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Anti-uniform semilattices

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An inverse semigroup which is a union of groups is called *Cliffordian*. A semilattice E is called *universally Cliffordian* if *every* inverse semigroup having E as semilattice of idempotents is Cliffordian. It is shown that E is universally Cliffordian if and only if it is *anti-uniform*, that is, if and only if no two distinct principal ideals of E are isomorphic.

A semilattice E satisfying the minimum condition is anti-uniform if and only if it is a well-ordered chain. Examples are given of anti-uniform semilattices of more complicated types.

A.H. Clifford [2] has given a complete description (in terms of groups and semilattices) of the structure of inverse semigroups that are unions of groups, and for this reason we shall (as in [7]) refer to such inverse semigroups as *Cliffordian*. If E is a given semilattice, there do of course exist Cliffordian inverse semigroups having E as semilattice of idempotents, the simplest such being E itself. Let us temporarily call a semilattice E *universally Cliffordian* if *every* inverse semigroup having E as semilattice of idempotents is Cliffordian. The content of Theorem 7.5 in [3] is that E is universally Cliffordian if it is finite and forms a chain under its natural ordering. More generally, one of us [7] has shown that E is universally Cliffordian if it is a well-ordered chain (under its natural ordering).

It was conjectured in [7] that the converse also holds: E is universally Cliffordian *only if* it is a well-ordered chain. The purpose of this note is to show that this conjecture is correct provided we restrict attention to semilattices with minimum condition. In general the conjecture

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is false.

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1. Anti-uniformity

Various basic definitions and results of semigroup theory, all to be found in Clifford and Preston [3], will be used without comment.

If E is a semilattice, we define an equivalence relation $\mathcal{U}$ on E by

$$(e,f) \in \mathcal{U} \text{ if and only if } Ee \cong Ef .$$

If $\mathcal{U}$ is the identical relation on E we shall say that E is *anti-uniform* (a term suggested by Munn's use of 'uniform' for semilattices in which $\mathcal{U}$ is the universal relation [6]).

If S is an inverse semigroup having E as its semilattice of idempotents, and if $\mathcal{D}$ denotes Green's equivalence, then $(e,f) \in \mathcal{D} \cap (E \times E)$ if and only if there exists a in S such that $aa^{-1} = e$ and $a^{-1}a = f$. (See, for example, the proof given of Lemma 8.34 in [3].) In this case, as remarked in [4], there is an isomorphism $\alpha_a : Ee \to Ef$ defined by

$$x\,\alpha_a = a^{-1}\,x\,a \qquad (x \in Ee) .$$

Thus

$$\mathcal{D} \cap (E \times E) \subseteq \mathcal{U} .$$

Now, an inverse semigroup is Cliffordian if and only if $aa^{-1} = a^{-1}a$ for every a, that is, if and only if

$$\mathcal{D} \cap (E \times E) = \iota_E ,$$

the identical relation on E. Thus we have established half of the following theorem, which reduces our problem to one purely in the theory of partially ordered sets.

THEOREM 1. *A semilattice* E *is universally Cliffordian if and only if it is anti-uniform.*

Proof. To establish the remaining half, we shall show that if E is *not* anti-uniform then there exists a non-Cliffordian inverse semigroup having E as its semilattice of idempotents. To do this, we use a construction due to Munn [6]. For each (e,f) in U, let $T_{e,f}$ be the set of all isomorphisms from Ee onto Ef, and let

$$T_E = \bigcup\{T_{e,f} : (e,f) \in U\} .$$

Munn has shown [6, Lemma 2.2] that, under the usual multiplication of one-one partial mappings, T_E is an inverse semigroup. Moreover, the set of idempotents of T_E is $E^* = \{\varepsilon_e : e \in E$, where ε_e is the identical mapping of Ee onto itself. Since $\varepsilon_e \varepsilon_f = \varepsilon_{ef}$ it is possible to *identify* ε_e with e and to say that E is the semilattice of idempotents of T_E.

If E is not anti-uniform, there exists $\alpha \in T_{e,f} \subseteq T_E$ with $e \neq f$. Clearly $\alpha\alpha^{-1} = e$, $\alpha^{-1}\alpha = f$, and so T_E is not Cliffordian. This completes the proof.

We can now prove the conjecture in [7] in the case where E has the minimum condition (by which we mean that every non-empty subset of E has at least one minimal element).

THEOREM 2. *If a semilattice has the minimum condition and is anti-uniform, then it is a well-ordered chain.*

Proof. We show (what is clearly sufficient) that if E is not totally ordered, then it is not anti-uniform.

Let us define a subset K of E by saying that $x \in K$ if there exist elements of E that are incomparable with x. Clearly $K \neq \phi$ if E is not totally ordered; let e be a minimal element of K, and let f be a minimal member of the non-empty set of elements of E that are incomparable with e. Then $ef < e$ and $ef < f$, since otherwise e and f would be comparable. In fact e *covers* ef, for if g is such that $ef \leqq g < e$, then $fef \leqq fg \leqq fe$ and so $fg = ef$. But g must be comparable with f, and so $fg = f$ or $fg = g$. The former alternative leads to $f = ef$, a contradiction; the latter alternative leads to $g = ef$. Thus e covers ef, and similarly f covers ef. It follows that

$$Ee = E\,ef \cup \{e\}\ , \qquad Ef = E\,ef \cup \{f\}\ ,$$

and so $(e,f) \in U$, an obvious isomorphism from Ee onto Ef being that which associates e with f and every other element with itself. This completes the proof.

2. Examples

We describe first a countable totally ordered set E which is anti-uniform but not well-ordered. The set in question has been described by Anne C. Morel [5, p.70] and is quoted by Chang and Ehrenfeucht [1, p.143], the property of interest there being not dissimilar to the property of anti-uniformity.[1]

Consider the set Q of rational numbers, and let ε be any injection of Q into $N = \{0, 1, 2, \ldots \}$. Let

$$E = \bigcup_{q\in Q} (\{q\} \times \{0, 1, \ldots, \varepsilon(q)\})\ ,$$

and define an order relation $\leqq$ on E lexicographically:

$$(q,m) \leqq (r,n)$$

if and only if either $q < r$ or $q = r$ and $m \leqq n$.

Note that the set of elements of E having no immediate predecessors is $\{(q,0) : q \in Q\}$, and the set of elements having no immediate successors is $\{(q,\varepsilon(q)) : q \in Q\}$.

To show that E is anti-uniform, suppose, by way of contradiction, that there exist distinct elements (q,m) , (r,n) in E for which there is an isomorphism $\phi : E(q,m) \to E(r,n)$. We distinguish two cases:

(i) $m \neq n$;

(ii) $m = n$ and $q \neq r$.

In case (i), certainly

$$(q,m)\phi = (r,n)\ , \quad (q,m-1)\phi = (r,n-1)\ , \ \ldots\ldots\ .$$

If, without essential loss of generality, we suppose that $m > n$, we eventually find that

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[1]We are indebted to Dr J.N. Crossley for these references.

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$$(q,m-n)\phi = (r,0) ,$$

a contradiction, since $(q,m-n)$ has an immediate predecessor and $(r,0)$ does not.

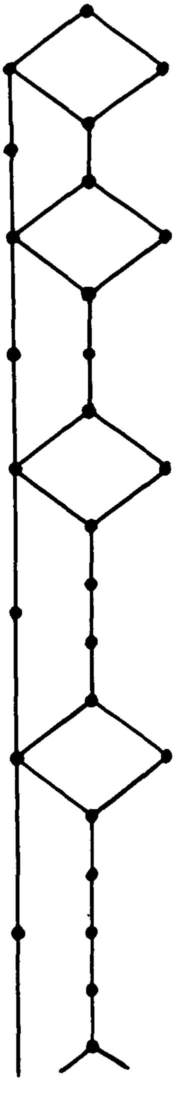
In case (ii) we may suppose without loss of generality that $q > r$. If s is such that $r < s < q$, then $(s,0) \in E(q,m)$, and has no immediate predecessor; hence $(s,0)\phi \in E(r,n)$, and has no immediate predecessor: that is, $(s,0)\phi = (t,0)$, where $t \leqq r$. Certainly $t \neq s$ and so $\varepsilon(t) \neq \varepsilon(s)$. We shall consider only the case in which $\varepsilon(s) > \varepsilon(t)$, since the other case is similar. Now

$$(s,1)\phi = (t,1) , \quad (s,2)\phi = (t,2) \ \ldots\ldots ,$$

and finally $(s,\varepsilon(t))\phi = (t,\varepsilon(t))$, a contradiction, since $(s,\varepsilon(t))$ has an immediate successor and $(t,\varepsilon(t))$ does not.

The set E is of course not well-ordered, since for each q in Q the non-empty subset $\{x \in E : x > (q,\varepsilon(q))\}$ does not have a least element.

An anti-uniform semilattice need not be totally ordered. If, for example, we take the union of the semilattice E described above with the semilattice $N = \{0, 1, 2, 3, \ldots\}$ under the natural ordering, and define $en = 0$ for all e in E and n in N, we obtain an anti-uniform semilattice. The diagram illustrates an example of an anti-uniform semilattice satisfying the maximum condition, suggested to us by Munn.



etc.

We are unable to give a complete classification of anti-uniform semilattices.

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