{ij}$, $\check{b}_{ij}$, $i, j = 1, 2, \ldots, n$, are all constants.

Or, equivalently given in matrix format

$$\begin{aligned}\dot{x}(t) &= -\sigma(x(t))x(t) + A(x(t))f(x(t)) \\ &\quad + B(x(t))g(x(t-\tau)), \\ &\quad t \geq 0, i = 1, 2, \ldots, N, \end{aligned} \tag{4}$$

where $\sigma(x(t)) = \mathrm{diag}(\sigma_1(x(t)), \sigma_2(x(t)), \ldots, \sigma_n(x(t)))$, $A(x(t)) = [a_{ij}(x_i(t))]_{n\times n}$, $B(x(t)) = [b_{ij}(x_i(t))]_{n\times n}$.

Remark 1. Memristor behavior is more and more shown its plentiful performance as new technology process nodes to be introduced because of this feature of pinched hysteresis, where it constructed memristive neural network consisting of three electronic neurons connected by two memristor emulator synapses that demonstrated experimentally the formation of associative memory. According to the analysis above, it is easy to see that the memristor-based neural network (3) is a state-dependent switching system, and it is not the same as the general class of neural network.

Remark 2. The network (4) which demonstrates plentiful characteristics represents a general class of memristor-based neural networks with constant or time-varying delays. When $\sigma(x(t)), A(x(t)), B(x(t))$ are all constants, network (4) becomes a general recurrent neural network [24]; when $\sigma(x(t))$ is constant, network (4) becomes a memristor-based recurrent neural network [2]; when $B(x(t)) \equiv 0$, network (4) becomes a memristor-based Hopfield network [25]. Besides that, when f is a sigmoid function and $g \equiv 0$, network (4) is a class memristor-based Hopfield neural network. Similarly, when $f = (|x+1| - |x-1|)/2$ and $g \equiv 0$ or $g \equiv f$, neural network (4) represents memristor-based cellular neural networks [26].

3 Preliminaries and problem formulation

As a matter of convenience, some preliminaries and notations are given as follows.

In this paper, solutions of all the systems considered in the following are intended in the Filippov's sense. In the Banach space $C([-\tau, 0], R^n)$, we define $\|V\|_c = [\sum_{i=1}^{n}(\sup_{-\tau\leq s\leq 0}|V_i(s)|)^2]^{1/2}$. For a vector $v \in R^n$, whose norm is denoted by $\|v\|_\infty = \max_{1\leq i\leq n}\{|v_i|\}$, and $\|\cdot\|$ denotes the Euclidean norm of the vector v. Let $\overline{\sigma_i} = \max\{\hat{\sigma}_i, \check{\sigma}_i\}$, $\underline{\sigma_i} = \min\{\hat{\sigma}_i, \check{\sigma}_i\}$, $\overline{a}_{ij} = \max\{\hat{a}_{ij}, \check{a}_{ij}\}$, $\underline{a}_{ij} = \min\{\hat{a}_{ij}, \check{a}_{ij}\}$, $\overline{b}_{ij} = \max\{\hat{b}_{ij}, \check{b}_{ij}\}$, $\underline{b}_{ij} = \min\{\hat{b}_{ij}, \check{b}_{ij}\}$. $co\{\underline{\zeta}_i, \overline{\zeta}_i\}$ denotes the closure of the convex hull generated by real numbers $\underline{\zeta}_i$ and $\overline{\zeta}_i$ or real matrices $\underline{\zeta}_i$ and $\overline{\zeta}_i$. Furthermore, $[\cdot,\cdot]$ represents an interval, and we define $[\underline{\zeta}_i, \overline{\zeta}_i] = co\{\underline{\zeta}_i, \overline{\zeta}_i\}$.

Definition 1 (see Ref. [27]). Suppose $E \subseteq R^n$, then $x \to F(x)$ is called a set-valued map from $E \to R^n$, if for each point $x \in E$, there exists a nonempty set $F(x) \subseteq R^n$. A set-valued map F with nonempty values is said to be upper semi-continuous at $x_0 \in E$, if for any open set N

containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subseteq N$. The map $F(x)$ is said to be have a closed (convex, compact) image if for each $x \in E$, $F(x)$ is a closed (convex, compact).

Definition 2 (see Ref. [28]). For the system $\dot{x}(t) = f(x), x \in R^n$, with discontinuous right-hand sides, a set-valued map is defined as

$$F(t,x) = \bigcap_{\varepsilon>0} \bigcap_{\mu(N)=0} co[f(B(x,\varepsilon))\backslash N],$$

where $co[E]$ is the closure of the convex hull of set E, $B(x,\varepsilon) = \{y : \|y - x\| \le \varepsilon\}$ and $\mu(N)$ is Lebesgue measure of set N. A solution in Filippov's sense of Cauchy problem for this system with initial condition $x(0) = x_0$ is an absolutely continuous function $x(t)$, which satisfies $x(0) = x_0$ and differential inclusion $\dot{x}(t) \in F(t,x)$.

In order to establish our main results, some necessary Assumptions and Lemmas are given in the following.

Assumption 1. The neuron activation functions $f(\cdot)$ and $g(\cdot)$ which are continuous with $f(0) = g(0) = 0$ are bounded, and there exist positive scalars l_1 and l_2 for all x, y satisfying

$$\|f(x) - f(y)\| = l_1\|x - y\|, \quad \|g(x) - g(y)\| = l_2\|x - y\|.$$

Assumption 2. For $i, j = 1, 2, \ldots, n$,

$$\begin{aligned} &co\{\hat{a}_i, \check{a}_i\}x_i(t) + co\{\hat{a}_i, \check{a}_i\}y_i(t) \subseteq co\{\hat{a}_i, \check{a}_i\}e_i(t), \\ &co\{\hat{a}_i, \check{a}_i\}f_j(x_i(t)) + co\{\hat{a}_i, \check{a}_i\}f_j(y_i(t)) \\ &\quad \subseteq co\{\hat{a}_i, \check{a}_i\}(f_j(x_i(t)) + f_j(y_i(t))), \\ &co\{\hat{a}_i, \check{a}_i\}g_j(x_i(t-\tau)) + co\{\hat{a}_i, \check{a}_i\}g_j(y_i(t-\tau)) \\ &\quad \subseteq co\{\hat{a}_i, \check{a}_i\}(g_j(x_i(t-\tau)) + g_j(y_i(t-\tau))). \end{aligned}$$

Assumption 3. Assume that the non-modeled dynamics are bounded and for all $t > 0$ there exist some positive constants $h > 0$ such that

$$\| \triangle h(t)\|_\infty \le h.$$

Assumption 4. The noise matrix $\phi(\cdot)$ is local Lipschitz continuous and satisfies the linear growth condition as well, where $\phi(t, 0, 0) = 0$. Moreover, there exist two real positive matrixes R_1, R_2 for all x, y such that

$$\text{trace}\left[\phi^T(t,e)\phi(t,e)\right] \le x^T R_1 x + y^T R_2 y.$$

Lemma 1 Gronwall's inequality (see Ref. [29]). Let $T > 0$ and $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0, T]$. If

$$u(t) \le c + v\int_0^t u(s)ds, \quad \forall 0 \le t \le T,$$

for some constants u, v, then

$$u(t) \le c\exp(vt), \quad \forall 0 \le t \le T.$$

Lemma 2. Assume that the stochastic differential equation is

$$d(x) = f(x(t), x(t-\tau), t)dt + g(x(t), x(t-\tau), t)dv(t),$$

then an operator $\mathscr{L}V(x(t), t)$ from $R^+ \times R^n$ along the trajectory of the error system is defined as

$$\begin{aligned} \mathscr{L}V(x(t),t) = &V_t(x(t),t) + V_x(x(t),t)f(x(t),x(t-\tau),t) \\ &+ \frac{1}{2}\text{trace}\Big\{g^T(x(t),x(t-\tau),t) \\ &\times V_{xx}g(x(t),x(t-\tau),t)\Big\}, \end{aligned}$$

where

$$\begin{aligned} V_t(x(t),t) &= \frac{\partial V(x(t),t)}{\partial t}, \\ V_x(x(t),t) &= \left(\frac{\partial V(x(t),t)}{\partial x_1}, \frac{\partial V(x(t),t)}{\partial x_2}, \ldots, \frac{\partial V(x(t),t)}{\partial x_n}\right), \\ V_{xx} &= \left(\frac{\partial^2 V(x(t),t)}{\partial x_i \partial x_j}\right)_{n\times n}. \end{aligned}$$

Lemma 3 (see Ref. [30]). If X and Y are real matrices with appropriate dimensions, then there exists a $\theta > 0$, such that

$$X^TY + Y^TX \le \theta X^TX + \frac{1}{\theta}Y^TY.$$

According to systems (3) and (4), and using the theory of differential inclusions, we have

$$\begin{aligned} \dot{x}_i(t) \in &-[\underline{\sigma}_i, \overline{\sigma}_i]x_i(t) + \sum_{j=1}^n [\underline{a}_{ij}, \overline{a}_{ij}]f_j(x_j(t)) \\ &+ \sum_{j=1}^n \left[\underline{b}_{ij}, \overline{b}_{ij}\right] g_j(x_j(t-\tau)), \\ &t \ge 0, \quad i = 1, 2, \ldots, N. \end{aligned} \tag{5}$$

The vector form is given by

$$\begin{aligned} &\dot{x}(t) \in -[\underline{\Sigma}, \overline{\Sigma}]x(t) + [\underline{A}, \overline{A}]f(x(t)) + [\underline{B}, \overline{B}]g(x(t-\tau)), \\ &\quad t \ge 0, \end{aligned} \tag{6}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $\underline{\Sigma} = \text{diag}\{\underline{\sigma}_1, \underline{\sigma}_2, \ldots, \underline{\sigma}_n\}$, $\overline{\Sigma} = \text{diag}\{\overline{\sigma}_1, \overline{\sigma}_2, \ldots, \overline{\sigma}_n\}$, $\underline{A} = (\underline{a}_{ij})_{n\times n}$, $\overline{A} = (\overline{a}_{ij})_{n\times n}$, $\underline{B} = (\underline{b}_{ij})_{n\times n}$, $\overline{B} = (\overline{b}_{ij})_{n\times n}$.

Applying the theories of set-valued maps and differential inclusions, system (6) is equivalent to:

$$\dot{x}(t) = -\Sigma(t)x(t) + A(t)f(x(t)) + B(t)g(x(t-\tau)), \tag{7}$$

where $t \ge 0$, $\Sigma(t) \in [\underline{\Sigma}, \overline{\Sigma}]$, $A(t) \in [\underline{A}, \overline{A}]$, $B(t) \in [\underline{B}, \overline{B}]$.

On the basis of the discussion above, for the drive systems (6) and (7) we can construct a general MNN as the response system which can be written as follows:

$$\begin{aligned} \dot{y}(t) \in &-\left[\underline{\Sigma}, \overline{\Sigma}\right] x(t) + \left[\underline{A}, \overline{A}\right] f(y(t)) \\ &+ \left[\underline{B}, \overline{B}\right] g(y(t-\tau)), \end{aligned} \tag{8}$$

or, there exists $\Sigma(t) \in [\underline{\Sigma}, \overline{\Sigma}]$, $A(t) \in [\underline{A}, \overline{A}]$, and $B(t) \in [\underline{B}, \overline{B}]$, such that

$$\dot{y}(t) = -\Sigma(t)y(t) + A(t)f(y(t)) + B(t)g(y(t-\tau)), \tag{9}$$

where the initial conditions are given by $y(t) = \psi(t), t \in [-\tau, 0]$, the initial condition $\psi(t) \in R^n$ is a continuous vector function.

Let $e(t) = y(t) + x(t)$ be the anti-synchronization error. Since there exist non-modeled dynamics and environmental noises in the real network system, model uncertainty and stochastic perturbations should be considered in the general neural networks models. Then according to Assumption 2, we obtain the following error system

$$\begin{aligned} de(t) \in (&-[\underline{\Sigma}, \overline{\Sigma}]e(t) + \triangle h(t) \\ &+ [\underline{A}, \overline{A}]f(e(t)) + [\underline{B}, \overline{B}]g(e(t-\tau)))dt \\ &+ \phi(t, e(t), e(t-\tau))d\omega(t), \end{aligned} \tag{10}$$

or

$$\begin{aligned} de(t) = &[-\Sigma(t)e_i(t) + \triangle h(t) + A(t)f(e(t)) \\ &+ B(t)g(e(t-\tau))]dt + \phi(t, e(t), e(t-\tau))d\omega(t), \end{aligned} \tag{11}$$

where $e(t, \varepsilon)$ denotes the state trajectory from the initial data $e(\theta) = \varepsilon(\theta)$, on $-\tau \le \theta \le 0$ in $C^2_{\mathscr{F}}([-\tau, 0]; R^n)$. $\Sigma(t) \in [\underline{\Sigma}, \overline{\Sigma}]$, $A(t) \in [\underline{A}, \overline{A}]$, $B(t) \in [\underline{B}, \overline{B}]$. $\triangle h(t)$ represents the non-modeled dynamics in the system and $f(e(t)) = f(y(t)) + f(x(t))$, $g(e(t)) = g(y(t)) + g(x(t))$. $\phi(t, e(t), e(t-\tau))$ is noise intensity. $w(t)$ is a m-dimensional Brownian motion defined on the probability space $(\Omega, \mathscr{F}, P)$ with $E\{\omega(t)\} = 0, E\{\omega^2(t)\} = 1, E\{\omega(s)\omega(t)\} = 0, s \ne t$, and $\phi(t, e(t), e(t-\tau)) \in R^n$ is the noise intensity satisfying the Assumption 4.

Definition 3. The error function $e(t)$ with non-modeled dynamics and stochastic perturbations can be exponentially converge to zero in mean square sense if there exist constants $\beta > 0$ and $\alpha > 0$, such that $E\|e(t)\|^2 \le \alpha \exp(-\beta t)$ and $t > 0$, where β is called the decay rate of convergence. System (11) are said to be exponentially anti-synchronization in mean square sense.

4 Main results

In this section, the adaptive exponential anti-synchronization criteria is given for a general memristive neural networks. Two corollaries are also derived for memristive neural networks. The adaptive controller is designed as

$$\begin{cases} u(t) = -k_1(t)e(t) - k_2(t) \cdot \mathrm{sign}(e(t)), \\ \dot{k}_1(t) = \gamma e^T(t)Pe(t), \\ \dot{k}_2(t) = \upsilon\|Pe(t)\|, \end{cases} \tag{12}$$

where

$$\mathrm{sign}(e(t)) = \begin{cases} -1, & e(t) < 0, \\ 0, & e(t) = 0, \\ 1, & e(t) > 0. \end{cases}$$

Theorem 1. Under the Assumptions 1−4, the system (11) with non-modeled dynamics and stochastic perturbations will be converged under the controller $u(t)$. If there exist two $n \times n$ symmetric matrices $P > 0$ and $Q > 0$, and four positive constants ρ, θ_1, θ_2 and δ satisfying some certain conditions, systems (11) can achieve exponentially anti-synchronization in the mean square sense. And such that the following conditions hold:

$$\text{(i) } P \le \rho I;$$
$$\text{(ii) } k_2 \ge h;$$
$$\text{(iii) } \begin{bmatrix} \Phi & P\bar{A} & P\bar{B} \\ \star & -\frac{1}{2\theta_1}I & 0 \\ \star & \star & -\frac{1}{2\theta_2}I \end{bmatrix} \le 0.$$
$$\frac{1}{\theta_2}l_2^2 - Q + \rho R_2 \le 0,$$

where

$$\Phi = -P\underline{\Sigma} - \underline{\Sigma}^T P + Q + \frac{1}{\theta_1}l_1^2 + 2k_1 P + \rho R_1 + \delta I.$$

For the proof of Theorem 1, please see the Appendix for details.

Furthermore, the analysis and expansion of the Theorem 1, then we have the following results.

Corollary 1. If $B(t) \equiv 0$ or $g(x) \equiv 0$. The memristor-based neural network (4) becomes the memristor-based Hopfield neural network, and if Assumptions 1−4 hold, the error system of the stochastic memristor-based Hopfield neural network with non-modeled dynamics will be convergent. If there exist two $n \times n$ symmetric matrices $P > 0$ and $Q > 0$, and three positive constants ρ, θ_1, and δ which satisfy some certain conditions, system (11) under the controller $u(t)$ (12) can achieve exponentially anti-synchronization in the mean square sense. Therefore, the following conditions hold:

$$\text{(i) } P \le \rho I;$$
$$\text{(ii) } k_2 > h;$$
$$\text{(iii) } \begin{bmatrix} \Phi & P\bar{A} \\ \star & -\frac{1}{2\theta_1}I \end{bmatrix} \le 0.$$
$$-Q + \rho R_2 \le 0,$$

where

$$\Phi = -P\underline{\Sigma} - \underline{\Sigma}^T P + Q + \frac{1}{\theta_1}l_1^2 + 2k_1 P + \rho R_1 + \delta I.$$

Corollary 2. Let Assumptions 1−4 hold. If there exist two $n \times n$ symmetric matrices $P > 0$ and $Q > 0$, and four positive constants ρ, θ_1, θ_2 and δ which satisfy some certain conditions, such that the following conditions hold:

$$\text{(i) } P \le \rho I;$$
$$\text{(ii) } \begin{bmatrix} \Phi & P\bar{A} & P\bar{B} \\ \star & -\frac{1}{2\theta_1}I & 0 \\ \star & \star & -\frac{1}{2\theta_2}I \end{bmatrix} \le 0.$$
$$\frac{1}{\theta_2}l_2^2 - Q + \rho R_2 \le 0,$$

where

$$\Phi = -P\underline{\Sigma} - \underline{\Sigma}^T P + Q + \frac{1}{\theta_1}l_1^2 + 2k_1 P + \rho R_1 + \delta I.$$

Then system (11) with $\triangle h(t) = 0$ can obtain exponentially anti-synchronization in the mean square sense under the action of the following adaptive controller

$$\begin{cases} u(t) = -k_1(t)e(t), \\ \dot{k}_1(t) = \gamma e^T(t)Pe(t). \end{cases} \tag{13}$$

5 Numerical simulations

In this section, we give some numerical simulations to verify our analysis by using the MATLAB simulink toolbox.

Example 1. We consider the following two-dimensional memristive neural networks without non-modeled dynamic and stochastic perturbations as drive system:

$$\begin{cases} \dot{x}_1(t) = -\sigma_1(x_1(t))x_1(t) + a_{11}(x_1(t))f(x_1(t)) \\ \qquad +a_{12}(x_1(t))f(x_2(t)) + b_{11}(x_1(t)) \\ \qquad \times g(x_1(t-\tau)) + b_{12}(x_1(t))g(x_2(t-\tau)), \\ \dot{x}_2(t) = -\sigma_2(x_2(t))x_2(t) + a_{21}(x_2(t))f(x_1(t)) \\ \qquad +a_{22}(x_2(t))f(x_2(t)) + b_{21}(x_2(t)) \\ \qquad \times g(x_1(t-\tau)) + b_{22}(x_2(t))g(x_2(t-\tau)), \end{cases} \tag{14}$$

where $\tau = 1$, the initial conditions are $x(s) = (0.1, 0.1)^T$, $s \in [-1, 0]$, $f(\vartheta) = g(\vartheta) = \tanh(\vartheta)$. From Assumption 1, we can select $l_1 = l_2 = \mathrm{diag}(1, 1)$, and the rest parameter matrices are given as follows:

$$\sigma_1(x_1(t)) = \begin{cases} 1, & |x_1(t)| < 1, \\ 1.2, & |x_1(t)| > 1, \end{cases}$$

$$\sigma_2(x_2(t)) = \begin{cases} 1.2, & |x_1(t)| < 1, \\ 1, & |x_1(t)| > 1, \end{cases}$$

$$a_{11}(x_1(t)) = \begin{cases} 2, & |x_1(t)| < 1, \\ 1.8, & |x_1(t)| > 1, \end{cases}$$

$$a_{12}(x_1(t)) = \begin{cases} -0.1, & |x_1(t)| < 1, \\ -0.08, & |x_1(t)| > 1, \end{cases}$$

$$a_{21}(x_2(t)) = \begin{cases} -4.8, & |x_2(t)| < 1, \\ -5, & |x_2(t)| > 1, \end{cases}$$

$$a_{22}(x_2(t)) = \begin{cases} 2.8, & |x_2(t)| < 1, \\ 3, & |x_2(t)| > 1, \end{cases}$$

$$b_{11}(x_1(t)) = \begin{cases} -1.5, & |x_1(t)| < 1, \\ -1.3, & |x_1(t)| > 1, \end{cases}$$

$$b_{12}(x_1(t)) = \begin{cases} -0.1, & |x_1(t)| < 1, \\ -0.05, & |x_1(t)| > 1, \end{cases}$$

$$b_{21}(x_2(t)) = \begin{cases} -0.15, & |x_2(t)| < 1, \\ -0.2, & |x_2(t)| > 1, \end{cases}$$

$$b_{22}(x_2(t)) = \begin{cases} -2.3, & |x_2(t)| < 1, \\ -2.5, & |x_2(t)| > 1. \end{cases}$$

Then the corresponding two-dimensional memristive neural network with non-modeled dynamic and stochastic perturbations under the controller $u(t)$ is given as response system

$$\begin{cases} \dot{y}_1(t) = -\sigma_1(y_1(t))y_1(t) + \triangle h_1(t) + a_{11}(y_1(t)) \\ \qquad \times f(y_1(t)) + a_{12}(y_1(t))f(y_2(t)) + b_{11}(y_1(t)) \\ \qquad \times g(y_1(t-\tau)) + b_{12}(y_1(t))g(e_2(t-\tau)) \\ \qquad +u_1(t) + \phi_1(t, e_1(t), e_1(t-\tau))\dot{\omega}_1(t), \\ \dot{y}_2(t) = -\sigma_2(y_2(t))y_2(t) + \triangle h_2(t) + a_{21}(y_2(t)) \\ \qquad \times f(y_1(t)) + a_{22}(y_2(t))f(y_2(t)) + y_{21}(y_2(t)) \\ \qquad \times g(y_1(t-\tau)) + b_{22}(y_2(t))g(y_2(t-\tau)) \\ \qquad +u_2(t) + \phi_2(t, e_2(t), e_2(t-\tau))\dot{\omega}_2(t), \end{cases} \tag{15}$$

where the initial conditions are $y(s) = (0.2, -0.2)^T, s \in [-\tau, 0]$. $\triangle h_1(t) = \triangle h_2(t) = 0.01 \times \sin(t)$, and the noise intensity are given as follows:

$$\begin{cases} \phi_1(t, e_1(t), e_1(t-\tau)) = \sqrt{0.005}e_1(t) \\ \qquad +\sqrt{0.25}e_1(t-\tau), \\ \phi_2(t, e_2(t), e_2(t-\tau)) = \sqrt{0.025}e_2(t) \\ \qquad +\sqrt{0.005}e_2(t-\tau). \end{cases}$$

$\omega(t) = (\omega_1(t), \omega_2(t))^T$ is a 2-dimensional Brownian motion satisfying $E\{d\omega(t)\} = 0$ and $E\{[d\omega(t)]^2\} = dt$. From Assumption 3, it is easy to get $R_1 = \mathrm{diag}(0.01, 0.05)$, $R_2 = \mathrm{diag}(0.5, 0.01)$. The remaining parameters of equation (15) are the same as equation (14). The state trajectory of response system with uncertainty terms can deviate from an ideal world.

Therefore, the error system $e(t) = x(t) + y(t)$ without the controller $u(t)$ is described as follows:

$$\begin{cases} \dot{e}_1(t) = -\sigma_1(e_1(t))e_1(t) + \triangle h_1(t) + a_{11}(e_1(t)) \\ \qquad \times f(e_1(t)) + a_{12}(e_1(t))f(e_2(t)) + b_{11}(e_1(t)) \\ \qquad \times g(e_1(t-\tau)) + b_{12}(e_1(t))g(e_2(t-\tau)) \\ \qquad +\phi_1(t, e_1(t), e_1(t-\tau))\dot{\omega}_1(t), \\ \dot{e}_2(t) = -\sigma_2(e_2(t))e_2(t) + \triangle h_2(t) + a_{21}(e_2(t)) \\ \qquad \times f(e_1(t)) + a_{22}(e_2(t))f(e_2(t)) + e_{21}(e_2(t)) \\ \qquad \times g(e_1(t-\tau)) + b_{22}(e_2(t))g(e_2(t-\tau)) \\ \qquad +\phi_2(t, e_2(t), e_2(t-\tau))\dot{\omega}_2(t), \end{cases} \tag{16}$$

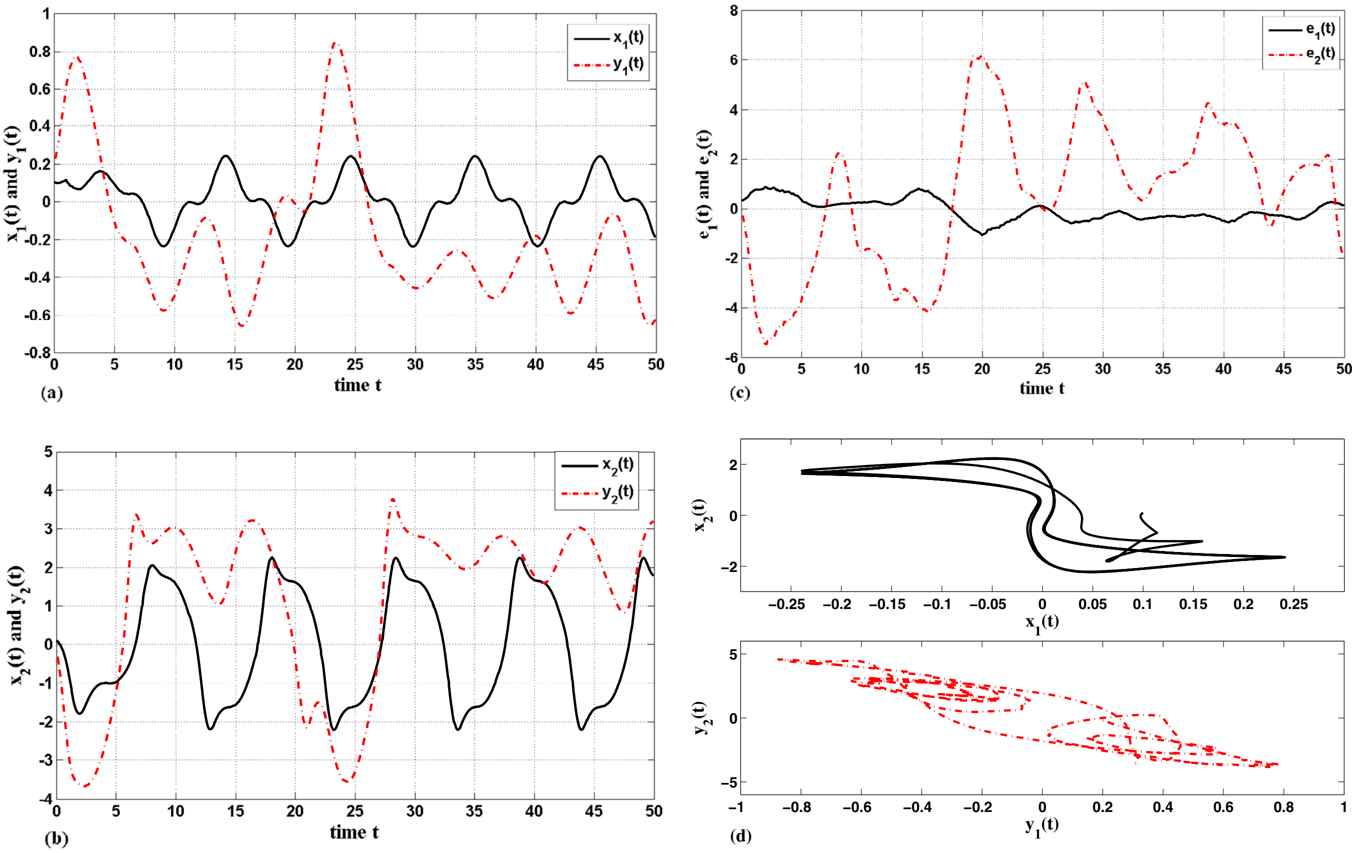

Fig. 3. (a, b) Time response curves of state variables, (c) the system error curves with non-model dynamical and stochastic perturbation, and (d) the phase curves of systems (14) and (15) without the adaptive controller $u(t)$.

Then, the system (16) with the controller $u(t)$ can be written as

$$\begin{cases} \dot{e}_1(t) = -\sigma_1(e_1(t))e_1(t) + \triangle h_1(t) + a_{11}(e_1(t)) \\ \qquad \times f(e_1(t)) + a_{12}(e_1(t))f(e_2(t)) + b_{11}(e_1(t)) \\ \qquad \times g(e_1(t-\tau)) + b_{12}(e_1(t))g(e_2(t-\tau)) \\ \qquad + u_1(t) + \phi_1(t, e_1(t), e_1(t-\tau))\dot{\omega}_1(t), \\ \dot{e}_2(t) = -\sigma_2(e_2(t))e_2(t) + \triangle h_2(t) + a_{21}(e_2(t)) \\ \qquad \times f(e_1(t)) + a_{22}(e_2(t))f(e_2(t)) + e_{21}(e_2(t)) \\ \qquad \times g(e_1(t-\tau)) + b_{22}(e_2(t))g(e_2(t-\tau)) \\ \qquad + u_2(t) + \phi_2(t, e_2(t), e_2(t-\tau))\dot{\omega}_2(t). \end{cases} \tag{17}$$

Using the adaptive controller $u(t)$ for the response system and the drive-response concept, the error system (17) quickly converge to zero and state trajectories of the response system anti-synchronized the drive system.

We choose

$$k_1 = k_2 = 0.01, \quad \upsilon = \gamma = 1$$

and

$$\underline{\Sigma} = \min(\hat{\Sigma}, \check{\Sigma}), \quad \bar{A} = \max(\hat{A}, \check{A}), \quad \bar{B} = \max(\hat{B}, \check{B}).$$

By using the MATLAB LMI toolbox, we obtain the following feasible solutions by Theorem 1:

$$P = \begin{pmatrix} 0.2364 & 0.0470 \\ 0.0470 & 0.0570 \end{pmatrix},$$

$$Q = \begin{pmatrix} 2.4208 & 0.0154 \\ 0.0154 & 2.0029 \end{pmatrix},$$

$$\rho = 1.1133, \theta_1 = 0.4313,$$

$$\theta_2 = 0.8450, \delta = 5.5094.$$

Thus, we can calculate the decay rate as

$$\beta = \delta/\lambda_{\min}(P) = 121.3642.$$

When $k_1(0) = 0.1$ and $k_2(0) = 0.2$, Figure 3 depicts the time responses of the state variables, the phase curves of systems (14) and (15), and the synchronization error (17) without the robust adaptive controller $u(t)$ designed in (12). Figure 4 shows the time response of the state variables, the phase curves of systems (14) and (15),

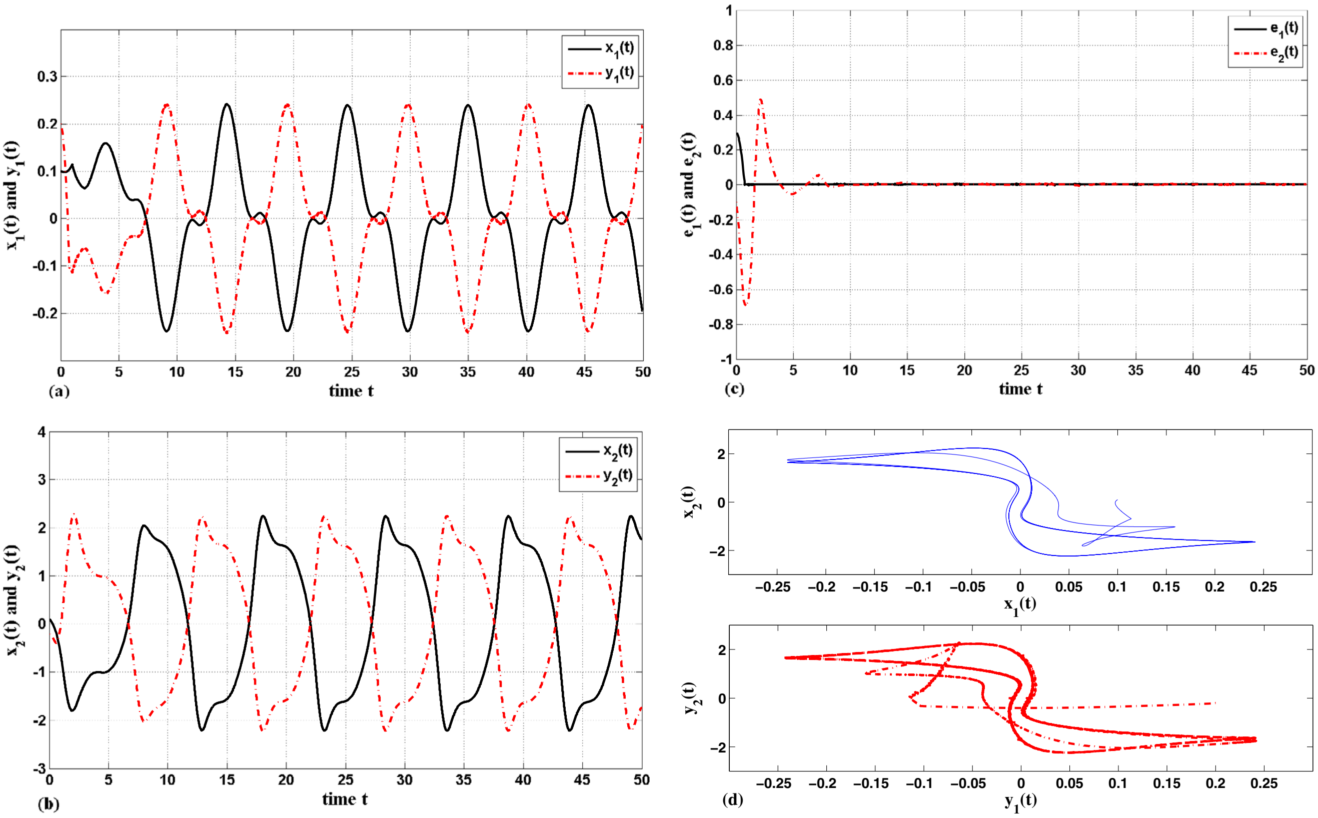

Fig. 4. (a, b) Time response curves of state variables, (c) the system error curves with non-model dynamical and stochastic perturbation, and (d) the phase curves of systems (14) and (15) with the adaptive controller $u(t)$.

and the synchronization errors (17) with the adaptive controller $u(t)$ designed in (12), where we can see that the drive system (14) and the response system (15) with the adaptive controller indeed become exponentially anti-synchronized in the mean square sense the synchronization error can also quickly converge to zero.

6 Conclusion

An adaptive controller which is constructed based on the differential inclusions theory, linear matrix inequalities, Gronwall's inequality, and adaptive control technique achieve the adaptive anti-synchronization of stochastic memristor-based neural network with non-modeled dynamics in this paper.

Two parts of the controller are designed that one is to facilitate the system itself synchronizing with the target drive system and the other is to eliminate the influence of the response system's uncertainties. In addition, by using LMIs approach and the Lyapunov functional method, the sufficient conditions for synchronization are obtained. A numerical example has shown that our method is effective.

The authors thank all the editor and the anonymous referees for their constructive comments and valuable suggestions, which are helpful to improve the quality of this paper. The work is supported by the National Natural Science Foundation of China (Grant Nos. 61170269, 61472045), the Beijing Higher Education Young Elite Teacher Project (Grant No. YETP0449), and the Beijing Natural Science Foundation (Grant No. 4142016).

All authors contributed equally to the paper.

Appendix

Proof of theorem 1. The system (11) under the controller $u(t)$ can be written as:

$$\begin{aligned} de(t) = [&-\Sigma(t)e(t) + \triangle h(t) + A(t)f(e(t)) \\ &+ B(t)g(e(t-\tau)) + u(t)]dt \\ &+ \phi(t, e(t), e(t-\tau))d\omega(t). \end{aligned} \tag{A.1}$$

Consider the controlled system (A.1) with controller (12), we construct a Lyapunov functional in the form of

$$\begin{aligned} V(t) = e^T(t)Pe(t) + \int_{t-\tau}^{t} e^T(s)Qe(s)ds \\ + \frac{1}{\gamma}(k_1(t) - k_1)^2 + \frac{1}{\mu}(k_2(t) - k_2)^2. \end{aligned} \tag{A.2}$$

Moreover, from Lemma 2, we obtain the operator

$$\begin{aligned}\mathscr{L}V(t) &= 2e^T(t)P[-\Sigma(t)e(t) + \triangle h(t) + A(t)f(e(t))\\&\quad + B(t)g(e(t-\tau)) + u(t)]\\&\quad + \text{trace}[\phi^T(t,e(t),e(t-\tau))\\&\quad \times P\phi(t,e(t),e(t-\tau))] + e^T(t)Qe(t)\\&\quad - e^T(t-\tau)Qe(t-\tau) + \frac{2}{\gamma}(k_1(t)-k_1)\dot{k_1}(t)\\&\quad + \frac{2}{\mu}(k_2(t)-k_2)\dot{k_2}(t),\\&= e^T(t)[-P\Sigma(t) - \Sigma^T(t)P + Q]e(t) + 2e^T(t)P\triangle h(t)\\&\quad + 2e^T(t)PA_tf(e(t)) + 2e^T(t)PB_tg(e(t-\tau))\\&\quad + 2e^T(t)Pu(t) + \text{trace}[\phi^T(t,e(t),e(t-\tau))\\&\quad \times P\phi(t,e(t),e(t-\tau))] - e^T(t-\tau)Qe(t-\tau)\\&\quad + \frac{2}{\gamma}(k_1(t)-k_1)\dot{k_1}(t) + \frac{2}{\mu}(k_2(t)-k_2)\dot{k_2}(t).\end{aligned}$$

It is clear from Lemma 3 that

$$\begin{aligned}2e^T(t)PA_tf(e(t)) &\le \theta_1 e^T(t)P\bar{A}\bar{A}^TPe(t)\\&\quad + \frac{1}{\theta_1}f^T(e(t))f(e(t))\\&\le \theta_1 e^T(t)P\bar{A}\bar{A}^TPe(t)\\&\quad + \frac{1}{\theta_1}e^T(t)l_1^2e(t),\\2e^T(t)PB_tg(e(t-\tau)) &\le \theta_2 e^T(t)P\bar{B}\bar{B}^TPe(t)\\&\quad + \frac{1}{\theta_2}g^T(e(t-\tau))g(e(t-\tau))\\&\le \theta_2 e^T(t)P\bar{B}\bar{B}^TPe(t)\\&\quad + \frac{1}{\theta_2}e^T(t-\tau)l_2^2e(t-\tau).\end{aligned}$$

From Assumptions 2 and 3, we have

$$2e^T(t)P\triangle h(t) \le 2\|e^T(t)P\|\cdot\|\triangle h(t)\|_\infty \le 2h\|Pe(t)\|,$$

and

$$\begin{aligned}&\text{trace}[\phi^T(t,e(t),e(t-\tau))P\phi(t,e(t),e(t-\tau)]\\&\quad\le \lambda_{\max}(P)\text{trace}[\phi^T(t,e(t),e(t-\tau))\phi(t,e(t),e(t-\tau))]\\&\quad\le \rho(e^T(t)R_1e(t) + e^T(t-\tau)R_2e(t-\tau)).\end{aligned}$$

Then we obtain

$$\begin{aligned}\mathscr{L}V(t) &\le e^T(t)\Big[-P\underline{\Sigma} - \underline{\Sigma}^TP + Q + \theta_1P\bar{A}\bar{A}^TP\\&\quad + \frac{1}{\theta_1}l_1^2 + \theta_2P\bar{B}\bar{B}^TP + 2k_1P + \rho R_1\Big]e(t)\\&\quad + e^T(t-\tau)\left[\frac{1}{\theta_2}l_2^2 - Q + \rho R_2\right]e(t-\tau)\\&\quad + 2(h-k_2)\|Pe(t)\|\\&\le e^T(t)\varPhi_1e(t) + e^T(t-\tau)\varPhi_2e(t-\tau)\\&\quad - \delta e^T(t)e(t),\end{aligned}\tag{A.3}$$

where

$$\begin{aligned}\varPhi_1 &= -P\underline{\Sigma} - \underline{\Sigma}^TP + Q + \theta_1P\bar{A}\bar{A}^TP + \frac{1}{\theta_1}l_1^2\\&\quad + \theta_2P\bar{B}\bar{B}^TP + 2k_1P + \rho R_1 + \delta I,\\\varPhi_2 &= \frac{1}{\theta_2}l_2^2 - Q + \rho R_2.\end{aligned}$$

Considering the condition of Theorem 1 and the Schur complement, we get

$$\mathscr{L}V(t) \le -\delta e^T(t)e(t).$$

It follows from Itôs formula that

$$EV(t) - EV(0) = \int_0^t E\mathscr{L}V \le -\delta\int_0^t E(e(s)e(s))ds,\tag{A.4}$$

which means that

$$\lambda_{\min}(P)E\|e(t)\|^2 \le EV(t) \le EV(0) - \delta\int_0^t E\|e(s)\|^2ds.$$

By using Lemma 1, we have

$$E\|e(t)\|^2 \le \alpha\exp(-\beta t),$$

where $\alpha = EV(0)/\lambda_{\min}(P)$, $\beta = \delta/\lambda_{\min}(P)$. On the other hand, we denote

$$\begin{aligned}EV(0) &= Ee^T(0)e(0) + \int_{-\tau}^0 Ee^T(s)e(s)ds\\&\quad + \frac{1}{\gamma}(k_1(0)-k_1)^2 + \frac{1}{\mu}(k_2(0)-k_2)^2\\&\le \lambda_{\max}(P)E\sup_{-\tau\le\theta\le0}\|\varepsilon(\theta)\|^2\\&\quad + \tau\lambda_{\max}(Q)E\sup_{-\tau\le\theta\le0}\|\varepsilon(\theta)\|^2\\&\quad + \frac{1}{\gamma}(k_1(0)-k_1)^2 + \frac{1}{\mu}(k_2(0)-k_2)^2.\end{aligned}\tag{A.5}$$

Through the above calculation, we finally obtain

$$E\|e(t)\|^2 \le \alpha\exp(-\beta t),$$

where

$$\alpha = [\lambda_{\max}(P) + \tau\lambda_{\max}(Q)E \sup_{-\tau\leq\theta\leq 0} \|\varepsilon(\theta)\|^2 + \frac{1}{\gamma}(k_1(0) - k_1)^2 + \frac{1}{\mu}(k_2(0) - k_2)^2] \times (\lambda_{\min}(P))^{-1}.$$

Therefore, by Definition 1 we see that systems (6) and (8) can be exponentially synchronized in the mean square sense with a decay rate β. The proof of Theorem 1 is completed.

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