Anti-Windup Design with Guaranteed Regions of Stability for Discrete-Time Linear Systems

J.M. Gomes da Silva Jr. and S. Tarbouriech

***Abstract*— The purpose of this paper is to study the determination of stability regions for discrete-time linear systems with saturating controls through anti-windup schemes. Considering that a linear dynamic output feedback has been designed to stabilize the linear discrete-time system (without saturation), a method is proposed for designing an anti-windup gain that maximizes an estimate of the basin of attraction of the closed-loop system in the presence of saturation. It is shown that the closed-loop system obtained from the controller plus the anti-windup gain can be locally modelled by a linear system with a deadzone nonlinearity. Then, based on the proposition of a new sector condition and quadratic Lyapunov functions, stability conditions in an LMI form are stated. These conditions are then considered in a convex optimization problem in order to compute an anti-windup gain that maximizes an estimate of the basin of attraction of the closed-loop system. Moreover, considering stable open-loop systems, it is shown that the conditions can be slightly modified in order to determine an anti-windup gain that ensures global stability.**

I. INTRODUCTION

The basic idea underlining anti-windup designs for linear systems with saturating actuators is to introduce control modifications in order to recover, as much as possible, the performance induced by a previous design carried out on the basis of the unsaturated system. First results on anti-windup consisted in ad-hoc methods intended to work with standard PID controllers [1], [2] which are commonly used in present commercial controllers. Nonetheless, major improvements in this field have been achieved in the last decade as it can be observed in [3], [4], [5], [6], [7], [8], [9], [10] among others.

Several results on the anti-windup problem are concerned with achieving global stability properties. Since global results cannot be achieved for open-loop unstable linear systems in the presence of actuator saturation, local results have to be developed. In this context, a key issue is the determination of domains of stability for the closed-loop system (estimates of the regions of attraction). With very few exceptions, most of the local results available in the literature of anti-windup do not provide explicit characterization of the domain of stability.

In [11] and [12], considering continuous-time systems, an attempt has being made to fill in this gap by providing design algorithms that explicitly optimize a criterion aiming at maximizing a stability domain of the closed-loop system. In [12], the modelling of the nonlinear behavior of the system under saturation is made by using a polytopic differential inclusion and quadratic Lyapunov functions. On the other hand, in [11], based on a transformation of the saturation term in a deadzone nonlinearity, classical sector condition and S-procedure techniques are used to derive stability conditions considering both quadratic and Lure type Lyapunov functions. The main drawback of the approaches above is that the conditions allowing to compute the anti-windup gains are given in terms of bilinear matrix inequalities (BMIs). In order to overcome this difficulty, iterative LMI algorithms are proposed to solve the synthesis problem. It is well-known that, in general, this kind of approach does not lead to global optimal solutions and are very sensitive to the initialization [13].

On the other hand, the anti-windup problem for discrete-time systems has received less attention in the literature. It has been addressed in [1], [14] (see references therein), in the scope of the conditioning technique, and in [15] in the context of constrained regulation. Recently, in [16], the anti-windup problem for discrete-time linear systems was addressed in an $\mathcal{L}_2$-norm performance context. Similarly as in the continuous-time case, the proposed designs do not explicitly address the problem of enlarging the domain of stability of the closed loop system.

Hence, considering discrete-time systems, the aim of this paper consists in providing a technique that allows the computation of anti-windup loops in order to enlarge the region of asymptotic stability of the closed-loop system. The theoretical conditions are obtained from the proposition of a modified sector condition and the use of quadratic Lyapunov functions. Thus, differently from [11] and [12], the stability conditions are directly formulated in an LMI form, avoiding the necessity of applying iterative algorithms. Furthermore it is shown that this new sector condition encompasses the classical one, largely applied in the literature (see for instance [11] and references therein). This fact introduces in the problem more degrees of freedom, which leads to less conservative solutions. On the other hand, considering stable open-loop systems, it is shown that the theoretical conditions can be slightly modified in order to determine anti-windup gains that ensure global stability.

The paper is organized as follows. In section II, we state the problem being considered and we provide the main definitions and concepts required in the paper. Stability conditions for the closed-loop system are provided in section III

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by employing quadratic Lyapunov functions. Based on the results of section III, an LMI-based convex optimization problem to synthesize the anti-windup gain is proposed in section IV. Section V provides numerical examples, illustrating the effectiveness of the proposed design technique. Concluding remarks are given in section VI.

Notations. For any vector $x \in \Re^n$, $x \succeq 0$ means that all the components of x, denoted $x_{(i)}$, are nonnegative. For two vectors x, y of $\Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0, \forall i = 1, \ldots, n$. The elements of a matrix $A \in \Re^{m \times n}$ are denoted by $A_{(i,j)}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. $A_{(i)}$ denotes the ith row of matrix A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A. $diag(x)$ denotes a diagonal matrix obtained from vector x. I_m denotes the m-order identity matrix. $Co\{\cdot\}$ denotes a convex hull.

II. Problem Statement

Consider the discrete-time linear system

$$\begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, the input and the measured output vectors, respectively, and $t = 0, 1, 2, \ldots$. Matrices A, B and C are real constant matrices of appropriate dimensions.

Considering system (1), we assume that an n_c-order dynamic output feedback stabilizing compensator

$$\begin{array}{rcl} x_c(t+1) & = & A_c x_c(t) + B_c y(t) \\ v_c(t) & = & C_c x_c(t) + D_c y(t) \end{array} \tag{2}$$

where $x_c(t) \in \Re^{n_c}$ is the controller state, $u_c = y(t)$ is the controller input and $v_c(t)$ is the controller output, was designed to guarantee some performance requirements and the stability of the closed-loop system in the absence of control saturation.

Suppose now that the input vector u is subject to amplitude limitations defined as follows:

$$-u_0 \preceq u \preceq u_0 \tag{3}$$

where $u_{0(i)} > 0, i = 1, ..., m$, denote the control amplitude bounds. In consequence of the control bounds, the actual control signal to be injected in the system is a saturated one, that is,

$$u(t) = sat(v_c(t)) = sat(C_c x_c(t) + D_c C x(t)) \tag{4}$$

where each component of $sat(v_c(t))$ is defined, $\forall i = 1, ..., m$, by:

$$sat(v_c(t))_{(i)} \triangleq \left\{ \begin{array}{ll} -u_{0(i)} & \text{if } v_{c(i)}(t) < -u_{0(i)} \\ v_{c(i)}(t) & \text{if } -u_{0(i)} \leq v_{c(i)}(t) \leq u_{0(i)} \\ u_{0(i)} & \text{if } v_{c(i)}(t) > u_{0(i)} \end{array} \right. \tag{5}$$

In order to mitigate the undesirable effects of the windup, caused by input saturation, an anti-windup term $E_c(sat(v_c(t)) - v_c(t))$ can be added to the controller [2], [5]. Thus, considering the dynamic controller and this anti-windup strategy, the closed-loop system reads:

$$\begin{array}{rcl} x(t+1) & = & Ax(t) + Bsat(v_c(t)) \\ y(t) & = & Cx(t) \\ x_c(t+1) & = & A_c x_c(t) + B_c y(t) + E_c(sat(v_c(t)) - v_c(t)) \\ v_c(t) & = & C_c x_c(t) + D_c y(t) \end{array} \tag{6}$$

Define now an extended state vector

$$\xi(t) = \left[\begin{array}{c} x(t) \\ x_c(t) \end{array} \right] \in \Re^{n+n_c} \tag{7}$$

and the following matrices

$$\mathbf{A} = \left[\begin{array}{cc} A + BD_cC & BC_c \\ B_cC & A_c \end{array} \right] ; \; \mathbf{B} = \left[\begin{array}{c} B \\ 0 \end{array} \right]$$

$$\mathbf{R} = \left[\begin{array}{c} 0 \\ I_{n_c} \end{array} \right] ; \; \mathbf{K} = \left[\begin{array}{cc} D_cC & C_c \end{array} \right]$$

Hence, from (7) and (6), the closed-loop system reads:

$$\xi(t+1) = \mathbf{A}\xi(t) - (\mathbf{B} + \mathbf{R}E_c)\psi(\mathbf{K}\xi(t)) \tag{8}$$

with the function $\psi(v) \triangleq v - sat(v)$. Note that, in this case, $\psi(v)$ corresponds to a decentralized deadzone nonlinearity $\psi(v) = \left[\begin{array}{ccc} \psi(v_{(1)}) & \ldots & \psi(v_{(m)}) \end{array} \right]'$, where

$$\psi(v_{(i)}) \triangleq \left\{ \begin{array}{lll} v_{(i)} - u_{0(i)} & \text{if} & v_{(i)} > u_{0(i)} \\ 0 & \text{if} & -u_{0(i)} \leq v_{0(i)} \leq u_{0(i)} \\ v_{(i)} + u_{0(i)} & \text{if} & v_{(i)} < -u_{0(i)} \end{array} \right. \tag{9}$$

$\forall i = 1, \ldots, m$

Since, by hypothesis, the controller (2) is supposed to stabilize system (1) in the absence of saturation, the matrix $\mathbf{A}$ is Schur-Cohn, i.e., in the absence of control bounds, the closed-loop system would be globally stable.

The basin of attraction of system (8) is defined as the set of all $\xi \in \Re^{n+n_c}$ such that for $\xi(0) = \xi$ the corresponding trajectory converges asymptotically to the origin. In particular, when the global stability of the system is ensured the basin of attraction corresponds to the whole state space. However, in the general case, the exact characterization of the basin of attraction is not possible. In this case, it is important to obtain estimates of the basin of attraction. Consider then the following definition:

Definition 1: A set $\mathcal{E}$ is said to be a region of asymptotic stability for the system (8) if for all $\xi(0) \in \mathcal{E}$ the corresponding trajectory converges asymptotically to the origin.

Hence, the idea is to use regions of stability in order to approximate the basin of attraction [17].

The problem we aim to solve throughout this paper is summarized as follows.

Problem 1: Determine the anti-windup gain matrix E_c and an associated region of asymptotic stability, as large as possible, for the closed-loop system (8).

Of course, the implicit objective in Problem 1 is to optimize the size of the basin of attraction for the closed-loop system (8) over the choice of the gain matrix E_c. This can be accomplished indirectly by searching for an anti-windup gain E_c that leads to a region of stability for the closed-loop system as large as possible or even that ensures global stability.

In order to address Problem 1, we propose to use quadratic Lyapunov functions and ellipsoidal regions of stability, as will be seen in the sequel.

III. Stability Conditions

Consider a matrix $G \in \Re^{m\times(n+n_c)}$ and define the following polyhedral set

$$\mathcal{S} \stackrel{\triangle}{=} \{\xi \in \Re^{n+n_c} \; ; \; -u_0 \preceq (\mathbf{K}-G)\xi \preceq u_0\} \qquad (10)$$

Lemma 1: Consider the function $\psi(v)$ defined in (9). If $\xi \in \mathcal{S}$ then the relation

$$\psi(\mathbf{K}\xi)'T[\psi(\mathbf{K}\xi) - G\xi] \leq 0 \qquad (11)$$

is verified for any matrix $T \in \Re^{m\times m}$ diagonal and positive definite.

Proof: Consider the three cases below.

(a): $-u_{0(i)} \leq \mathbf{K}_{(i)}\xi \leq u_{0(i)}$
In this case, by definition, $\psi(\mathbf{K}_{(i)}\xi) = 0$ and then

$$\psi(\mathbf{K}_{(i)}\xi)T_{(i,i)}[\psi(\mathbf{K}_{(i)}\xi) - G_{(i)}\xi] = 0$$

(b): $\mathbf{K}_{(i)}\xi > u_{0(i)}$

In this case, $\psi(\mathbf{K}_{(i)}\xi) = \mathbf{K}_{(i)}\xi - u_{0(i)}$. If $\xi \in \mathcal{S}$ it follows that $\mathbf{K}_{(i)}\xi - G_{(i)}\xi \leq u_{0(i)}$. Hence, it follows that:

$$\psi(\mathbf{K}_{(i)}\xi) - G_{(i)}\xi = \mathbf{K}_{(i)}\xi - u_{0(i)} - G_{(i)}\xi \leq 0$$

and, since in this case $\psi(\mathbf{K}_{(i)}\xi) > 0$, one gets

$$\psi(\mathbf{K}_{(i)}\xi)T_{(i,i)}[\psi(\mathbf{K}_{(i)}\xi) - G_{(i)}\xi] \leq 0$$

for all $T_{(i,i)} > 0$.

(c): $\mathbf{K}_{(i)}\xi < -u_{0(i)}$
In this case, $\psi(\mathbf{K}_{(i)}\xi) = \mathbf{K}_{(i)}\xi + u_{0(i)}$. If $\xi \in \mathcal{S}$ it follows that $\mathbf{K}_{(i)}\xi - G_{(i)}\xi \geq -u_{0(i)}$. Hence, it follows that:

$$\psi(\mathbf{K}_{(i)}\xi) - G_{(i)}\xi = \mathbf{K}_{(i)}\xi + u_{0(i)} - G_{(i)}\xi \geq 0$$

and, since in this case $\psi(\mathbf{K}_{(i)}\xi) < 0$, one gets

$$\psi(\mathbf{K}_{(i)}\xi)T_{(i,i)}[\psi(\mathbf{K}_{(i)}\xi) - G_{(i)}\xi] \leq 0$$

for all $T_{(i,i)} > 0$.

From the 3 cases above, once $\xi \in \mathcal{S}$ we can conclude that $\psi(\mathbf{K}_{(i)}\xi)T_{(i,i)}[\psi(\mathbf{K}_{(i)}\xi) - G_{(i)}\xi] \leq 0$, $\forall T_{(i,i)} > 0$, $\forall i = 1, \ldots, m$, whence follows (11). □

Consider now as Lyapunov candidate function, the quadratic function

$$V(\xi(t)) = \xi(t)'P\xi(t) \qquad (12)$$

where $P = P' > 0, \quad P \in \Re^{(n+n_c)\times(n+n_c)}$.

Theorem 1: If there exist a symmetric positive definite matrix $W \in \Re^{(n+n_c)\times(n+n_c)}$, a matrix $Y \in \Re^{m\times(n+n_c)}$ and a matrix $Z \in \Re^{n_c\times m}$, a diagonal positive definite matrix $S \in \Re^{m\times m}$ satisfying:

$$\begin{bmatrix} W & -Y' & -W\mathbf{A}' \\ -Y & 2S & S\mathbf{B}' + Z'\mathbf{R}' \\ -\mathbf{A}W & \mathbf{B}S + \mathbf{R}Z & W \end{bmatrix} > 0 \qquad (13)$$

$$\begin{bmatrix} W & W\mathbf{K}'_{(i)} - Y'_{(i)} \\ \mathbf{K}_{(i)}W - Y_{(i)} & u^2_{0(i)} \end{bmatrix} \geq 0, \; i = 1, ..., m \qquad (14)$$

then for the gain matrix $E_c = ZS^{-1}$ the ellipsoid $\mathcal{E}(P) = \{\xi \in \Re^{n+n_c}; \xi'P\xi \leq 1\}$, with $P = W^{-1}$, is a region of stability for the system (8).

Proof. The satisfaction of relations (14) implies that the set $\mathcal{E}(P)$ is included in the polyhedral set $\mathcal{S}$ defined as in (10) with $G = YP$ [18]. Hence, from Lemma 1, for all $\xi(t) \in \mathcal{E}(P)$ it follows that $\psi(\mathbf{K}\xi(t)) = \mathbf{K}\xi(t) - sat(\mathbf{K}\xi(t))$ satisfies the sector condition (11). By considering the quadratic Lyapunov function as defined in (12) and by computing the variation of $V(\xi(t))$ along the trajectories of system (8) one gets:

$$\begin{aligned} \Delta V(\xi(t)) = V(\xi(t)) - V(\xi(t+1)) = \\ \xi(t)'P\xi(t) - \xi(t)'(\mathbf{A}'P\mathbf{A})\xi(t) \\ +2\xi(t)'\mathbf{A}'P(\mathbf{B}+\mathbf{R}E_c)\psi(\mathbf{K}\xi(t)) \\ -\psi(\mathbf{K}\xi(t))'(\mathbf{B}+\mathbf{R}E_c)'P(\mathbf{B}+\mathbf{R}E_c)\psi(\mathbf{K}\xi(t)) \end{aligned} \qquad (15)$$

Thus, by using the sector condition (11) it follows that [1]:

$$\begin{aligned} \Delta V(\xi(t)) \geq \xi'P\xi - \xi'(\mathbf{A}'P\mathbf{A})\xi + 2\xi'\mathbf{A}'P(\mathbf{B}+\mathbf{R}E_c)\psi \\ - \psi'(\mathbf{B}+\mathbf{R}E_c)'P(\mathbf{B}+\mathbf{R}E_c)\psi + 2\psi'T[\psi - G\xi] \end{aligned} \qquad (16)$$

$\forall T > 0$, T diagonal, or equivalently

$$\Delta V(\xi(t)) \geq \begin{bmatrix} \xi' & \psi' \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_2' & X_3 \end{bmatrix} \begin{bmatrix} \xi \\ \psi \end{bmatrix} \qquad (17)$$

where

$$\begin{aligned} X_1 &= P - \mathbf{A}'P\mathbf{A} \\ X_2 &= \mathbf{A}'P(\mathbf{B}+\mathbf{R}E_c) - G'T \\ X_3 &= 2T - (\mathbf{B}+\mathbf{R}E_c)'P(\mathbf{B}+\mathbf{R}E_c) \end{aligned}$$

Note now that, by Schur's complement, relation (13) is equivalent to

$$\begin{bmatrix} W & -Y' \\ -Y & 2S \end{bmatrix} - X_4'PX_4 > 0 \qquad (18)$$

with $X_4 = [-\mathbf{A}W \quad (\mathbf{B}S + \mathbf{R}Z)]$

Considering now $T = S^{-1}$ and pre and post-multiplying (18) by $\begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix}$ it follows that

$$\begin{bmatrix} X_1 & X_2 \\ X_2' & X_3 \end{bmatrix} > 0$$

[1]For notational simplicity we drop the time dependence and consider $\xi(t) = \xi$ and $\psi(\mathbf{K}\xi(t)) = \psi$.

As a result, the quadratic form in (17) is positive definite implying $\Delta V(\xi(t)) > 0$ (i.e, $V(\xi(t+1) < V(\xi(t)))$. Since this reasoning is valid $\forall \xi(t) \in \mathcal{E}(P)$, $\xi(t) \neq 0$, it follows that the function $V(\xi(t))$ is strictly decreasing along the trajectories of system (8). Hence, we can conclude that $\mathcal{E}(P)$ is a stability region for system (8) which means that for any $\xi(0) \in \mathcal{E}(P)$, the corresponding trajectory converges asymptotically to the origin. □

Theorem 1 provides stability conditions for the system (8) in a local context. Considering the global stability, the following corollary can be stated.

Corollary 1: If there exist a symmetric positive definite matrix $W \in \Re^{(n+n_c)\times(n+n_c)}$, a diagonal positive definite matrix $S \in \Re^{m\times m}$ and a matrix $Z \in \Re^{n_c \times m}$ satisfying:

$$\begin{bmatrix} W & -W\mathbf{K}' & -W\mathbf{A}' \\ -\mathbf{K}W & 2S & S\mathbf{B}' + Z'\mathbf{R}' \\ -\mathbf{A}W & \mathbf{B}S + \mathbf{R}Z & W \end{bmatrix} > 0 \qquad (19)$$

then, for $E_c = ZS^{-1}$, system (8) is globally asymptotically stable.

Proof: Consider $G = \mathbf{K}$. It follows that (11) is verified for all $\xi \in \Re^{n+n_c}$. In this case, (19) corresponds to (13) and the global asymptotic stability follows. □

It should be recalled that the global stability can be achieved only when the matrix A has all its eigenvalues in the closed unit disc [16]. Hence, the global stability condition proposed in the Corollary 1 should be used only when A satisfies this assumption. Otherwise, only the local stability can be ensured.

Remark 1: The results in [11] for the continuous-time case are stated considering a classical sector condition:

$$\psi(\mathbf{K}\xi)'T[\psi(\mathbf{K}\xi) - \Lambda\mathbf{K}\xi] \leq 0, \quad \forall \xi \in S(\mathbf{K}, u_0^\lambda) \qquad (20)$$

where Λ is a positive diagonal matrix and the set $S(\mathbf{K}, u_0^\lambda)$ is a polyhedral set defined as follows:

$$S(\mathbf{K}, u_0^\lambda) = \{\xi \in \Re^{n+n_c}; -u_0^\lambda \preceq \mathbf{K}\xi \preceq u_0^\lambda\} \qquad (21)$$

with $u_{0(i)}^\lambda \triangleq \frac{u_{0(i)}}{1-\Lambda_{(i,i)}}, \ i = 1, ..., m$

Considering this sector condition and following a similar procedure to the one applied in the proof of Theorem 1, the following conditions are obtained for the discrete-time case:

$$\begin{bmatrix} W & -W\mathbf{K}'\Lambda & -W\mathbf{A}' \\ -\Lambda\mathbf{K}W & 2S & S\mathbf{B}' + Z'\mathbf{R}' \\ -\mathbf{A}W & \mathbf{B}S + \mathbf{R}Z & W \end{bmatrix} > 0 \qquad (22)$$

$$\begin{bmatrix} W & (1-\Lambda_{(i,i)})W\mathbf{K}'_{(i)} \\ (1-\Lambda_{(i,i)})\mathbf{K}_{(i)}W & \gamma u_{0(i)}^2 \end{bmatrix} \geq 0 \qquad (23)$$

$$0 < \Lambda_{(i,i)} \leq 1, \qquad i = 1, ..., m$$

Note that these matrix inequalities are bilinear in variables W and Λ. It is easy to see that (22) and (23) correspond to the conditions of Theorem 1 by taking $G = \Lambda\mathbf{K}$. Hence all the solutions obtained considering (22) and (23) are also feasible solutions for (13) and (14) which means that the new proposed condition is more generic and less conservative than the classical one. On the other hand, note that in (1) it appears a constant γ and, in this case, the domain of stability corresponds to a set $\mathcal{E}(P, \gamma^{-1}) = \{\xi \in \Re^{n+n_c} \ ; \ \xi' P \xi \leq \gamma^{-1}\}$. Considering the result obtained with the new sector condition, γ can be normalized as 1 without loss of generality.

IV. Numerical Anti-Windup Gain Design

Based on the result stated in Theorem 1, in this section we aim to present a numerical procedure in order to solve Problem 1. The main idea is to obtain an anti-windup gain matrix that ensures the local stability of the closed-loop system in a region of the state space $\Re^{n+n_c}$. We are then interested in one of the following cases:

1) A set of admissible initial conditions, $\Xi_0 \subset \Re^{n+n_c}$, for which asymptotic stability must be ensured, is given. In this case, E_c should be computed in order to ensure the stability in a set $\mathcal{E}(P)$ containing Ξ_0.
2) We aim to design the anti-windup gain in order to maximize the estimate of the basin of attraction associated to it. In other words, we want to compute E_c such that the associated region of asymptotic stability is as large as possible considering some size criterion.

Both cases can be addressed if we consider a set Ξ_0 with a given shape and a scaling factor β. For example, let Ξ_0 be defined as a polyhedral set described by its vertices:

$$\Xi_0 \triangleq Co\{v_r \in \Re^{n+n_c} \ ; \ r = 1, \ldots, n_r\}$$

Recalling Theorem 1, we aim at searching for matrices W, Y, S, Z in order to satisfy

$$\beta \ \Xi_0 \subset \mathcal{E}(P) \qquad (24)$$

In case 1, mentioned above, this problem reduces to a feasibility problem with $\beta = 1$ whereas in case 2, the goal will be to maximize β. Note that in the last case, Ξ_0 defines the directions in which we want to maximize $\mathcal{E}(P)$. The problem of maximizing β can be accomplished by solving the following optimization problem:

$$\begin{array}{c} \min\limits_{W,Z,S,Y,\mu} \mu \\ \text{subject to} \\ (i) \ \begin{bmatrix} W & -Y' & -W\mathbf{A}' \\ -Y & 2S & S\mathbf{B}' + Z'\mathbf{R}' \\ -\mathbf{A}W & \mathbf{B}S + \mathbf{R}Z & W \end{bmatrix} > 0 \\ (ii) \ \begin{bmatrix} W & W\mathbf{K}'_{(i)} - Y'_{(i)} \\ \mathbf{K}_{(i)}W - Y_{(i)} & u_{0(i)}^2 \end{bmatrix} \geq 0, \\ i = 1, ..., m \\ (iii) \ \begin{bmatrix} \mu & v_r' \\ v_r & W \end{bmatrix} \geq 0 \ , \quad r = 1, \ldots, n_v \end{array} \qquad (25)$$

Considering $\beta = 1/\sqrt{\mu}$, the minimization of μ implies the maximization of β. The satisfaction of the inclusion relation (24) is ensured by the LMI (iii). Note that (25) is an *eigenvalue problem* [18].

A. Gain Constraints

A constraint of anti-windup gain limitation can be added to the optimization problem (25) as follows. Note that, since $E_c = ZS^{-1}$ it follows that $E_{c(i,j)} = Z_{(i,j)} S^{-1}_{(j,j)}$. Hence, if

$$\begin{bmatrix} S_{(j,j)}\sigma & Z_{(i,j)} \\ Z_{(i,j)} & S_{(j,j)} \end{bmatrix} \geq 0$$

by the Schur's complement one has

$$\sigma - Z_{(i,j)} S^{-1}_{(j,j)} Z_{(i,j)} S^{-1}_{(j,j)} \geq 0$$

which ensures that $(E_{c(i,j)})^2 \leq \sigma$

By the same reasoning, structural constraints on E_c can be taken into account in (25) by fixing some of the elements of matrix $Z_{(i,j)}$ as zero.

V. Illustrative examples

Example 1: Consider the following linear open-loop unstable system:

$$\begin{array}{rcl} x(t+1) & = & 1.2x(t) + u(t) \\ y(t) & = & x(t) \end{array}$$

and the stabilizing PI controller

$$\begin{array}{rcl} x_c(t+1) & = & x_c(t) - 0.05y(t) \\ v_c(t) & = & x_c(t) - y(t) \\ u(t) & = & sat(v_c(t)) \end{array}$$

Let the shape set Ξ_0 be defined by as a square region in the space $\Re^2$:

$$\Xi_0 = Co\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \begin{bmatrix} -1 \\ 1 \end{bmatrix}; \begin{bmatrix} -1 \\ -1 \end{bmatrix} \}$$

Considering, the control bound $u_0 = 1$ and a scaling factor β, we aim to compute an anti-windup gain E_c in order to obtain a region of stability $\beta\Xi_0 \subset \mathcal{E}(P)$ with β as large as possible.

Using the optimization problem (25), the obtained optimal solution is $\beta = 1.9165$ with

$$P = \begin{bmatrix} 0.0497 & -0.0377 \\ -0.0377 & 0.1472 \end{bmatrix} \text{ and } E_c = 0.0920$$

The figure 1 depicts several trajectories of the closed-loop system as an attempt to illustrate its basin of attraction. Regarding the state of the plant, it can be seen that the domain of stability is confined to the interval $x(0) \in (-5,\ 5)$. In fact, the closed-loop system presents two additional equilibrium points in $\pm \begin{bmatrix} 5 \\ 1.2814 \end{bmatrix}$.

On the other hand, the ellipsoidal estimate includes points that are close to the boundaries of the basin of attraction, especially in the direction of the state of the plant, thus providing a reasonable estimate of the basin of attraction. In this regard, it is important to remark that the optimization criterion and the choice of Ξ_0 are degrees of freedom that influence the ellipsoidal estimate of the basin of attraction.

It should be pointed out that without anti-windup gain (i.e., $E_c = 0$) the maximal value of β is 1.7562 which is smaller than the previous one. On the other hand, if we consider the classical sector condition one obtains $E_c = -0.0011$ and $\beta = 1.5729$, which shows that the proposed approach is less conservative.

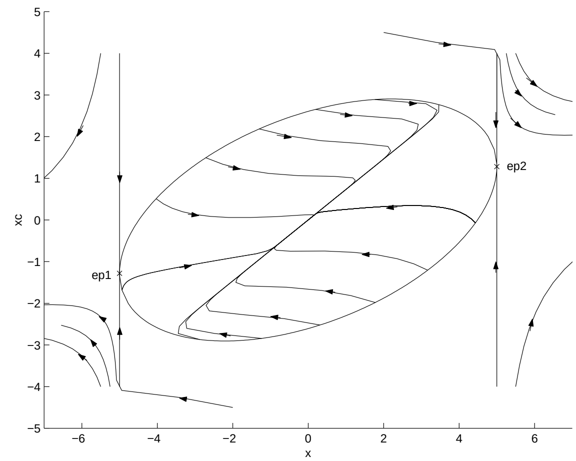

Fig. 1. stability region and trajectories for $E_c = 0.0920$

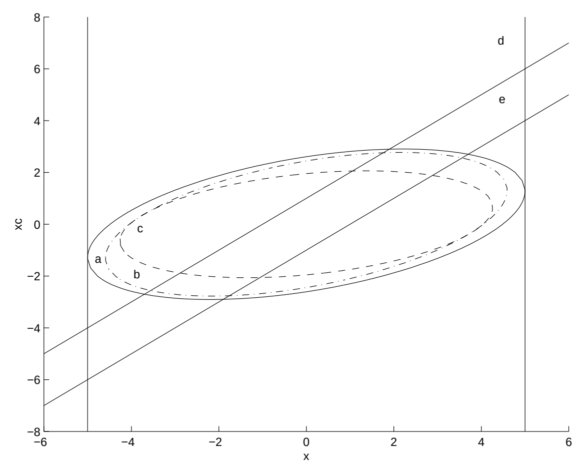

Fig. 2. a: domain obtained with $E_c = 0.0920$; b: domain obtained with $E_c = 0$; c: domain obtained with the classical sector condition; d: $\mathcal{S}$; e: region of linearity

The obtained domains of stability are shown in figure 2. The ellipsoidal estimates of the domain of stability are seen to span beyond the region of linearity meaning that saturation does effectively occur for certain initial conditions inside the estimated domain of stability.

Example 2: Consider the model of an aircraft borrowed from [19]. The matrices of the discrete-time system obtained with a sampling period of $0.001s$ are the following:

$$A = \begin{bmatrix} 1.0000 & 0.0010 & 0.0000 \\ 0 & 0.9992 & 0.0432 \\ 0 & 0.0010 & 0.9987 \end{bmatrix};$$

$$B = \begin{bmatrix} -0.0000 & -0.0000 \\ -0.0172 & -0.0016 \\ -0.0002 & -0.0003 \end{bmatrix};\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The matrix A has unstable eigenvalues. The following matrices corresponds to a stabilizing controller.

$$A_c = -0.0087 \; ; \; B_c = \begin{bmatrix} 2.2633 & -0.3088 \end{bmatrix} \; ;$$

$$C_c = \begin{bmatrix} -173.4958 \\ -17.5120 \end{bmatrix} \; ; \; D_c = \begin{bmatrix} 393.2203 & -53.3798 \\ 38.6827 & -5.4587 \end{bmatrix}$$

Consider now the control bounds given by $u_0 = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$ and the shape set defined as an hypercube in the space defined by the states of the plant:

$$\Xi_0 = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} \}$$

Solving (25) with the data above one obtains:

$$E_c = \begin{bmatrix} 0.0052 & 0.0004 \end{bmatrix} \; ; \; \beta = 3.0801$$

On the other hand, considering a similar optimization problem based on conditions (22) and (23), one obtains as optimal solution $\beta = 1.7498$, which corresponds only to 56.8% of the β obtained with the new proposed condition. Furthermore, it should be pointed out that, in this case, the solution is not directly obtained. The optimal solution for the BMI problem has been obtained by solving interactively LMI problems with Λ fixed.

VI. Concluding Remarks

We have provided a method to design an anti-windup gain aiming at enlarging the region of asymptotic stability of discrete-time linear control systems with saturated inputs. The method considers a given output linear feedback designed for the original systems in the absence of saturation, and provide a design of an anti-windup gain in order to improve its region of asymptotic stability. Such an improvement is always possible since the trivial solution (zero gain) is part of the set of solutions encompassed by the method.

Stability conditions, in both local and global contexts have been stated. These conditions are based on the proposition of a new modified sector condition The main advantage of the proposed approach is that the stability conditions are directly in an LMI form. Considering a criterion associated to the maximization of the stability region (estimate of the basin of attraction), it is then possible to formulate the anti-windup synthesis problem directly as a convex optimization problem, avoiding the iterative schemes present in the previous approaches. Furthermore, it has been shown that the results obtained with a classical sector condition are particular cases of the present one.

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