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ANTICANONICAL RATIONAL SURFACES

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Abstract. A determination of the fixed components, base points and irregularity is made for arbitrary numerically effective divisors on any smooth projective rational surface having an effective anticanonical divisor. All of the results are proven over an algebraically closed field of arbitrary characteristic. Applications, to be treated in separate papers, include questions involving: points in good position, birational models of rational surfaces in projective space, and resolutions for 0-dimensional subschemes of $\mathbf{P}^2$ defined by complete ideals.

I. Introduction

This paper is concerned with complete linear systems on smooth projective rational surfaces, over an algebraically closed field of arbitrary characteristic. The focus is on *anticanonical* rational surfaces, i.e., those rational surfaces supporting an effective anticanonical divisor. Such surfaces include all Del Pezzo surfaces, all blowings up of relatively minimal models of rational surfaces at 8 or fewer points, and all smooth complete toric surfaces, but also include surfaces for which there is an effective but highly nonreduced anticanonical divisor; indeed, there is no bound to the multiplicities of fixed components of effective anticanonical divisors on anticanonical surfaces.

Anticanonical rational surfaces have received attention from numerous authors [Co], [D], [F], [H1], [H2], [H5], [M], [Sk], [U] for various reasons, and share much of the behavior of $K3$ surfaces [H4], [Ma], [PS], [SD], [St]. However, partly because the rational surfaces need not be relatively minimal, and partly because their anticanonical classes are nontrivial, the behavior of the rational surfaces is more complicated. Thus previous work has been carried out under one or another special hypothesis, to tame this anticanonical refractoriness. For example, [D] considers blowings up of $\mathbf{P}^2$ at 9 or fewer points which are *en position presque général*, and [F] assumes that there is a reduced anticanonical divisor, and only considers classes of effective and numerically effective divisors which have no fixed component in common with the anticanonical divisor; [H1] handles arbitrary classes, but considers only blowings up of $\mathbf{P}^2$ for which there is a reduced and irreducible anticanonical divisor. Likewise, [Sk] mainly considers linear systems related to the anticanonical

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divisor itself. In this paper we make no assumptions beyond the existence of an effective anticanonical divisor.

The structure of this paper is as follows. Section II gathers results about rational surfaces generally, without necessarily assuming effectivity of an anticanonical divisor. In Section III we prove our main result, Theorem III.1, which fully determines the behavior (that is, dimensions, base points and fixed components) of complete linear systems for numerically effective classes on smooth projective rational surfaces, under no special hypotheses beyond effectivity of an anticanonical divisor. Our approach is partly that of [H1], and partly a modification of [F] using an idea based on [A] (see Lemma II.9).

As a consequence of results in Section III, we have for example the following theorem. (Given a divisor class $\mathcal{L}$ on a surface X, we recall that a class $\mathcal{L}$ is *numerically effective* if it meets every effective divisor nonnegatively. We will also find it convenient to identify an invertible sheaf with its corresponding divisor class, and employ the additive notation customary for the group of divisor classes. We will usually reserve $\otimes$ to denote restriction, as in the restriction $\mathcal{O}_C \otimes \mathcal{F}$ of a divisor class $\mathcal{F}$ on a surface X to a curve $C \subset X$.)

Theorem I.1. *Let $\mathcal{F}$ be a numerically effective divisor class on a smooth projective rational anticanonical surface X and let D be a general section of $-K_X$. Then $h^0(X, \mathcal{F}) > 0$; moreover, $h^1(X, \mathcal{F}) > 0$ if and only if $\mathcal{F} \cdot K_X = 0$ and a general section of $\mathcal{F}$ has a connected component disjoint from D. In fact, $1 + h^1(X, \mathcal{F})$ is the number of connected components of a general section of $\mathcal{F} - K_X$.*

A proof is given at the end of Section III, but a remark here may be helpful. Although $h^0(X, \mathcal{F}) > 0$ is considered in Section II, the main interest is in h^1. That there be a topological sufficient condition for h^1 to be positive is easy to see. If a section of $\mathcal{F}$ has a connected component disjoint from D, then $\mathcal{F} - K_X$ has a section C which is not connected, so $h^0(C, \mathcal{O}_C) > 1$. But $h^1(X, -(\mathcal{F} - K_X)) = h^1(X, \mathcal{F})$ by duality, so from $0 \to -(\mathcal{F} - K_X) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$, we see $h^1(X, \mathcal{F}) = -1 + h^0(C, \mathcal{O}_C)$ is positive.

It is more difficult to see that the question is purely topological, since C need not be reduced; hence *a priori*, $h^0(C, \mathcal{O}_C)$ could exceed the number of connected components of C. That it does not is a consequence of the detailed analysis of the fixed components of numerically effective divisors on anticanonical rational surfaces, carried out in Section III, showing, among other things, that the fixed part N of the linear system of sections of a numerically effective class is reduced and no components of N can be components of an anticanonical divisor, unless N contains an entire anticanonical divisor.

Some readers may be interested in classes that are not necessarily numerically effective; a word to them may be in order. Given an arbitrary class $\mathcal{F}$ on a smooth projective rational anticanonical surface X, to determine $h^0(X, \mathcal{F})$, and fixed components and base points of the linear system of sections of $\mathcal{F}$ when $\mathcal{F}$ is the class of an effective divisor, one essentially needs to know the monoid EFF of classes of effective divisors. However, this is reasonably accessible and is discussed in more detail in the remark at the end of Section III.

Applications, to questions involving points in good position, to birational models of rational surfaces in projective space, and to computing resolutions for 0-dimensional subschemes of $\mathbf{P}^2$ defined by complete ideals will be considered in separate papers.

II. Results not assuming $-K$ effective

In this section we consider smooth projective rational surfaces, stating some results which hold in general, whether the surfaces are anticanonical or not. We begin with the homomorphism on Picard groups induced by a morphism of schemes. In the cases of interest here, this homomorphism is well-known to behave very nicely. (Given a surface X, a curve $C \subset X$, and a divisor class $\mathcal{F}$ on X, we may in place of the more accurate but more complicated $h^i(C, \mathcal{F} \otimes \mathcal{O}_C)$ write $h^i(C, \mathcal{F})$ for the dimension of the i^{th} cohomology group of the restriction of $\mathcal{F}$ to C.)

Lemma II.1. *Let $\pi : Y \to X$ be a birational morphism of smooth projective surfaces, $\pi^* : Pic(X) \to Pic(Y)$ the corresponding homomorphism on Picard groups, and let $\mathcal{L}$ be a divisor class on X.*

(a) *The map π^* is an injective intersection-form preserving map (of free abelian groups of finite rank if X is rational).*
(b) *The map π^* preserves dimensions of cohomology groups; i.e., $h^i(X, \mathcal{L}) = h^i(Y, \pi^*\mathcal{L})$ for every i.*
(c) *The map π^* preserves EFF; i.e., $\mathcal{L}$ is the class of an effective divisor if and only if $\pi^*\mathcal{L}$ is.*
(d) *The map π^* preserves numerical effectivity; i.e., $\mathcal{L} \cdot F \geq 0$ for every effective divisor F on X if and only if $(\pi^*\mathcal{L}) \cdot F' \geq 0$ for every effective divisor F' on Y.*

Proof. (a) See [Ha, V].
(b) Use [Ha, V.3.4] and the Leray spectral sequence.
(c) This follows from (b) with $i = 0$.
(d) If $\pi^*\mathcal{L}$ is numerically effective, then numerical effectivity for $\mathcal{L}$ follows from (a) and (c). The converse follows from [Ha, V.5] and induction. □

Given an appropriate scheme X, we recall that K_X denotes its canonical class.

Lemma II.2. *Let X be a smooth projective rational surface, and let $\mathcal{F}$ be a divisor class on X.*

(a) *We have: $h^0(X, \mathcal{F}) - h^1(X, \mathcal{F}) + h^2(X, \mathcal{F}) = (\mathcal{F}^2 - K_X \cdot \mathcal{F})/2 + 1$.*
(b) *If $\mathcal{F}$ is the class of an effective divisor, then $h^2(X, \mathcal{F}) = 0$.*
(c) *If $\mathcal{F}$ is numerically effective, then $h^2(X, \mathcal{F}) = 0$ and $\mathcal{F}^2 \geq 0$.*

Proof. Item (a) is just the Riemann-Roch formula in the case of a rational surface. Item (b) follows by duality, while item (c) is elementary. □

Corollary II.3. *On a smooth projective rational surface, a numerically effective divisor meeting the anticanonical class nonnegatively is in EFF. In particular, effectivity of an anticanonical divisor implies effectivity of all numerically effective divisors.*

Proof. This follows from Lemma II.2(a) and (c). □

The next lemma is a well-known consequence of the Hodge Index Theorem.

Lemma II.4. *Let X be a smooth projective rational surface, and denote by $K^\perp$ the subspace of $Pic(X)$ perpendicular to the canonical class K_X. Then $K^\perp$ is negative definite if and only if $K_X^2 > 0$. Also, $K^\perp$ is negative semidefinite if and only if $K_X^2 = 0$; in this case, if $x \in K^\perp$, then $x^2 = 0$ if and only if x is a multiple of K_X.*

In spite of the title of this section, the next fact involves anticanonical surfaces, but it is convenient to state it now.

Lemma II.5. *Let X be a smooth projective anticanonical rational surface. Let D be an effective anticanonical divisor on X and let $\mathcal{F}$ be any divisor class on X. Then $h^0(D, \mathcal{O}_D) = h^1(D, O_D) = 1$, and $h^0(D, \mathcal{F}) - h^1(D, \mathcal{F}) = -K_X \cdot \mathcal{F}$.*

Proof. By duality, $h^2(X, K_X) = h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, K_X) = h^1(X, \mathcal{O}_X)$ vanish, while $h^0(X, K_X) = 0$ since X is rational. Thus from $0 \to K_X \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ we see that $h^0(D, \mathcal{O}_D) = 1 = h^1(D, \mathcal{O}_D)$. From $0 \to \mathcal{F} + K_X \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_D \to 0$, comparing $h^0 - h^1$ for $\mathcal{F} \otimes \mathcal{O}_D$ against $h^0 - h^1 + h^2$ for $\mathcal{F}$ and $\mathcal{F} + K_X$, using Riemann-Roch for the latter two and simplifying, we see that $h^0(D, \mathcal{F}) - h^1(D, \mathcal{F}) = -K_X \cdot \mathcal{F}$. □

We will need a Bertini-type theorem. Recall that a fixed component free linear system is said to be *composed with a pencil*, if away from the base points it defines a morphism whose image has dimension 1.

Lemma II.6. *Let X be a smooth projective rational surface, and let $\mathcal{F}$ be the class of a nontrivial effective divisor on X without fixed components.*

(a) *If the linear system of sections of $\mathcal{F}$ is composed with a pencil, then there is an $r > 0$ such that every section of $\mathcal{F}$ is the divisorial sum of r sections of some class $\mathcal{C}$ whose general section is reduced and irreducible and whose sections comprise a pencil. Moreover, if $\mathcal{F} \cdot K_X \leq 0$, then either: $r = 1$, $\mathcal{F}^2 = 2$, $\mathcal{F} \cdot K_X = 0$, $h^1(X, \mathcal{F}) = 0$ and $-K_X$ is not effective; $r = \mathcal{F}^2 = -K_X \cdot \mathcal{F} = 1$ and $h^1(X, \mathcal{F}) = 0$; $\mathcal{F}^2 = 0$, $-K_X \cdot \mathcal{F} = 2r$ and $h^1(X, \mathcal{F}) = 0$; or $\mathcal{F}^2 = K_X \cdot \mathcal{F} = 0$ and $h^1(X, \mathcal{F}) = r$.*

(b) *If the linear system of sections of $\mathcal{F}$ is not composed with a pencil, then $\mathcal{F}^2 > 0$ and the general section of $\mathcal{F}$ is reduced and irreducible.*

Proof. We first point out that the general section of the class $\mathcal{F}$ of a fixed component free divisor is always reduced. For if not, then by [J, Corollaire 6.4.2] the morphism defined by the sections away from the base points factors through Frobenius. This cannot happen for complete linear systems on rational surfaces, since it would imply that the characteristic p is nonzero, that $\mathcal{F} = p\mathcal{G}$ for some class $\mathcal{G}$, and (since X is rational) that $h^0(X, \mathcal{G}) = h^0(X, p\mathcal{G})$. Since the sections of $\mathcal{F}$ are fixed component free, $h^0(X, \mathcal{G}) = h^0(X, p\mathcal{G}) > 1$; hence $h^0(X, p\mathcal{G}) \geq \dim \operatorname{Sym}^p(H^0(X, \mathcal{G})) = \binom{d-1+p}{p} > d$, where $d = h^0(X, \mathcal{G})$.

For (b), note that $\mathcal{F}^2 = 0$ would imply the linear system of sections of $\mathcal{F}$ is composed with a pencil; for the irreducibility, use [J, Théorème 6.3].

Now consider (a). Since X is rational, the linear system of sections of $\mathcal{F}$ is composed with a rational pencil. By appropriately blowing up the base points of the sections of $\mathcal{F}$, we obtain a class $\mathcal{F}'$ on a surface Z whose sections are base point free, whose general section is the proper transform of a general section of $\mathcal{F}$. The sections of $\mathcal{F}'$ define a morphism $Z \to R$, where R is isomorphic to $\mathbf{P}^1$ since the pencil is rational. By Stein factorization [Ha, III.11.5], there is a morphism $Z \to \mathbf{P}^1$, with connected fibers, factoring $Z \to R$. If r is the degree of $\mathbf{P}^1 \to R$ and $\mathcal{C}'$ is the class of a fiber of $Z \to \mathbf{P}^1$, then $\mathcal{F}' = r\mathcal{C}'$, where the sections of $\mathcal{C}'$ are connected. Taking $\mathcal{C}$ to be the class of a curve on X whose proper transform is a general section of $\mathcal{C}'$, we see that $\mathcal{F} = r\mathcal{C}$, and hence (comparing sections of $\mathcal{F}$ with sections of $\mathcal{F}'$) that every section of $\mathcal{F}$ is the divisorial sum of r sections of $\mathcal{C}$. Thus it is enough to check that a general section of $\mathcal{C}'$ is integral, but this is well-known to hold (cf. [J, Théorème 4.10] and [J, Théorème 3.11]) since $Z \to \mathbf{P}^1$ induces a separable algebraically closed extension of function fields. This proves the first part of (a).

For the rest, assume $\mathcal{F} \cdot K_X \leq 0$. Because $\mathcal{C}$ gives a pencil, $h^0(X, \mathcal{C}) = 2$. Because the sections of $\mathcal{F}$ are composed with this pencil, $h^0(X, \mathcal{F}) = h^0(X, r\mathcal{C}) = r+1$. By Riemann-Roch, $r+1 = h^0(X, \mathcal{F}) \geq (r^2\mathcal{C}^2 - r\mathcal{C} \cdot K_X)/2 + 1$. Now suppose $\mathcal{C}$ has positive self-intersection. If $\mathcal{C} \cdot K_X = 0$, then $\mathcal{C}^2$ must be even and thus is at least 2, so $r+1 \geq r^2\mathcal{C}^2/2 + 1 \geq r^2+1$; hence $r = 1$ and $\mathcal{C}^2 = 2$. Also, from Riemann-Roch we can now calculate that $h^1(X, \mathcal{F}) = 0$. But if $-K_X$ were effective, then, since the sections of $\mathcal{C}$ are fixed component free, $-K_X \cdot \mathcal{C} = 0$ means D is disjoint from a general section of $\mathcal{C}$, where D is some effective anticanonical divisor. Thus $\mathcal{O}_D \otimes \mathcal{C} = \mathcal{O}_D$. By Lemma II.5, $h^1(D, \mathcal{O}_D) = 1$. Since $h^2(X, K_X + \mathcal{C}) = h^0(X, -\mathcal{C}) = 0$, from $0 \to K_X + \mathcal{C} \to \mathcal{C} \to \mathcal{O}_D \otimes \mathcal{C} \to 0$ we get the contradiction that $h^1(X, \mathcal{C}) > 0$. Thus $-K_X$ cannot be effective.

If, on the other hand, $\mathcal{C}^2 > 0$ and $\mathcal{C} \cdot K_X < 0$, then $2r \geq r^2 + r$ by Riemann-Roch so $r = 1$ (and $\mathcal{F} = \mathcal{C}$) whence $2 = (\mathcal{C}^2 - \mathcal{C} \cdot K_X)/2 + 1 + h^1(X, \mathcal{C}) = 2 + h^1(X, \mathcal{C})$, and thus $\mathcal{C}^2 = \mathcal{C} \cdot (-K_X) = 1$ and $h^1(X, \mathcal{C}) = 0$.

Finally, say $\mathcal{C}^2 = 0$. Then $-K_X \cdot \mathcal{C}$ is even, but $2r \geq -r\mathcal{C} \cdot K_X$ by Riemann-Roch, so $-K_X \cdot \mathcal{C}$ is either 2 or 0. From Riemann-Roch we have in the first case $h^0(X, \mathcal{F}) = r + 1 + h^1(X, \mathcal{F})$, and we have in the second $h^0(X, \mathcal{F}) = 1 + h^1(X, \mathcal{F})$, so from $r+1 = h^0(X, \mathcal{F})$ follow $h^1(X, \mathcal{F}) = 0$ and $h^1(X, \mathcal{F}) = r$, respectively. □

Lemma II.7. *Let X be a smooth projective rational surface, and let $\mathcal{C}$ be the class of an effective divisor without fixed components. If $\mathcal{C} \cdot K_X < 0$, then $h^1(X, \mathcal{C}) = 0$.*

Proof. Say $\mathcal{C}$ is the class of a reduced and irreducible divisor C on X, with $C \cdot K_X < 0$. Then $h^1(C, \mathcal{O}_X(rC) \otimes \mathcal{O}_C) = 0$ for all $r > 0$. (If C is smooth this follows by duality, since by adjunction $\mathcal{O}_X(-rC) \otimes K_C$ has negative degree. In general, $h^1(C, \mathcal{O}_C) = h^2(X, \mathcal{O}_X(-C)) = 1 + (C^2 + C \cdot K_X)/2$ follows from checking the cohomology groups for the sequence $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$, and, for the second equality, applying Riemann-Roch. Thus $\mathcal{O}_X(rC) \otimes \mathcal{O}_C$ has degree greater than $2h^1(C, \mathcal{O}_C) - 2$; $h^1(C, \mathcal{O}_X(rC) \otimes \mathcal{O}_C) = 0$ for all $r > 0$ now follows by Proposition 7 on page 59 of [D].) Since X is rational, $h^1(X, \mathcal{O}_X) = 0$. Induction on r applied to the cohomology groups of $0 \to \mathcal{O}_X((r-1)C) \to \mathcal{O}_X(rC) \to \mathcal{O}_C(rC) \to 0$ now gives the result.

In general, if the linear system of sections of $\mathcal{C}$ has no reduced and irreducible member, then, by Lemma II.6, we find that $\mathcal{C}$ is the class of a multiple of a reduced irreducible divisor C' which moves in a pencil at least. The result now follows as above for any positive multiple of C' and hence for any positive multiple of C. □

The following lemma, essentially Theorem 1.7 of [A], turns out to be the key to avoiding special assumptions (such as are assumed in [F]) regarding components in common with an anticanonical divisor when handling numerically effective classes on anticanonical surfaces.

Lemma II.8. *Let X be a smooth projective surface supporting an effective divisor N and a divisor class $\mathcal{F}$ which meets every component of N nonnegatively. If $h^1(N, \mathcal{O}_N) = 0$, then $h^0(N, \mathcal{F}) > 0$ and $h^1(N, \mathcal{F}) = 0$.*

Proof. Modifying the proof of [A, Theorem 1.7] gives the result. See Lemma II.6 of [H6] for an explicit proof. □

Corollary II.9. *Let X be a smooth projective rational surface and let $\mathcal{N}$ be the class of a nontrivial effective divisor N on X. If $\mathcal{N} + K_X$ is not the class of an effective divisor and $\mathcal{F}$ meets every component of N nonnegatively (in particular, if $\mathcal{F}$ is numerically effective), then $h^0(N, \mathcal{F}) > 0$, $h^1(N, \mathcal{F}) = 0$, $N^2 + N \cdot K_X < -1$, and every component M of N is a smooth rational curve (of negative self-intersection, if M does not move).*

Proof. Clearly, $h^0(X, -\mathcal{N}) = 0$, while $h^2(X, -\mathcal{N}) = h^0(X, \mathcal{N} + K_X) = 0$ follows by duality and by hypothesis. By Riemann-Roch, $-2 \geq -2 - 2h^1(X, -\mathcal{N}) = N^2 + N \cdot K_X$. Next, taking cohomology of $0 \to -\mathcal{N} \to \mathcal{O}_X \to \mathcal{O}_N \to 0$, we see $h^1(N, \mathcal{O}_N) = 0$. Since $\mathcal{F}$ meets every component of N nonnegatively, applying Lemma II.8 shows $h^0(N, \mathcal{F}) > 0$ and $h^1(N, \mathcal{F}) = 0$. To finish, apply the foregoing to a component M of N. Then $M^2 + M \cdot K_X = -2$, so M is smooth and rational. Substituting into Riemann-Roch gives $h^0(X, \mathcal{O}_X(M)) \geq M^2 + 2$; hence, if M does not move, then $1 \geq M^2 + 2$ so $M^2 \leq -1$. □

III. Results assuming $-K$ effective

We now develop results concerning anticanonical rational surfaces. The results of this section give a complete determination of $h^1(X, \mathcal{F})$, the class $\mathcal{N}$ of the fixed part of the linear system of sections of $\mathcal{F}$ and the base points of the linear system of sections of $\mathcal{F}$ (note that $\mathcal{F} \in \text{EFF}$ is automatic by Corollary II.3), for any numerically effective class $\mathcal{F}$ on a smooth projective rational anticanonical surface X.

Briefly, what we find is this. Behavior of the linear system of sections of a numerically effective class $\mathcal{F}$ is controlled by $-K_X \cdot \mathcal{F}$. If this is at least 2, then $\mathcal{F}$ is regular and its linear system has empty base locus.

If $-K_X \cdot \mathcal{F} = 1$, $\mathcal{F}$ is still regular, but its linear system has a base point, and possibly a base divisor. (The base divisor is always the class of a reduced but not necessarily irreducible exceptional curve. The pencil of cubics through eight general points gives an example of a numerically effective class $\mathcal{F}$ on a blow up of $\mathbf{P}^2$ at eight points whose sections have a base point; by blowing up the base point and pulling back to the blown up surface, the base point converts to a divisorial base locus consisting of the exceptional curve E of the last blow up. In this example, $\mathcal{F} \cdot E = 0$; this example is not completely general, since it is also possible to have a divisorial base locus E with $\mathcal{F} \cdot E > 0$, as happens if X is the blow up of $\mathbf{P}^2$ at the base points of a general pencil of cubics and $\mathcal{F}$ is $-rK_X + \mathcal{E}$, where $\mathcal{E}$ is the class of the exceptional curve E coming from one of the blow ups.)

Regardless of what else happens, if $-K_X \cdot \mathcal{F} = 1$, then the sections of $\mathcal{F}$ have a base point at a smooth point of any effective anticanonical divisor D. Blowing this base point up and considering $\mathcal{G} = \mathcal{F} - \mathcal{E}'$ on the blown up surface (where $\mathcal{E}'$ is the class of the exceptional divisor of the blow up) shows that numerically effective classes $\mathcal{G}$ can occur with $-K_X \cdot \mathcal{G} = 0$, such that $h^1(X, \mathcal{G}) > 0$ and either the sections of $\mathcal{G}$ are base point free (which happens if the sections of $\mathcal{F}$ are fixed component free) or the base locus consists of a smooth rational curve N with $N^2 = -2$ (which happens, for example, if we take $\mathcal{F} = -rK_X + \mathcal{E}$ with $r > 1$, as in the preceding paragraph; note that $r = 1$ does not work since then $\mathcal{G}$ is not numerically effective). It turns out that the only other possibility that can occur for a numerically effective class perpendicular to K_X is that the base locus of its linear system of sections can contain an anticanonical divisor, and understanding that this may happen in essentially only one way (this being that of a numerically effective class perpendicular to $-K_X$ whose restriction to an effective anticanonical divisor is nontrivial—although a precise description of the base locus in this case depends on whether or not the numerically effective class has self-intersection 0) completes our understanding of linear systems of numerically effective classes on anticanonical surfaces.

The following theorem gives a rigorous statement of the preceding discussion, summarizing our results in this section:

Theorem III.1. *Let X be a smooth projective rational anticanonical surface with a numerically effective class $\mathcal{F}$ and let D be a nonzero section of $-K_X$. Let $\mathcal{N}$ be the class of the fixed part of the linear system of sections of $\mathcal{F}$, and let $\mathcal{H} = \mathcal{F} - \mathcal{N}$ be the class of the free part.*

(a) *If $-K_X \cdot \mathcal{F} \geq 2$, then $h^1(X, \mathcal{F}) = 0$ and the sections of $\mathcal{F}$ are base point (and thus fixed component) free (i.e., $\mathcal{N} = 0$).*
(b) *If $-K_X \cdot \mathcal{F} = 1$, then $h^1(X, \mathcal{F}) = 0$. If the sections of $\mathcal{F}$ are fixed component free, then the sections of $\mathcal{F}$ have a unique base point, which is on D. Moreover, the linear system of sections of $\mathcal{F}$ has a fixed component if and only if $\mathcal{H} = r\mathcal{C}$ and $\mathcal{N} = \mathcal{N}_1 + \cdots + \mathcal{N}_t$, where $\mathcal{C} \in K_X^{\perp}$ is a class with $h^1(X, \mathcal{C}) = 1$ whose general section is reduced and irreducible, $r = h^1(X, \mathcal{H})$ with $r > 1$ only if $\mathcal{C}^2 = 0$, $\mathcal{N}_i$ is a smooth rational curve for every i, $\mathcal{N}_i^2 = -2$ and $\mathcal{N}_i \cdot \mathcal{N}_{i+1} = 1$ for $i < t$, $\mathcal{N}_t^2 = -1$, $\mathcal{N}_i \cdot \mathcal{N}_j = 0$ for $j > i + 1$, $\mathcal{C} \cdot \mathcal{N}_1 = 1$, and $\mathcal{C} \cdot \mathcal{N}_i = 0$ for $i > 1$.*
(c) *If $-K_X \cdot \mathcal{F} = 0$, then either $\mathcal{N} = 0$ (in which case the sections of $\mathcal{F}$ are base point free, $\mathcal{F} \otimes \mathcal{O}_D$ is trivial and either $\mathcal{F}^2 > 0$ and $h^1(X, \mathcal{F}) = 1$ or $\mathcal{F} = r\mathcal{C}$ and $h^1(X, \mathcal{F}) = r$, where $r > 0$ and $\mathcal{C}$ is a class of self-intersection 0 whose general section is reduced and irreducible), or $\mathcal{N}$ is a smooth rational curve of self-intersection -2 (in which case $h^1(X, \mathcal{F}) = 1$, $\mathcal{N} \otimes \mathcal{O}_D$ is trivial, and $\mathcal{H} = r\mathcal{C}$, where $r > 1$ and $\mathcal{C}$ is reduced and irreducible with $\mathcal{C}^2 = 0$, $\mathcal{C} \cdot \mathcal{N} = 1$ and $\mathcal{C} \otimes \mathcal{O}_D$ being trivial), or $\mathcal{N} + K_X \in EFF$.*
(d) *We have $\mathcal{N} + K_X \in EFF$ if and only if $\mathcal{F} \cdot D = 0$ but $\mathcal{F} \otimes \mathcal{O}_D$ is nontrivial. In this case, there exists a birational morphism of X to a smooth projective rational anticanonical surface Y, and either: $K_Y^2 < 0$, there is a numerically effective class $\mathcal{F}'$ on Y, $\mathcal{F}$ is the pullback of $\mathcal{F}' - K_Y$, and $0 = h^1(Y, \mathcal{F}') = h^1(X, \mathcal{F})$; or $K_Y^2 = 0$, $\mathcal{H}$ and $\mathcal{N}$ are the pullbacks of $-sK_Y$ and $-rK_Y$ for some integers $s \geq 0$ and $r > 0$ respectively, and $h^1(X, \mathcal{F}) = \sigma$, where $\sigma = 0$ if $s = 0$, and otherwise $r < \tau$ and $\sigma = s/\tau$, where τ is the least positive integer such that the restriction of $-\tau K_X$ to D is trivial.*

At the end of this section we show how this summary statement follows from the lemmas we will by then have accumulated.

On a K3 surface, one never needs to worry about a linear system's having a fixed component which is also a component of an effective anticanonical divisor. As a consequence, the analysis of complete linear systems on K3 surfaces is less complicated than on rational surfaces with an effective anticanonical divisor. That this analysis can, nevertheless, be carried out for rational surfaces is perhaps partly due to the occurrence of such behavior being very restricted, as this next result shows:

Corollary III.2. *Let X be a smooth projective rational anticanonical surface with a numerically effective class $\mathcal{F}$ and let $\mathcal{N}$ be the class of the fixed part of the linear system of sections of $\mathcal{F}$. Then either no fixed component of the linear system of sections of $\mathcal{F}$ is a component of any section of $-K_X$, or $\mathcal{N}$ contains an anticanonical divisor (i.e., $\mathcal{N} + K_X \in$ EFF).*

Thus this corollary implies that no numerically effective divisor on X can have a fixed component in common with the anticanonical linear system if $h^0(X, -K_X) > 1$ (and so in particular if $K_X^2 > 0$). Note that this corollary can be obtained from Theorem III.1, but for convenience we put off the proof until the end of this section, obtaining it as a consequence of other results later in this section.

The next corollary gives some geometrical insight into the occurrence of irregular numerically effective classes; we give its proof now as an example of applying Theorem III.1.

Corollary III.3. *Let X be a smooth projective rational anticanonical surface with a numerically effective class $\mathcal{F}$ and let D be a nonzero section of $-K_X$.*

(a) *We have $h^1(X, \mathcal{F}) > 0$ if and only if the general section of $\mathcal{F}$ has a connected component disjoint from D.*
(b) *Say $\mathcal{F}^2 - K_X \cdot \mathcal{F} > 0$; then $h^1(X, \mathcal{F}) > 0$ if and only if the general section of $\mathcal{F}$ is disjoint from D.*

Proof. Let $\mathcal{N}$ denote the class of the fixed part of the linear system of sections of $\mathcal{F}$, and $\mathcal{H}$ the rest. Suppose $h^1(X, \mathcal{F}) > 0$. By Theorem III.1, we see that $\mathcal{F} \cdot D = 0$. If $\mathcal{N} + K_X \notin$ EFF, then the linear systems of sections of $\mathcal{F}$ and $-K_X$ have no common fixed components by Corollary III.2, so $\mathcal{F} \cdot D = 0$ implies that the general section of $\mathcal{F}$ is disjoint from D. If $\mathcal{N} + K_X \in$ EFF, then by Theorem III.1(d) $h^1(X, \mathcal{F}) > 0$ implies that $\mathcal{H}$ and $\mathcal{N}$ are the pullbacks of $-sK_Y$ and $-rK_Y$, where $K_Y^2 = 0$, and thus that $\mathcal{F}^2 - K_X \cdot \mathcal{F} = 0$. But $\mathcal{H}$ is numerically effective and its linear system of sections is without fixed components, so $\mathcal{H} \cdot \mathcal{N} = -sK_Y \cdot (-rK_Y) = 0$ and $\mathcal{H} \cdot (-K_X) = -sK_Y \cdot (-K_Y) = 0$ imply that $\mathcal{H}$ has a section containing a connected component of a general section of $\mathcal{F}$ disjoint from D. This establishes the forward implications of (a) and (b).

For the converses, note that if a general section of $\mathcal{F}$ has a connected component disjoint from D, then some section C of $\mathcal{F} - K_X$ is not connected; hence $h^1(C, K_C) = h^0(C, \mathcal{O}_C) > 1$. Now $h^1(X, \mathcal{F}) > 0$ follows from $K_C = \mathcal{F} \otimes \mathcal{O}_C$ and from $0 \to K_X \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_C \to 0$. □

We now begin to develop the tools we will need to prove Theorem III.1. The following lemma comes from [F], included here for convenience, and in order to clarify the statement of the lemma.

Lemma III.4. *Let $\mathcal{L}$ be a divisor class on a smooth projective rational anticanonical surface X with a reduced connected section L no component of which is a fixed component of the linear system of sections of $-K_X$.*

(a) *If $-K_X \cdot \mathcal{L} > 0$, then $h^1(X, \mathcal{L}) = 0$.*
(b) *If $-K_X \cdot \mathcal{L} = 0$, then $h^1(X, \mathcal{L}) = 1$.*

Proof. (a) This is essentially [F, 5.2], although the statement there does not make it clear that it is not enough that L be reduced with each of its connected components meeting $-K_X$ positively. The following proof is taken from [F].

From $0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{O}_L \otimes \mathcal{L} \to 0$, we obtain $h^1(X, \mathcal{L}) = h^1(L, \mathcal{L})$. By adjunction, $\mathcal{O}_L \otimes \mathcal{L} = K_L \otimes (-K_X)$, so by duality $h^1(L, \mathcal{L}) = h^0(L, K_X)$. But the restriction of $-K_X$ to L is the class of an effective divisor (since no component of L is a fixed component of the linear system of sections of $-K_X$) and nontrivial (since $-K_X$ meets L positively); since L is reduced and connected, it follows that $h^0(L, K_X) = 0$.

(b) This is [F, 5.3]. The proof is similar to (a), except here the hypotheses imply that the restriction of $-K_X$ to L is trivial, so now $\mathcal{O}_L \otimes \mathcal{L} = K_L \otimes (-K_X) = K_L$. Thus $h^1(L, \mathcal{L}) = h^0(L, \mathcal{O}_L)$, and $h^0(L, \mathcal{O}_L) = 1$ since L is reduced and connected. □

The next lemma is a modification of Lemma 9 of [F], in which the requirement there that $-K_X$ have a section meeting $\mathcal{L}_2$ in just a finite set of points is weakened here by only assuming that $h^0(X, \mathcal{L}_2 + K_X) = 0$. Thus in the following lemma, the fixed part of the linear system of sections of $\mathcal{L}$ contains a section of $\mathcal{L}_2$, and *a priori* could contain fixed components of the linear system of sections of $-K_X$, but merely cannot contain an entire anticanonical divisor.

Lemma III.5. *Let X be a smooth projective rational anticanonical surface. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be classes of effective divisors on X such that $h^0(X, \mathcal{L}_2 + K_X) = 0$ and such that $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ and $\mathcal{L}_1$ are numerically effective with $h^0(X, \mathcal{L}) = h^0(X, \mathcal{L}_1)$ and $\mathcal{L}_2 \neq 0$; then $\mathcal{L}_1 \cdot \mathcal{L}_2 > 0$.*

Proof. Suppose on the contrary that $\mathcal{L}_1 \cdot \mathcal{L}_2 = 0$. Since $\mathcal{L}_1$ is numerically effective, this means $\mathcal{L}_1 \cdot C = 0$ for every component C of the linear system of sections of $\mathcal{L}_2$. Since $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ is numerically effective, it follows that $\mathcal{L}_2 \cdot C \geq 0$ for every component C of the linear system of sections of $\mathcal{L}_2$, and hence that $\mathcal{L}_2$ is numerically effective; thus $\mathcal{L}_2^2 \geq 0$ and $-K_X \cdot \mathcal{L}_2 \geq 0$. But $\mathcal{L}_2$ is the class of a unique effective divisor, so from Riemann-Roch we have $1 \geq (\mathcal{L}_2^2 - K_X \cdot \mathcal{L}_2)/2 + 1$; hence $\mathcal{L}_2^2 = -K_X \cdot \mathcal{L}_2 = 0$, so $\mathcal{L}_2 = 0$ by Corollary II.9. □

Lemma III.6. *Let $\mathcal{F}$ be a nontrivial numerically effective class on a smooth projective rational anticanonical surface X, such that $\mathcal{F}^2 = 0$ and the class of the fixed part of the linear system of sections of $\mathcal{F}$ is $\mathcal{N}$. If $h^0(X, \mathcal{N} + K_X) = 0$, then the sections of $\mathcal{F}$ are base point free and composed with a pencil.*

Proof. Let $\mathcal{F} - \mathcal{N} = \mathcal{H}$; then $0 = \mathcal{F}^2 = \mathcal{H}^2 + \mathcal{H} \cdot \mathcal{N} + \mathcal{N} \cdot \mathcal{F}$; hence by numerical effectivity of $\mathcal{F}$ and $\mathcal{H}$, we have $\mathcal{H} \cdot \mathcal{N} = 0$, so $\mathcal{N} = 0$ by Lemma III.5. Thus the sections of $\mathcal{F}$ are not only fixed component free, but, since $\mathcal{F}^2 = 0$, base point free and composed with a pencil. □

Lemma III.7. *Let $\mathcal{F}$ be a numerically effective class on a smooth projective rational anticanonical surface X, such that $\mathcal{F}^2 > 0$, the class of the fixed part of the linear system of sections of $\mathcal{F}$ is $\mathcal{N}$ and $\mathcal{F} - \mathcal{N} = \mathcal{H}$. Let D be a nonzero section of $-K_X$. If $h^0(X, \mathcal{N} + K_X) = 0 = h^1(X, \mathcal{H})$, then $\mathcal{F} \cdot D > 0$, $\mathcal{F}$ is regular, and the linear system of sections of $\mathcal{F}$ is fixed component free.*

Proof. First we show $\mathcal{N} = 0$. Suppose not. Let N be the nontrivial section of $\mathcal{N}$. Then by Corollary II.9, $h^1(N, \mathcal{F}) = 0$ and $h^0(N, \mathcal{F}) > 0$. Using regularity of $\mathcal{H}$

we see $0 \to \mathcal{H} \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_N \to 0$ is exact on global sections, so $h^0(N, \mathcal{F}) > 0$ guarantees that $\mathcal{H}$ cannot be all of the class of the nonfixed part of $\mathcal{F}$, contradiction. Thus $\mathcal{N}$ is trivial, and the sections of $\mathcal{F} = \mathcal{H}$ are fixed component free, and regular by assumption. To see that $\mathcal{F} \cdot D > 0$, consider $0 \to \mathcal{F} + K_X \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_D \to 0$. By duality, $h^2(X, \mathcal{F} + K_X) = 0$, so $h^1(X, \mathcal{F}) \geq h^1(D, \mathcal{F} \otimes \mathcal{O}_D)$. If $\mathcal{F} \cdot D = 0$, then the general section of $\mathcal{F}$ is disjoint from D, so $\mathcal{F} \otimes \mathcal{O}_D = \mathcal{O}_D$, but this contradicts regularity of $\mathcal{F}$, since $h^1(D, \mathcal{O}_D) = 1$ by Lemma II.5. □

The argument for the next result is essentially that of [F, 10.5], modified to fit the current more general situation.

Lemma III.8. *Let $\mathcal{F}$ be a numerically effective class on a smooth projective rational anticanonical surface X, with $\mathcal{F}^2 > 0$, where $\mathcal{N}$ is the class of the fixed part N of the linear system of sections of $\mathcal{F}$ and $\mathcal{F} - \mathcal{N} = \mathcal{H}$ is the class of the free part. Let D be an arbitrary anticanonical divisor on X. Suppose $h^0(X, \mathcal{N} + K_X) = 0$ but $h^1(X, \mathcal{H}) > 0$.*

(a) *If $\mathcal{N} = 0$, then $\mathcal{F} \cdot K_X = 0$ and $h^1(X, \mathcal{F}) = 1$.*

(b) *If $\mathcal{N} \neq 0$, then $0 \leq -K_X \cdot \mathcal{F} \leq 1$, and there is a class $\mathcal{C} \in K_X^{\perp}$ with $h^1(X, \mathcal{C}) = 1$ whose general section is reduced and irreducible such that $\mathcal{H} = r\mathcal{C}$, where $h^1(X, \mathcal{H}) = r$. Moreover, no component of N is a component of D and either: $h^1(X, \mathcal{F}) = 1$, in which case $\mathcal{F} \cdot K_X = 0$, $\mathcal{C}^2 = 0$, $r > 1$, $\mathcal{N} \cdot \mathcal{C} = 1$ and $\mathcal{N}$ is the class of a smooth rational curve of self-intersection -2; or $h^1(X, \mathcal{F}) = 0$, in which case $\mathcal{N} = \mathcal{N}_1 + \cdots + \mathcal{N}_t$, where $\mathcal{N}_i$ is a smooth rational curve for every i, $\mathcal{N}_i^2 = -2$ and $\mathcal{N}_i \cdot \mathcal{N}_{i+1} = 1$ for $i < t$, $\mathcal{N}_t^2 = -1$, $\mathcal{N}_i \cdot \mathcal{N}_j = 0$ for $j > i + 1$, $\mathcal{C} \cdot \mathcal{N}_1 = 1$, $\mathcal{C} \cdot \mathcal{N}_i = 0$ for $i > 1$, and either $r = 1$ or $\mathcal{C}^2 = 0$.*

Proof. (a) By Lemma II.7, we see $\mathcal{F} \cdot K_X = 0$. By Lemma II.6, $\mathcal{F}$ is the class of a reduced and irreducible divisor F, and the restriction of K_X to F is trivial, since the sections of $\mathcal{F}$ are fixed component free, $-K_X$ is in EFF and $\mathcal{F} \cdot K_X = 0$. From adjunction we thus have $K_F = \mathcal{F} \otimes \mathcal{O}_F$, so $h^1(F, \mathcal{F} \otimes \mathcal{O}_F) = 1$. Now $h^1(X, \mathcal{F}) = 1$ follows from $0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_F \to 0$.

(b) By Lemma II.7, we see $\mathcal{H} \cdot K_X = 0$, and by Lemma III.5, $\mathcal{H} \cdot \mathcal{N} > 0$.

Also, $\mathcal{H} = r\mathcal{C}$ and $h^1(X, \mathcal{H}) = r$ for some positive integer r, where $\mathcal{C}$ is the class of some reduced and irreducible curve C with $h^1(X, \mathcal{C}) = 1$, and either $r = 1$ or $\mathcal{C}^2 = 0$. [This follows if the linear system of sections of $\mathcal{H}$ is composed with a pencil by Lemma II.6(a). If not, then $\mathcal{H}^2 > 0$, $r = 1$ by Lemma II.6(b), and $\mathcal{H} = \mathcal{C}$ is irregular by Lemma III.7, so $h^1(X, \mathcal{H}) = h^1(X, \mathcal{C}) = 1$ by (a).] Thus $h^0(X, \mathcal{H}) = r^2 C^2/2 + r + 1$ by Riemann-Roch.

By Corollary II.9, the components of N are smooth and rational, of negative self-intersection, and each component M which is not also a fixed component of the linear system of sections of $-K_X$ has by adjunction either $M^2 = M \cdot K_X = -1$ or $M^2 = -2$ and $M \cdot K_X = 0$. Moreover, each connected component of N meets $\mathcal{H}$ positively, by Lemma III.5. Note that an irreducible component of N having positive intersection product with $\mathcal{H}$ cannot be a component of D, since $-K_X \cdot \mathcal{H} = 0$ means D is disjoint from a general section of $\mathcal{H}$. In particular, each connected component of N is either disjoint from D (and hence each irreducible component of such a connected component has self-intersection -2) or contains a string of components $N_1, \ldots, N_t$, none being a component of D, with (we may assume) $N_i \cdot N_{i+1} > 0$, $C \cdot N_1 > 0$, $-K_X \cdot N_i = 0$ for $i < t$, and $-K_X \cdot N_t = 1$.

Consider an irreducible component M of N which meets $\mathcal{H}$, and hence which is not a component of D. Clearly, M is a fixed component of $|rC+M|$ (so $rC+M$ and rC move in complete linear systems of the same dimension). Also the general section of $\mathcal{O}_X(rC+M)$ is connected (since $C\cdot M>0$), reduced, and has no fixed components in common with D. Thus $h^1(X,\mathcal{O}_X(rC+M))=0$ by Lemma III.4(a) if $M^2=-1$, and $h^1(X,\mathcal{O}_X(rC+M))=1$ by Lemma III.4(b) if $M^2=-2$. We can in either case now compute $h^0(X,\mathcal{O}_X(rC+M))$ by plugging into Riemann-Roch; setting the result equal to $h^0(X,\mathcal{H})=r^2C^2/2+r+1$ and solving give $C\cdot M=1$.

We now consider two cases. First say the connected components of N are not all disjoint from D; thus there is a string $N_1,\dots,N_t$, as above, with N_1 meeting $\mathcal{H}$ and N_t meeting D. Note that $L=rC+N_1+\cdots+N_t$ is numerically effective, since it meets each N_i nonnegatively, and regular by Lemma III.4(a). If $N_1+\cdots+N_t$ is not all of $\mathcal{N}$, then the difference N' is nontrivial. Since $h^0(X,\mathcal{O}_X(L))=h^0(X,\mathcal{F})$, we see from $0\to\mathcal{O}_X(L)\to\mathcal{F}\to\mathcal{F}\otimes\mathcal{O}_{N'}\to 0$ that $h^0(N',\mathcal{F})=0$, contradicting Corollary II.9. I.e., N' must be trivial, $\mathcal{N}$ is precisely the class of $N_1+\cdots+N_t$, and $\mathcal{F}$ is the class of L, hence regular. If $t=1$, we are done with this case. If $t>1$, to see that $C\cdot N_i=0$ for $i>1$, that $N_i\cdot N_j=0$ for $j>i+1$, and that $N_i\cdot N_{i+1}=1$ for each i, note by Lemma III.4 that $h^1(X,\mathcal{O}_X(rC+N_1+N_2))$ is 1 if $N_2^2=-2$ and 0 if $N_2^2=-1$. But $h^0(X,\mathcal{O}_X(rC+N_1+N_2))=h^0(X,\mathcal{H})=r^2C^2/2+r+1$. Using Riemann-Roch and our knowledge of h^1 to compute $h^0(X,\mathcal{O}_X(rC+N_1+N_2))$ explicitly by setting it equal to $r^2C^2/2+r+1$ and simplifying give $(rC+N_1)\cdot N_2=1$. Since $N_1\cdot N_2>0$, we see $N_1\cdot N_2=1$ and $C\cdot N_2=0$. This argument can be repeated with $rC+N_1+\cdots+N_i$ for each $i\le t$, to give the result.

Consider now the case that the connected components of N are all disjoint from D. This means that $\mathcal{F}\otimes\mathcal{O}_D=\mathcal{O}_D$, and so from $0\to\mathcal{F}+K_X\to\mathcal{F}\to\mathcal{O}_D\to 0$ and Lemma II.5 we see $\mathcal{F}$ is irregular. Let M be an irreducible component of N meeting $\mathcal{H}$. Then $M^2=-2$, and $h^1(X,\mathcal{O}_X(rC+M))=1$ (as observed above). If M is not all of $\mathcal{N}$, the difference N' is nontrivial. Thus we can consider $0\to\mathcal{O}_X(rC+M)\to\mathcal{F}\to\mathcal{F}\otimes\mathcal{O}_{N'}\to 0$. Since $\mathcal{F}$ is irregular and $h^1(N',\mathcal{F})=0$ (Corollary II.9), the 1-dimensional space $H^1(X,\mathcal{O}_X(rC+M))$ maps in the long exact sequence isomorphically to $H^1(X,\mathcal{F})$. Thus $h^0(N',\mathcal{F})=0$, contradicting Corollary II.9. I.e., N' must be trivial and $\mathcal{N}$ is precisely the class of M. Thus $\mathcal{F}$ is the class of $rC+M$; hence $h^1(X,\mathcal{F})=1$, and $r>1$ (since otherwise $\mathcal{F}\cdot M<0$). $\square$

The following result is needed to deal with cases in which the base locus contains an anticanonical divisor.

Lemma III.9. *Let $\mathcal{F}$ be a numerically effective class on a smooth projective rational anticanonical surface X. Let $\mathcal{N}$ be the class of the fixed part of the linear system of sections of $\mathcal{F}$.*

(a) *There is a birational morphism $X\to Y$ to a smooth surface Y such that $\mathcal{F}$ is the pullback of a numerically effective class $\mathcal{L}$ on Y, and $\mathcal{L}\cdot\mathcal{E}>0$ for every class $\mathcal{E}$ of an irreducible exceptional divisor on Y.*

(b) *Let $X\to Y$ be a birational morphism to a smooth surface Y such that $\mathcal{F}$ is the pullback of a class $\mathcal{L}$ on Y. Let $\mathcal{M}$ be the class of the fixed part of the linear system of sections of $\mathcal{L}$. Then $h^0(X,\mathcal{N}+K_X)>0$ if and only if $h^0(Y,\mathcal{M}+K_Y)>0$.*

(c) *If $h^0(X,\mathcal{F}+K_X)>0$ and if $\mathcal{F}\cdot\mathcal{E}>0$ for every class $\mathcal{E}$ of an irreducible exceptional curve, then $\mathcal{F}+K_X$ is numerically effective.*

Proof. (a) If $\mathcal{E}$ is the class of an irreducible exceptional divisor E with $\mathcal{F} \cdot E = 0$, let $X \to Y'$ be the birational morphism contracting E. Then the corresponding homomorphism $\mathrm{Pic}(Y') \to \mathrm{Pic}(X)$ is an isomorphism of $\mathrm{Pic}(Y')$ to $E^{\perp}$. Thus there is a class $\mathcal{F}'$ on Y' whose pullback to X is $\mathcal{F}$. Note that $\mathcal{F}'$ is numerically effective by Lemma II.1. We can repeat this process unless Y' has no exceptional classes perpendicular to $\mathcal{F}$, which must eventually occur since the rank of $\mathrm{Pic}(Y')$ is less than the rank of $\mathrm{Pic}(X)$.

(b) Since any birational morphism of surfaces factors into a sequence of contractions of irreducible exceptional divisors [Ha, V.3.2], it is enough by induction to show $h^0(X, \mathcal{N} + K_X) > 0$ if and only if $h^0(Y, \mathcal{M} + K_Y) > 0$ in the case that $X \to Y$ is obtained by contracting a single irreducible exceptional curve E, whose class we denote $\mathcal{E}$. Thus we have $\mathcal{F} \cdot \mathcal{E} = 0$. Also, the pullback of K_Y is $K_X - \mathcal{E}$ [Ha, V.3.3]. Thus, by Lemma II.1, $h^0(X, \mathcal{F} + K_X - \mathcal{E}) = h^0(Y, \mathcal{L} + K_Y)$ and $h^0(Y, \mathcal{L}) = h^0(X, \mathcal{F})$.

Say $h^0(X, \mathcal{N} + K_X) > 0$. Since $\mathcal{F} - \mathcal{N} \in \mathrm{EFF}$, so is $\mathcal{F} + K_X$. But $(\mathcal{F} + K_X) \cdot \mathcal{E} = -1$ so E is in the fixed part of the linear system of sections of $\mathcal{F} + K_X$. Of course, a section of $-K_X$ is in the fixed part of the linear system of sections of $\mathcal{F}$. Thus $h^0(X, \mathcal{F}) = h^0(X, \mathcal{F} + K_X) = h^0(X, \mathcal{F} + K_X - \mathcal{E})$ and hence $h^0(Y, \mathcal{L}) = h^0(Y, \mathcal{L} + K_Y)$, which means that a section of $-K_Y$ is in the fixed part of the linear system of sections of $\mathcal{L}$; i.e., $h^0(Y, \mathcal{M} + K_Y) > 0$.

Conversely, say $h^0(Y, \mathcal{M} + K_Y) > 0$. Then $h^0(Y, \mathcal{L}) = h^0(Y, \mathcal{L} + K_Y)$, so $h^0(X, \mathcal{F}) = h^0(X, \mathcal{F} + K_X - \mathcal{E})$. Thus a section of $-K_X + \mathcal{E}$ is in the fixed part of the linear system of sections of $\mathcal{F}$ so certainly a section of $-K_X$ is in the fixed part, whence $h^0(X, \mathcal{N} + K_X) > 0$.

(c) Let C be a reduced and irreducible curve on X. If $C^2 \geq 0$ or $C \cdot K_X \geq 0$, then clearly $C \cdot (\mathcal{F} + K_X) \geq 0$. So say $C^2 < 0$ and $C \cdot K_X < 0$. By adjunction, C is an exceptional curve, i.e., a smooth rational curve with $C^2 = C \cdot K_X = -1$. By assumption, $\mathcal{F} \cdot C > 0$, so again $C \cdot (\mathcal{F} + K_X) \geq 0$. $\square$

By the following lemma we see that there are two ways a numerically effective divisor can contain an anticanonical divisor in its fixed part. One of these is described explicitly; the other case reduces (essentially by subtracting off the anticanonical divisor) to cases previously worked out, in which the fixed part does not contain an anticanonical divisor. An explicit description could be made in this second case, too. We chose not to include it here, since the statement of the lemma is already somewhat complicated and since it is not hard using Theorem III.1 to work out an explicit description if one chooses.

Lemma III.10. *Let $\mathcal{F}$ be a numerically effective class on a smooth projective rational anticanonical surface X, let D be a nonzero section of $-K_X$, let $\mathcal{N}$ denote the class of the fixed part of the linear system of sections of $\mathcal{F}$ and let $\mathcal{H} = \mathcal{F} - \mathcal{N}$. Then $\mathcal{N} + K_X \in$ EFF if and only if $\mathcal{F} \cdot D = 0$ but $\mathcal{F} \otimes \mathcal{O}_D$ is nontrivial. In this case, there exists a birational morphism of X to a smooth projective rational anticanonical surface Y, and either: $K_Y^2 < 0$, there is a numerically effective class $\mathcal{F}'$ on Y, $\mathcal{F}$ is the pullback of $\mathcal{F}' - K_Y$, and $0 = h^1(Y, \mathcal{F}') = h^1(X, \mathcal{F})$; or $K_Y^2 = 0$, $\mathcal{H}$ and $\mathcal{N}$ are the pullbacks of $-sK_Y$ and $-rK_Y$ for some integers $s \geq 0$ and $r > 0$ respectively, and $h^1(X, \mathcal{F}) = \sigma$, where $\sigma = 0$ if $s = 0$, and otherwise $r < \tau$ and $\sigma = s/\tau$, where τ is the least positive integer such that the restriction of $-\tau K_X$ to D is trivial.*

Proof. Suppose a section of $-K_X$ occurs in the fixed part of the linear system of sections of $\mathcal{F}$; i.e., $h^0(X, \mathcal{N}+K_X) > 0$. In particular, this means that $h^0(X, -K_X) = 1$, and thus that $(-K_X)^2 \leq 0$ (otherwise, by Riemann-Roch, $h^0(X, -K_X) > 1$).

We wish to prove that $\mathcal{F} \cdot K_X = 0$. Denote X by X_1, $\mathcal{F}$ by $\mathcal{F}_1$ and $\mathcal{N}$ by $\mathcal{N}_1$. By Lemma III.9(a), there is a birational morphism $X \to Y$ such that $\mathcal{F}$ is the pullback of a class $\mathcal{L}_1$ (numerically effective by Lemma II.1(d)) on Y which meets every irreducible exceptional class on Y positively; by Lemma III.9(b), $h^0(X, \mathcal{M}_1+K_Y) > 0$, where M_1 is the fixed part of the linear system of sections of $\mathcal{L}_1$. By Lemma III.9(c), $\mathcal{L}_1 + K_Y$ is numerically effective. Denote Y by X_2, $\mathcal{L}_1 + K_Y$ by $\mathcal{F}_2$ and the class of the fixed part of the linear system of sections of $\mathcal{F}_2$ by $\mathcal{N}_2$.

Thus given a numerically effective $\mathcal{F}_i$ on X_i such that $h^0(X_i, \mathcal{N}_i + K_{X_i}) > 0$, where $\mathcal{N}_i$ is the class of the fixed part of the linear system of sections of $\mathcal{F}_i$, we obtain a numerically effective class $\mathcal{L}_i$ on a surface X_{i+1} such that $h^0(X_{i+1}, \mathcal{M}_i+K_{X_{i+1}}) > 0$, where $\mathcal{M}_i$ is the class of the fixed part of the linear system of sections of $\mathcal{L}_i$ and every irreducible exceptional class on X_{i+1} meets $\mathcal{L}_i$ positively, and from $\mathcal{L}_i$ we obtain a numerically effective class $\mathcal{F}_{i+1}$ on X_{i+1}. Since $-K_{X_i} \in$ EFF for each i, it meets $\mathcal{F}_i$ nonnegatively. Since this process of going from i to $i+1$ reduces the number of integral divisors in the fixed part, it eventually must stop. I.e., for some j, $\mathcal{F}_j$ is numerically effective but $h^0(X_j, \mathcal{N}_j + K_{X_j}) = 0$. Since for each $i \leq j$ we have $h^0(X_i, \mathcal{M}_{i-1}+K_{X_i}) > 0$ (which, as remarked above, is impossible if $K^2_{X_i} > 0$), we see that $K^2_{X_i} \leq 0$ for all $i \leq j$.

Denote K_{X_i} by K_i. Then for $i < j$, $\mathcal{F}_i \cdot K_i = \mathcal{L}_i \cdot K_{i+1}$, and $\mathcal{F}_{i+1} = \mathcal{L}_i + K_{i+1}$ so $\mathcal{F}_{i+1} \cdot K_{i+1} = \mathcal{F}_i \cdot K_i + K^2_{i+1}$. But $0 \geq K^2_{i+1} \geq K^2_i$ (with $K^2_{i+1} = K^2_i$ precisely if $X_i = X_{i+1}$, which would just mean that $\mathcal{F}_i = \mathcal{L}_i$ already meets every irreducible exceptional class on X_i positively); thus $0 \geq \mathcal{F} \cdot K_X > \mathcal{F}_j \cdot K_j$ unless $X_2 = X_j$ and $K^2_2 = 0$, in which case $\mathcal{F}_j = \mathcal{L}_1 + (j-1)K_2$. We see it is enough to consider two cases: either $0 > \mathcal{F}_j \cdot K_j$, or $0 = \mathcal{F}_j \cdot K_j$. In the latter case, $X_2 = X_j$, $K^2_2 = 0$ and by Lemma II.4, $\mathcal{F}_j = -sK_2$ for some $s \geq 0$, so for $r = j-1$, $\mathcal{L}_1 = -sK_2 - rK_2$.

Consider first the contingency $0 > \mathcal{F}_j \cdot K_j$. Let D_j be a nontrivial section of $-K_j$ (e.g., take the image of D under $X_1 \to X_j$). Consider $0 \to \mathcal{F}_j \to \mathcal{L}_{j-1} \to \mathcal{L}_{j-1} \otimes \mathcal{O}_{D_j} \to 0$. From Lemma III.6 and Lemma II.6(a), Lemma III.7, and Lemma III.8, we see that a numerically effective divisor not containing an anticanonical divisor in the fixed part of its linear system of sections but meeting the anticanonical class positively is regular; i.e., $\mathcal{F}_j$ is regular, so the sequence is exact on global sections and on h^1. Since $\mathcal{F}_j$ and $\mathcal{L}_{j-1}$ differ only in the fixed components of their linear systems of sections, they have isomorphic H^0's. From Riemann-Roch, $h^0(X_j, \mathcal{F}_j) = (\mathcal{F}^2_j - K_j \cdot \mathcal{F}_j)/2 + 1$ and $h^0(X_j, \mathcal{L}_{j-1}) = (\mathcal{L}^2_{j-1} - K_j \cdot \mathcal{L}_{j-1})/2 + 1 + h^1(X_j, \mathcal{L}_{j-1})$. Setting these equal and simplifying using $\mathcal{F}_j = \mathcal{L}_{j-1} + K_j$ give $\mathcal{L}_{j-1} \cdot K_j = h^1(X_j, \mathcal{L}_{j-1})$, but $\mathcal{L}_{j-1} \cdot K_j \leq 0$ since $\mathcal{L}_{j-1}$ is numerically effective, so h^1 vanishes for $\mathcal{L}_{j-1}$. Thus $(\mathcal{F}_j - K_j) \cdot K_j = \mathcal{L}_{j-1} \cdot K_j = 0$, or $\mathcal{F}_j \cdot K_j = K^2_j$. Since $\mathcal{F} \cdot K_1 + K^2_2 = \mathcal{F}_2 \cdot K_2 \geq \mathcal{F}_j \cdot K_j = K^2_j$, we have $\mathcal{F} \cdot K_X = \mathcal{F} \cdot K_1 \geq K^2_j - K^2_2 \geq 0$. Since $\mathcal{F}$ is numerically effective, we see that $\mathcal{F} \cdot K_X = 0$ as desired. We have also proved that in this case $K^2_j = K^2_2$, hence that $X_j = X_2$, so $\mathcal{F}_j = \mathcal{L}_1 + (j-1)K_2$. Substituting into $\mathcal{F}_j \cdot K_j = K^2_j$ gives $(\mathcal{L}_1+(j-1)K_2) \cdot K_2 = K^2_2$ or $\mathcal{L}_1 \cdot K_2 = (2-j)K^2_2$ and so we have $0 = \mathcal{F} \cdot K_X = \mathcal{L}_1 \cdot K_2 = (2-j)K^2_2$; i.e., either $j = 2$ or $K^2_2 = 0$. But $K^2_2 = 0$ implies $\mathcal{F}_j \cdot K_j = 0$, contradicting the hypothesis $0 > \mathcal{F}_j \cdot K_j$. Thus $j = 2$, $K^2_2 < 0$, and Y in the statement of Lemma III.10 is X_2, while $\mathcal{F}'$ is $\mathcal{F}_2$.

We note that in the course of the argument above we found that $\mathcal{L}_{j-1} = \mathcal{L}_1$ is regular; thus $\mathcal{F}$, being a pullback of $\mathcal{L}_1$, is regular too. But $h^2(X, \mathcal{F} + K_X) = 0$ by duality, so $h^1(D, \mathcal{F}) = 0$ follows from regularity of $\mathcal{F}$ and from $0 \to \mathcal{F} + K_X \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_D \to 0$. From Lemma II.5 we therefore see that $\mathcal{F} \otimes \mathcal{O}_D$ is nontrivial.

Now consider the remaining contingency: $X_2 = X_j$, $K_2^2 = 0$, $\mathcal{F}_j = -sK_2$ where $s \geq 0$, and $\mathcal{L}_1 = -sK_2 - rK_2$ for $r = j-1$. Since $\mathcal{F}_j$ is numerically effective of self-intersection 0 and the fixed part of its linear system of sections does not contain a section of $-K_2$, either Lemma III.6 applies or $\mathcal{F}_j = 0$. Either way, $\mathcal{N}_j = 0$, with $h^1(X_2, \mathcal{F}_j) = \sigma$, where $\sigma = 0$ if $\mathcal{F}_j = 0$ and otherwise by Lemma II.6(a) $\mathcal{F}_j = \sigma\mathcal{C}$ for some positive integer σ, where $\mathcal{C}$ is the class of some reduced and irreducible curve C moving in a pencil.

Clearly, if $s = 0$, then $\sigma = 0$ and $\mathcal{L}_1 = -rK_2$; i.e., we can take $Y = X_2$, and then $\mathcal{H}$ is the pullback of $-sK_2$ with $s = 0$, and $\mathcal{N}$ is the pullback of $-rK_2$, as the statement of Lemma III.10 requires. Moreover, in this case $h^0(X_2, \mathcal{L}_1) = 1$, so $h^1(X_2, \mathcal{L}_1) = 0$ by Riemann-Roch, so $h^1(X, \mathcal{F}) = h^1(X_2, \mathcal{L}_1) = \sigma = 0$, as claimed.

If $s > 0$, then $\mathcal{C} = -\tau K_2$, where $\sigma\tau = s$. Thus the sections of $\mathcal{F}_j = \sigma\mathcal{C} = -sK_2$ are base point free and we have $h^1(X_2, \mathcal{F}_j) = \sigma$. But $h^0(X_2, \mathcal{F}_j) = h^0(X_2, \mathcal{L}_1)$ and by Riemann-Roch $h^0 - h^1$ is the same for $\mathcal{F}_j$ and $\mathcal{L}_1$, and hence h^1 is the same too, so $h^1(X_2, \mathcal{L}_1) = h^1(X, \mathcal{F}_j) = \sigma$. Moreover, since the pullback of $\mathcal{F}_j$ to X has the same h^0 as $\mathcal{F}$, we see that $\mathcal{H}$ is the pullback of $\mathcal{F}_j = -sK_2$ and hence $\mathcal{N}$ is the pullback of $-rK_2$, and so again $Y = X_2$.

We also have $r < \tau$, since $-rK_2$ is the class of a unique effective divisor but the sections of $\mathcal{C} = -\tau K_2$ move in a pencil. Let $t < \tau$; then $\mathcal{C} + tK_2 = -(\tau - t)K_2 \in \mathrm{EFF}$. Since the sections of $\mathcal{C} = -\tau K_2$ are fixed component free and hence $D_2 \cdot \mathcal{C} = 0$, the restriction of $-K_2$ to C is trivial, so from $0 \to -\mathcal{C} - tK_2 \to -tK_2 \to -tK_2 \otimes \mathcal{O}_C \to 0$ we see $h^0(X_2, -tK_2) = 1$ and, by Riemann-Roch, $h^1(X_2, -tK_2) = 0$. But, denoting the pullback of K_2 to X also by K_2, we have $0 = h^1(X_2, -tK_2) = h^1(X, -tK_2)$, so from $0 \to -tK_2 + K_X \to -tK_2 \to -tK_2 \otimes \mathcal{O}_D \to 0$, we see $h^1(D, -tK_2) = 0$; i.e., for $t < \tau$, the restriction of $-K_2$ to D is, by Lemma II.5, nontrivial. But C is disjoint from D_2, and hence the pullback of $\mathcal{C}$ has trivial restriction to D; thus τ is the least positive multiple of $-K_2$ whose restriction to D is trivial. Moreover, it follows that the restriction of $\mathcal{H}$ to D is trivial, and hence that $\mathcal{F} \otimes \mathcal{O}_D = -rK_2 \otimes \mathcal{O}_D$ is nontrivial.

Conversely, say $\mathcal{F} \cdot D = 0$. By Lemma III.6, Lemma III.7, and Lemma III.8, if $\mathcal{N} + K_X \notin \mathrm{EFF}$ (i.e., if the fixed part of the linear system of sections of $\mathcal{F}$ does not contain an anticanonical divisor), then no fixed component of the linear system of sections of $\mathcal{F}$ is a fixed component of the linear system of sections of $-K_X$. In this case $\mathcal{F} \cdot D = 0$ means that $\mathcal{F}$ has a section which is disjoint from D, and hence that $\mathcal{F} \otimes \mathcal{O}_D$ is indeed trivial. Thus, if $\mathcal{F} \otimes \mathcal{O}_D$ is nontrivial, it must be that $\mathcal{N} + K_X \in \mathrm{EFF}$. □

Our results on base points depend on the following lemma.

Lemma III.11. *Let C be an integral projective curve whose dualizing sheaf K_C is locally free rank 1. If $h^0(C, K_C) > 0$ (i.e., C is not smooth and rational), then K_C is generated by global sections.*

Proof. This is a special case of Theorem D of [Cn]. □

The proof in [F] corresponding to the next result assumes that $-K_X$ has a reduced section, but this can be avoided.

Lemma III.12. *Let $\mathcal{F}$ be the class of an effective fixed component free divisor on a smooth projective rational anticanonical surface X. Let D be a nonzero section of $-K_X$. Then the sections of $\mathcal{F}$ have a base point if and only if $\mathcal{F} \cdot D = 1$, in which case the base point is unique and lies on D.*

Proof. First we check that there is no base point if $\mathcal{F} \cdot D \neq 1$. Since $\mathcal{F}$ is numerically effective, we have $\mathcal{F} \cdot D \geq 0$. Say $\mathcal{F} \cdot D = 0$; if also $\mathcal{F}^2 = 0$ then, being fixed component free, the sections of $\mathcal{F}$ are also obviously base point free (since distinct sections of $\mathcal{F}$ are disjoint). So say $\mathcal{F}^2 > 0$. A general section F of $\mathcal{F}$ is reduced and irreducible by Lemma II.6. Thus we may assume F and D are disjoint, and so $\mathcal{O}_F \otimes \mathcal{F} = K_F$ by adjunction. From $0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_F \to 0$, we see the sections of $\mathcal{F}$ surject to those of $\mathcal{F} \otimes \mathcal{O}_F = K_F$. By Lemma III.11, K_F is generated by global sections (and hence the sections of $\mathcal{F}$ are base point free) unless F is rational, i.e., unless $F^2 = -2$, contrary to assumption.

Now say $\mathcal{F} \cdot D > 1$. Thus $\mathcal{O}_F \otimes \mathcal{F} = -K_X \otimes K_F$ has degree at least $2g$, where g is the genus of F. By [D, Proposition 7, p. 59], $\mathcal{O}_F \otimes \mathcal{F}$ is generated by global sections, and hence as before the sections of $\mathcal{F}$ are base point free.

Conversely, suppose $\mathcal{F} \cdot D = 1$. Then $\mathcal{F}^2$ is odd by adjunction, so positive, so a general section F of $\mathcal{F}$ is integral by Lemma II.6, and we may assume F and D have no common components and hence meet at a single point, x, which must be smooth on both F and D. Let $Y \to X$ be the morphism resulting from blowing up x and let $\mathcal{E}$ be the class of the exceptional locus E. Then $\mathcal{F}' = \mathcal{F} - \mathcal{E}$ is numerically effective and its sections fixed component free on Y, and the proper transform D' of D is anticanonical, but $\mathcal{F}' \cdot D' = 0$. Thus, by our foregoing argument, the sections of $\mathcal{F}'$ are base point free. By Lemma II.6(a) (if ${\mathcal{F}'}^2 = 0$) or Lemma III.7 and Lemma III.8(a) (if ${\mathcal{F}'}^2 > 0$), we see $h^1(Y, \mathcal{F}') = 1$. But $\mathcal{F}' + \mathcal{E}$ is numerically effective (since $\mathcal{F}'$ is and since it meets $\mathcal{E}$ nonnegatively) of positive self-intersection and meets the anticanonical class, so from Lemma III.7 and Lemma III.8, we see it must be regular. Thus from $0 \to \mathcal{F}' \to \mathcal{F}' + \mathcal{E} \to (\mathcal{F}' + \mathcal{E}) \otimes \mathcal{O}_E \to 0$ we see that $h^0(Y, \mathcal{F}') = h^0(Y, \mathcal{F}' + \mathcal{E})$, so $\mathcal{E}$ is the class of a fixed component of the linear system of sections of $\mathcal{F}' + \mathcal{E}$. Since $\mathcal{F}' + \mathcal{E}$ is the pullback of $\mathcal{F}$ to Y, we see that the sections of $\mathcal{F}$ have a base point at x. $\square$

Proof of Theorem III.1. (a) By Lemma III.10, we see that the fixed part of the linear system of sections of $\mathcal{F}$ cannot contain an anticanonical divisor. By Lemma III.6, Lemma II.6, Lemma III.7 and Lemma III.8, we see that $\mathcal{F}$ is regular and its linear system of sections is fixed component free, and so base point free by Lemma III.12.

(b) Here $\mathcal{F}$ must have positive self-intersection (to satisfy adjunction), so, by Lemma III.7 and Lemma III.8, $\mathcal{F}$ is regular. If its linear system of sections is also fixed component free, then by Lemma III.12 it has a unique base point, on D. If the sections of $\mathcal{F}$ have a fixed component, then the result follows by Lemma III.7 and Lemma III.8, again.

Conversely, say $\mathcal{F} = \mathcal{H} + \mathcal{N}$, with $\mathcal{H} = r\mathcal{C}$, $h^1(X, \mathcal{H}) = r$ and $\mathcal{N} = \mathcal{N}_1 + \cdots + \mathcal{N}_t$, as in the statement of (b). It is easy to check that $\mathcal{N}^2 = \mathcal{N} \cdot K_X = -1$. Since $\mathcal{F}$ is regular, we have using $\mathcal{F} = \mathcal{H} + \mathcal{N}$, $\mathcal{C} \cdot \mathcal{N} = 1$ and Riemann-Roch that $h^0(X, \mathcal{F}) = (\mathcal{H}^2 - \mathcal{H} \cdot K_X)/2 + r + 1$. But $h^1(X, \mathcal{H}) = r$ by hypothesis so we now see $h^0(X, \mathcal{F}) = h^0(X, \mathcal{H})$. Thus $\mathcal{N}$ is the class of the fixed part of the linear system of sections of $\mathcal{H} + \mathcal{N}$, as we needed to show.

(c) If the sections of $\mathcal{F}$ are fixed component free, then first they are base point free by Lemma III.12, and second either Lemma III.8(a) applies and gives the result or Lemma III.6 applies and, with Lemma II.6, gives the result. If the sections of $\mathcal{F}$ are not fixed component free but the fixed part does not contain an anticanonical divisor, then only Lemma III.8(b) applies, giving the result.

(d) This is just Lemma III.10. $\square$

Proof of Corollary III.2. Let $\mathcal{F}$ be a numerically effective class such that the fixed part of its linear system of sections is nontrivial but does not contain an anticanonical divisor. Of Lemma III.6, Lemma III.7 and Lemma III.8, only Lemma III.8(b) applies, whence no fixed component of the linear system of sections of $\mathcal{F}$ is a component of an arbitrary anticanonical divisor D. $\square$

We now prove Theorem I.1, of the Introduction:

Proof. So $\mathcal{F}$ is a numerically effective divisor class on a smooth projective rational surface X with an effective anticanonical divisor D. Then $h^0(X,\mathcal{F}) > 0$ by Corollary II.3. Moreover, by Corollary III.3(a), $h^1(X,\mathcal{F}) > 0$ if and only if $\mathcal{F}\cdot K_X = 0$ and a general section of $\mathcal{F}$ has a connected component disjoint from D. Thus we only need to show that $1 + h^1(X,\mathcal{F})$ is the number of connected components of a general section of $\mathcal{F} - K_X$.

By Lemma II.5, anticanonical divisors are connected; hence the result follows for $\mathcal{F} = \mathcal{O}_X$, so we may assume that $\mathcal{F}$ is nontrivial. Suppose that $h^1(X,\mathcal{F}) = 0$. Then $h^1(X, -(\mathcal{F} - K_X)) = 0$ by duality, so from $0 \to -(\mathcal{F} - K_X) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$, where C is a general section of $\mathcal{F} - K_X$, we see C is connected, as desired. Suppose $h^1(X,\mathcal{F}) > 0$; then, by Theorem III.1 and Corollary III.2, $-K_X\cdot\mathcal{F} = 0$ and if the general section of $\mathcal{F}$ is not disjoint from D then, as we shall first assume, there is a birational morphism $X \to Y$ to a smooth surface Y with $K_Y^2 = 0$ and (regarding pullbacks from Y as classes on X) $\mathcal{F} = -\tau\sigma K_Y - rK_Y$, where $1 \le \sigma$, $1 \le r < \tau$, $-rK_Y$ is the class of the fixed part of the linear system of sections of $\mathcal{F}$, and $-\tau K_Y$ is a pencil with which the linear system of sections of $-\tau\sigma K_Y$ is composed. Suppose $X = Y$; if $r+1 < \tau$ then $\mathcal{F} - K_X = -\tau\sigma K_Y - (r+1)K_Y$, and $h^0(Y, -(r+1)K_Y) = 1$, so a general section of $\mathcal{F} - K_X$ has $1+\sigma$ connected components, while $h^1(X,\mathcal{F}) = \sigma$ by Theorem III.1(c), as desired. If $r+1 = \tau$, then the linear system of sections of $\mathcal{F} - K_X = -\tau(\sigma+1)K_Y$ is fixed component free, and a general section again has $1+\sigma$ connected components, as desired. If $X \to Y$ is not the identity, then X is a blowing up of points (possibly infinitely near) of Y. In this case, $\mathcal{F} - K_X = -\tau\sigma K_Y - rK_Y - K_X$; hence a general section once more has $1+\sigma$ connected components, as needed, if we check that h^0 of $-rK_Y - K_X$ is 1. Since in any case h^0 of $-rK_Y$ is 1, we have $r < \tau$; thus h^0 of $-rK_Y - K_X$ is 1 if $r+1 < \tau$, since h^0 of $-rK_Y - K_X + (-K_Y + K_X) = -rK_Y - K_Y$ is 1 and $-K_Y + K_X$, being a sum of classes of exceptional curves, is the class of an effective divisor. If $r+1 = \tau$, then the linear system of sections of $-rK_Y - K_Y$ is a pencil, and adding $-(-K_Y + K_X)$, which is the class of an antieffective divisor, we get the class $-rK_Y - K_X$, which has a unique effective section.

So now we may assume that the general section of $\mathcal{F}$ is disjoint from D. If the linear system of sections of $\mathcal{F}$ is fixed component free, then, by Theorem III.1(c) and Lemma II.6, $h^1(X,\mathcal{F})$ equals the number of components of a general section F of $\mathcal{F}$, and F is disjoint from D. Thus $\mathcal{O}_F\otimes(\mathcal{F} - K_X) = \mathcal{O}_F\otimes\mathcal{F}$, so from $0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{O}_F\otimes\mathcal{F} \to 0$ and $0 \to -K_X \to \mathcal{F} - K_X \to \mathcal{O}_F\otimes(\mathcal{F} - K_X) \to 0$

we see that the sections of $\mathcal{F} - K_X$ are just sums of sections of $\mathcal{F}$ and $-K_X$; hence the number of components of a general section of $\mathcal{F}$ is one less than the number of components of a general section of $\mathcal{F} - K_X$, as desired. If the fixed part N of the linear system of sections of $\mathcal{F}$ is not trivial but does not contain an anticanonical divisor, then by Theorem III.1(c) $h^1(X, \mathcal{F}) = 1$ and the general section of $\mathcal{F}$ is connected but, by Corollary III.2, disjoint from D. Arguing as in the previous case shows that the general section of $\mathcal{F} - K_X$ has two connected components, as required. □

Remark III.13. Our results have mostly concerned numerically effective classes, but one may be interested in arbitrary classes. In general, to determine dimensions of complete linear systems, fixed components and base points, given arbitrary divisor classes on a smooth projective rational anticanonical surface, one essentially needs to know the monoid EFF of effective classes on X.

Since EFF determines the cone NEF of numerically effective classes, given an arbitrary class $\mathcal{F}$ and knowing EFF one can determine if $\mathcal{F} \in$ EFF and, if so, find a decomposition $\mathcal{F} = \mathcal{L} + \mathcal{M}$, where $\mathcal{L}$ is maximal with respect to $\mathcal{L} \in$ NEF and $\mathcal{L} \leq \mathcal{F}$ (where $\mathcal{L} \leq \mathcal{F}$ means $\mathcal{F} - \mathcal{L} \in$ EFF). Then $h^0(X, \mathcal{F}) = h^0(X, \mathcal{L})$, and the results obtained herein for numerically effective classes apply to $\mathcal{L}$.

To make this discussion concrete, consider the case that X is a blowing up of $\mathbf{P}^2$. (Blowings up of other relatively minimal models can with minor changes be handled similarly.) In this case the following data suffice to determine EFF: an *exceptional configuration* $\{\mathcal{E}_0, \ldots, \mathcal{E}_n\} \subset \mathrm{Pic}(X)$ (i.e., a basis $\mathcal{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$ of $\mathrm{Pic}(X)$, where $\mathcal{E}_0$ is the pullback of the class of a line with respect to some birational morphism $X \to \mathbf{P}^2$ and the other classes $\mathcal{E}_i$ are the classes of the exceptional loci corresponding to a factorization of $X \to \mathbf{P}^2$ into a sequence of monoidal transformations); the kernel $\Lambda \subset \mathrm{Pic}(X)$ of the functorial homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$, where D is an effective anticanonical divisor; and the fixed components of $|D|$.

In fact, EFF is generated by: (a) the fixed components of $|D|$; (b) classes $\mathcal{F}$ such that $\mathcal{F}^2 = \mathcal{F} \cdot K_X = -1$ and $\mathcal{F} \cdot \mathcal{E}_0 \geq 0$; (c) classes $\mathcal{F}$ such that $\mathcal{F} = \mathcal{E}_i - \mathcal{E}_j$ where $0 < i < j$ and $\mathcal{F} \in \Lambda$; (d) classes $\mathcal{F}$ such that $\mathcal{F}^2 = -2$, $\mathcal{F} \cdot K_X = 0$, $\mathcal{F} \cdot \mathcal{E}_0 > 0$ and $\mathcal{F} \in \Lambda$; and (e) classes $\mathcal{F}$ such that $\mathcal{F}^2 - \mathcal{F} \cdot K_X \geq 0$ and $\mathcal{F} \cdot \mathcal{E}_0 \geq 0$.

To see this, we first check that classes of each of these types are in EFF. This is obvious for classes of type (a). For a class $\mathcal{F}$ of one of the remaining types note that $h^2(X, \mathcal{F}) = h^0(X, K_X - \mathcal{F}) = 0$ since $\mathcal{E}_0$ is numerically effective but $(K_X - \mathcal{F}) \cdot \mathcal{E}_0 < 0$. Now effectivity follows in cases (b) and (e) by Lemma II.2. In cases (c) and (d), by Lemma II.2 we have $h^0(X, \mathcal{F}) = h^1(X, \mathcal{F})$; we also have $h^2(X, K_X + \mathcal{F}) = 0$: in case (d) since $h^2(X, K_X + \mathcal{F}) = h^0(X, -\mathcal{F})$ and $-\mathcal{F} \cdot \mathcal{E}_0 < 0$, and in case (c) since, if $h^2(X, K_X + \mathcal{F}) > 0$, then $-\mathcal{F} \in$ EFF, which is impossible since $0 < i < j$ implies $\mathcal{E}_j - \mathcal{E}_i$ cannot be in EFF (as exceptional divisors, E_j may be contained in E_i but not vice versa). In either case (c) or (d), since $\mathcal{F} \in \Lambda$, the restriction of $\mathcal{F}$ to D is trivial so we have $0 \to \mathcal{F} + K_X \to \mathcal{F} \to \mathcal{O}_D \to 0$; hence $h^0(X, \mathcal{F}) = h^1(X, \mathcal{F}) \geq h^1(D, \mathcal{O}_D) = 1$.

Conversely, if $\mathcal{F}$ is the class of a reduced irreducible curve of negative self-intersection, then $\mathcal{F} \cdot \mathcal{E}_0 \geq 0$ and either $\mathcal{F}$ is a fixed component of $|D|$ (which is case (a)), or $\mathcal{F}$ is not the class of a fixed component of $|D|$ and so meets $-K_X$ nonnegatively. Hence by adjunction either $\mathcal{F}^2 = \mathcal{F} \cdot K_X = -1$ (case (b)), or $\mathcal{F}^2 = -2$, $\mathcal{F} \cdot K_X = 0$ and $\mathcal{F}$ is disjoint from D, and thus in Λ (which is case (c) if $\mathcal{F} \cdot \mathcal{E}_0 = 0$, and case (d) if $\mathcal{F} \cdot \mathcal{E}_0 > 0$). In general, if $\mathcal{F}$ is in EFF, we can write

$\mathcal{F} = \mathcal{L} + \mathcal{N}$, where $\mathcal{N}$ is a nonnegative sum of curves of negative self-intersection (and hence in the monoid generated by types (a) through (d)), and $\mathcal{L}$ is numerically effective (and hence of type (e)).

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