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ANTICIPATED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS[1]

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In this paper we discuss new types of differential equations which we call anticipated backward stochastic differential equations (anticipated BSDEs). In these equations the generator includes not only the values of solutions of the present but also the future. We show that these anticipated BSDEs have unique solutions, a comparison theorem for their solutions, and a duality between them and stochastic differential delay equations.

1. Introduction. Consider these types of stochastic differential delay equations (SDDEs):

$$\text{(1)}\qquad \begin{cases} dX_t = (\mu_t X_t + \bar{\mu}_{t-\theta} X_{t-\theta})\,dt + (X_t \sigma_t^T + X_{t-\theta}\bar{\sigma}_{t-\theta}^T)\,dW_t, \\ \qquad\qquad t \in [t_0, T+\theta]; \\ X_t = x_t, \qquad t \in [t_0-\theta, t_0], \end{cases}$$

where W is a d-dimensional Brownian motion, $\theta > 0$, x_t is a deterministic function, and Q is a given $\mathscr{F}_T^W$-measurable random variable. In the case where $\bar{\mu} = \bar{\sigma} \equiv 0$, this model is very typical in finance as the price of a stock. Then $Y_{t_0} = E[X_T Q | \mathscr{F}_{t_0}]$ can be the price of an option valued Q at maturity time T if $x_t \equiv 1$. It is easy to prove that (see, e.g., El Karoui, Peng and Quenez [7]) $Y_\bullet$ is a solution to the following backward stochastic differential equation (BSDE):

$$-dY_t = (\mu_t Y_t + Z_t \sigma_t)\,dt - Z_t\,dW_t, \qquad Y_T = Q.$$

This SDE with delay, in which $\bar{\mu}$ and $\bar{\sigma}$ are nonzero, has a solution. An interesting question is whether it can be expressed in the form of equation (1). The answer is positive if we can solve the following new type of "anticipated" BSDE:

$$\text{(2)}\qquad \begin{cases} -dY_t = (\mu_t Y_t + \bar{\mu}_t E^{\mathscr{F}_t}[Y_{t+\theta}] + Z_t\sigma_t \\ \qquad\qquad + E^{\mathscr{F}_t}[Z_{t+\theta}]\bar{\sigma}_t + l_t)\,dt - Z_t\,dW_t, & t \in [t_0, T]; \\ Y_t = Q_t, & t \in [T, T+\theta]; \\ Z_t = P_t, & t \in [T, T+\theta]. \end{cases}$$

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We observe that the generator, that is, the dt part of the BSDE, contains the values $(Y_\bullet, Z_\bullet)$ for present time t as well as for future time $t+\theta$. This is a new duality phenomenon for SDEs and BSDEs.

In this paper we consider a more general form of this new type of BSDE:

$$\begin{cases} -dY_t = f\big(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\zeta(t)}\big)\,dt - Z_t\,dW_t, & t \in [0,T]; \\ Y_t = \xi_t, & t \in [T, T+K]; \\ Z_t = \eta_t, & t \in [T, T+K]. \end{cases}$$

The paper is organized as follows. In Section 2 we consider the duality between SDDEs and anticipated BSDEs. After a brief presentation of some known results that we will use in Section 3, we prove an existence and uniqueness result for anticipated BSDEs in Section 4. In Section 5 we give an important result for anticipated BSDEs: a comparison theorem. In Section 6 we use the duality between SDDEs and anticipated BSDEs mentioned in Section 2 to solve a stochastic control problem.

2. Duality between SDDEs and anticipated BSDEs. It is well known that there is perfect duality between SDEs and BSDEs (see El Karoui, Peng and Quenez [7]). In this section we consider duality between the SDDEs and the anticipated BSDEs mentioned above. We will use this duality to solve a stochastic control problem in Section 6.

THEOREM 2.1. *Suppose $\theta > 0$ is a given constant and $\mu_\bullet, \bar{\mu}_\bullet \in L^2_{\mathscr{F}}(t_0 - \theta, T+\theta)$, $l_\bullet \in L^2_{\mathscr{F}}(t_0, T)$, $\sigma_\bullet, \bar{\sigma}_\bullet \in L^2_{\mathscr{F}}(t_0 - \theta, T+\theta; \mathbb{R}^{d\times 1})$, $\mu_\bullet, \bar{\mu}_\bullet, \sigma_\bullet, \bar{\sigma}_\bullet$ are uniformly bounded. Then for all $Q_\bullet \in S^2_{\mathscr{F}}(T, T+\theta)$, $P_\bullet \in L^2_{\mathscr{F}}(T, T+\theta; \mathbb{R}^d)$, the solution $Y_\bullet$ of the anticipated BSDE* (2) *can be given by the closed formula*

$$Y_t = E^{\mathscr{F}_t}\biggl[X_T Q_T + \int_t^T X_s l_s\,ds + \int_T^{T+\theta} (Q_s \bar{\mu}_{s-\theta} + P_s \bar{\sigma}_{s-\theta}) X_{s-\theta}\,ds\biggr], \qquad a.e.,\ a.s.,$$

where X_s is the solution to the SDDE

$$\begin{cases} dX_s = (\mu_s X_s + \bar{\mu}_{s-\theta} X_{s-\theta})\,ds + (X_s \sigma_s^T + X_{s-\theta}\bar{\sigma}_{s-\theta}^T)\,dW_s, \\ \qquad\qquad s \in [t, T+\theta]; \\ X_t = 1, \\ X_s = 0, \qquad s \in [t-\theta, t]. \end{cases} \tag{3}$$

PROOF. First, we show that (3) has a unique solution. When $s \in [t, t+\theta]$, (3) becomes

$$\begin{cases} dX_s = \mu_s X_s\,ds + X_s \sigma_s^T\,dW_s, & s \in [t, t+\theta]; \\ X_t = 1. \end{cases} \tag{4}$$

We can then easily obtain a unique continuous solution $\varsigma_{\bullet}$ for (4). When $s \in [t+\theta, T+\theta]$, (3) becomes

$$
\begin{cases} dX_s = (\mu_s X_s + \bar{\mu}_{s-\theta} X_{s-\theta})\,ds + (X_s \sigma_s^T + X_{s-\theta}\bar{\sigma}_{s-\theta}^T)\,dW_s, \\ \qquad\qquad\qquad s \in [t+\theta, T+\theta], \\ X_s = \varsigma_s, \qquad s \in [t, t+\theta]. \end{cases} \tag{5}
$$

Equation (5) is a classical SDDE, thus, it has a unique solution. Applying Itô's formula to $X_s Y_s$ for $s \in [t, T]$ and taking conditional expectations under $\mathscr{F}_t$, we get

$$
\begin{aligned}
&E^{\mathscr{F}_t}[X_T Y_T] - X_t Y_t \\
&\quad = E^{\mathscr{F}_t}\biggl[\int_t^T (Y_s \bar{\mu}_{s-\theta} X_{s-\theta} - E^{\mathscr{F}_s}[Y_{s+\theta}]\bar{\mu}_s X_s \\
&\qquad\qquad + Z_s \bar{\sigma}_{s-\theta} X_{s-\theta} - E^{\mathscr{F}_s}[Z_{s+\theta}]\bar{\sigma}_s X_s - X_s l_s)\,ds\biggr].
\end{aligned}
$$

Because $X_t = 1$ and $X_s = 0$, $s \in [t-\theta, t)$, we have

$$
\begin{aligned}
Y_t &= E^{\mathscr{F}_t}\biggl[X_T Y_T + \int_t^T X_s l_s\,ds\biggr] - E^{\mathscr{F}_t}\biggl[\int_t^T (Y_s \bar{\mu}_{s-\theta} X_{s-\theta} - Y_{s+\theta}\bar{\mu}_s X_s)\,ds\biggr] \\
&\quad - E^{\mathscr{F}_t}\biggl[\int_t^T (Z_s \bar{\sigma}_{s-\theta} X_{s-\theta} - Z_{s+\theta}\bar{\sigma}_s X_s)\,ds\biggr] \\
&= E^{\mathscr{F}_t}\biggl[X_T Y_T + \int_t^T X_s l_s\,ds - \int_t^T Y_s \bar{\mu}_{s-\theta} X_{s-\theta}\,ds + \int_{t+\theta}^{T+\theta} Y_s \bar{\mu}_{s-\theta} X_{s-\theta}\,ds\biggr] \\
&\quad - E^{\mathscr{F}_t}\biggl[\int_t^T Z_s \bar{\sigma}_{s-\theta} X_{s-\theta}\,ds - \int_{t+\theta}^{T+\theta} Z_s \bar{\sigma}_{s-\theta} X_{s-\theta}\,ds\biggr] \\
&= E^{\mathscr{F}_t}\biggl[X_T Q_T + \int_t^T X_s l_s\,ds + \int_T^{T+\theta} (Q_s \bar{\mu}_{s-\theta} X_{s-\theta} + P_s \bar{\sigma}_{s-\theta} X_{s-\theta})\,ds\biggr]. \qquad \square
\end{aligned}
$$

3. Preliminaries. Let $(\Omega, \mathscr{F}, P, \mathscr{F}_t, t \geq 0)$ be a complete stochastic basis such that $\mathscr{F}_0$ contains all P-null elements of $\mathscr{F}$ and suppose that the filtration is generated by a d-dimensional standard Brownian motion $W = (W_t)_{t \geq 0}$. Given $T > 0$, denote the norm in $\mathbb{R}^m$ by $|\cdot|$. We will use the following notation:

- $L^2(\mathscr{F}_T; \mathbb{R}^m) = \{\mathbb{R}^m$-valued $\mathscr{F}_T$-measurable random variables such that $E[|\xi|^2] < \infty\}$;
- $L^2_{\mathscr{F}}(0, T; \mathbb{R}^m) = \{\mathbb{R}^m$-valued and $\mathscr{F}_t$-adapted stochastic processes such that $E[\int_0^T |\varphi_t|^2\,dt] < \infty\}$;
- $S^2_{\mathscr{F}}(0, T; \mathbb{R}^m) = \{$continuous processes in $L^2_{\mathscr{F}}(0, T; \mathbb{R}^m)$ such that $E[\sup_{0 \leq t \leq T} |\varphi_t|^2] < \infty\}$.

If $m = 1$, we denote them by $L^2(\mathscr{F}_T)$, $L^2_{\mathscr{F}}(0, T)$ and $S^2_{\mathscr{F}}(0, T)$. The above L^2 are all separable Hilbert spaces.

The following lemmas can be found in Peng [13], Section 3. For their originalities we refer to the notes of [13] or [7]. Our Lemma 3.1 is Lemma 3.1 of Peng [13]. Lemma 3.2, which is Theorem 3.2 of Peng [13], is a basic result of BSDEs: an existence and uniqueness theorem. Both Lemmas 3.3 and 3.4 are comparison theorems for solutions of BSDEs. Lemma 3.3 is Theorem 3.3 of Peng [13] and can also be found in El Karoui, Peng and Quenez [7]. Lemma 3.4 can be easily obtained from Lemma 3.3.

LEMMA 3.1. *For a fixed* $\xi \in L^2(\mathscr{F}_T)$ *and* $g_0(\cdot)$ *which is an* $\mathscr{F}_t$*-adapted process satisfying* $E[(\int_0^T |g_0(t)|\,dt)^2] < +\infty$, *there exists a unique pair of processes* $(y_\cdot, z_\cdot) \in L^2_{\mathscr{F}}(0, T; \mathbb{R}^{1+d})$ *satisfying the following BSDE*:

$$y_t = \xi + \int_t^T g_0(s)\,ds - \int_t^T z_s\,dW_s, \qquad t \in [0, T].$$

If $g_0(\cdot) \in L^2_{\mathscr{F}}(0, T)$, *then* $(y_\cdot, z_\cdot) \in S^2_{\mathscr{F}}(0, T) \times L^2_{\mathscr{F}}(0, T; \mathbb{R}^d)$. *We have the following basic estimate*:

$$\begin{aligned} &|y_t|^2 + E^{\mathscr{F}_t}\left[\int_t^T \left(\frac{\beta}{2}|y_s|^2 + |z_s|^2\right) e^{\beta(s-t)}\,ds\right] \\ &\quad \le E^{\mathscr{F}_t}\left[|\xi|^2 e^{\beta(T-t)}\right] + \frac{2}{\beta} E^{\mathscr{F}_t}\left[\int_t^T |g_0(s)|^2 e^{\beta(s-t)}\,ds\right]. \end{aligned} \tag{6}$$

In particular,

$$\begin{aligned} &|y_0|^2 + E\left[\int_0^T \left(\frac{\beta}{2}|y_s|^2 + |z_s|^2\right) e^{\beta s}\,ds\right] \\ &\quad \le E[|\xi|^2 e^{\beta T}] + \frac{2}{\beta} E\left[\int_0^T |g_0(s)|^2 e^{\beta s}\,ds\right], \end{aligned} \tag{7}$$

where $\beta > 0$ *is an arbitrary constant. We also have*

$$E\left[\sup_{0 \le t \le T} |y_t|^2\right] \le k E\left[|\xi|^2 + \int_0^T |g_0(s)|^2\,ds\right], \tag{8}$$

where the constant k *depends only on* T.

We assume that $g = g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}^m$ satisfies the following conditions:

(a) $g(\cdot, y, z)$ is an $\mathbb{R}^m$-valued and $\mathscr{F}_t$-adapted process satisfying the Lipschitz condition in (y, z), that is, there exists $\rho > 0$ such that, for each $y, y' \in \mathbb{R}^m$ and $z, z' \in \mathbb{R}^{m \times d}$, $|g(t, y, z) - g(t, y', z')| \le \rho(|y - y'| + |z - z'|)$.

(b) $g(\cdot, 0, 0) \in L^2_{\mathscr{F}}(0, T; \mathbb{R}^m)$.

LEMMA 3.2. *Assume that g satisfies* (a) *and* (b), *then for any given terminal condition $\xi \in L^2(\mathscr{F}_T;\mathbb{R}^m)$, BSDE*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\,ds - \int_t^T Z_s\,dW_s, \qquad 0 \le t \le T, \tag{9}$$

has a unique solution, that is, there exists a unique pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot, Z_\cdot) \in S^2_{\mathscr{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^{m\times d})$ satisfying equation (9).

LEMMA 3.3. *Assume $g_j(\omega,t,y,z): \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfies* (a) *and* (b), $j = 1,2$. *Let $(Y^{(1)}_\cdot, Z^{(1)}_\cdot)$ and $(Y^{(2)}_\cdot, Z^{(2)}_\cdot)$ be respectively the solutions of BSDEs as follows*:

$$Y_t^{(j)} = \xi^{(j)} + \int_t^T g_j\bigl(s, Y_s^{(j)}, Z_s^{(j)}\bigr)\,ds - \int_t^T Z_s^{(j)}\,dW_s, \qquad 0 \le t \le T,$$

where $j = 1,2$. If $\xi^{(1)} \ge \xi^{(2)}$ and $g_1(t, Y_t^{(1)}, Z_t^{(1)}) \ge g_2(t, Y_t^{(1)}, Z_t^{(1)})$, a.e., a.s., then

$$Y_t^{(1)} \ge Y_t^{(2)}, \qquad \text{a.e., a.s.}$$

We also have strict comparison: *under the above conditions*,

$$Y_0^{(1)} = Y_0^{(2)} \iff \xi^{(1)} = \xi^{(2)}, \qquad \text{a.s.},$$
$$g_1\bigl(t, Y_t^{(1)}, Z_t^{(1)}\bigr) = g_2\bigl(t, Y_t^{(1)}, Z_t^{(1)}\bigr), \qquad \text{a.e., a.s.}$$

LEMMA 3.4. *We make the same assumption as in Lemma* 3.3. *If $\xi^{(1)} \ge \xi^{(2)}$, $g_1(t,y,z) \ge g_2(t,y,z)$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, then*

$$Y_t^{(1)} \ge Y_t^{(2)}, \qquad \text{a.e., a.s.}$$

4. Existence and uniqueness theorem. We consider a new form of BSDEs as follows:

$$\begin{cases} -dY_t = f\bigl(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\zeta(t)}\bigr)\,dt - Z_t\,dW_t, & t \in [0,T]; \\ Y_t = \xi_t, & t \in [T, T+K]; \\ Z_t = \eta_t, & t \in [T, T+K], \end{cases} \tag{10}$$

where $\delta(\cdot)$ and $\zeta(\cdot)$ are two $\mathbb{R}^+$-valued continuous functions defined on $[0,T]$ such that:

(i) There exists a constant $K \ge 0$ such that, for all $s \in [0,T]$,

$$s + \delta(s) \le T + K; \qquad s + \zeta(s) \le T + K.$$

(ii) There exists a constant $L \geq 0$ such that, for all $t \in [0, T]$ and for all nonnegative and integrable $g(\cdot)$,

$$\int_t^T g(s + \delta(s))\, ds \leq L \int_t^{T+K} g(s)\, ds;$$
$$\int_t^T g(s + \zeta(s))\, ds \leq L \int_t^{T+K} g(s)\, ds.$$

We call equation (10) the anticipated BSDE.

The setting of our problem is as follows: to find a pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot, Z_\cdot) \in S^2_{\mathscr{F}}(0, T + K; \mathbb{R}^m) \times L^2_{\mathscr{F}}(0, T + K; \mathbb{R}^{m\times d})$ satisfying anticipated BSDE (10).

Assume that for all $s \in [0, T]$, $f(s, \omega, y, z, \xi, \eta) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times L^2(\mathscr{F}_r; \mathbb{R}^m) \times L^2(\mathscr{F}_{r'}; \mathbb{R}^{m\times d}) \longrightarrow L^2(\mathscr{F}_s, \mathbb{R}^m)$, where $r, r' \in [s, T + K]$, and f satisfies the following conditions:

(H1) There exists a constant $C > 0$, such that for all $s \in [0, T]$, $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^{m\times d}$, $\xi_\cdot, \xi'_\cdot \in L^2_{\mathscr{F}}(s, T + K; \mathbb{R}^m)$, $\eta_\cdot, \eta'_\cdot \in L^2_{\mathscr{F}}(s, T + K; \mathbb{R}^{m\times d})$, $r, \bar{r} \in [s, T + K]$, we have

$$\begin{aligned}
&|f(s, y, z, \xi_r, \eta_{\bar{r}}) - f(s, y', z', \xi'_r, \eta'_{\bar{r}})| \\
&\qquad \leq C(|y - y'| + |z - z'| + E^{\mathscr{F}_s}[|\xi_r - \xi'_r| + |\eta_{\bar{r}} - \eta'_{\bar{r}}|]).
\end{aligned}$$

(H2) $E[\int_0^T |f(s, 0, 0, 0, 0)|^2\, ds] < \infty$.

REMARK 4.1. 1. Note that $f(s, \cdot, \cdot, \cdot, \cdot)$ is $\mathscr{F}_s$-measurable ensures the solution to the anticipated BSDE is $\mathscr{F}_s$-adapted.

2. We give examples of $\delta(s)$ and f. Both examples of $\delta(s)$ satisfy (i) and (ii). Example 1: Let $\delta(s) \equiv c$, where $c > 0$ is a constant. Example 2: Let $s + \delta(s)$ be a monotone nonnegative function whose converse function has a continuous differential function. We give examples of functions that satisfy (H1) and (H2): Let g satisfy (a) and (b) and let δ, ζ be two positive constants. For each $t \in [0, T]$ and $(\xi_\cdot, \eta_\cdot) \in L^2_{\mathscr{F}}(t, T + (\delta \vee \zeta); \mathbb{R}^m \times \mathbb{R}^{m\times d})$, define f_1, f_2 such that

$$f_1(t, \xi_{t+\delta}, \eta_{t+\zeta}) = g(t, E^{\mathscr{F}_t}[\xi_{t+\delta}], E^{\mathscr{F}_t}[\eta_{t+\zeta}]),$$
$$f_2(t, \xi_{t+\delta}, \eta_{t+\zeta}) = g(t, \mathcal{E}_{t,t+\delta}[\xi_{t+\delta}], \mathcal{E}_{t,t+\zeta}[\eta_{t+\zeta}]),$$

where $\mathcal{E}_{s,t}[\cdot] : L^2(\mathscr{F}_t) \longrightarrow L^2(\mathscr{F}_s)$, $0 \leq s \leq t \leq T + K$, is a $\mathscr{F}_t$-consistent nonlinear evaluation (see Peng [13]). Then f_1, f_2 satisfy (i) and (ii).

The following is the main result of this section: an existence and uniqueness theorem for adapted solutions for anticipated BSDEs.

THEOREM 4.2. *Suppose that f satisfies* (H1) *and* (H2), *and δ, ζ satisfy* (i) *and* (ii). *Then for any given terminal conditions $\xi_\bullet \in S^2_{\mathscr{F}}(T, T+K; \mathbb{R}^m)$ and $\eta_\bullet \in L^2_{\mathscr{F}}(T, T+K; \mathbb{R}^{m\times d})$, the anticipated BSDE* (10) *has a unique solution, that is, there exists a unique pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot, Z_\cdot) \in S^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m) \times L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^{m\times d})$ satisfying* (10).

PROOF. We fix $\beta = 12C^2(2L+1)+2$, where C is the Lipschitz constant of f given in (H1), and introduce a norm in the Banach space $L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m)$:

$$\|\nu(\cdot)\|_\beta = \left(E\left[\int_0^{T+K} |\nu_s|^2 e^{\beta s}\, ds\right]\right)^{1/2}.$$

Clearly, it is equivalent to the original norm of $L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m)$. But it is more convenient to use this norm to construct a contraction mapping that allows us to apply the Fixed Point Theorem. Set

$$\begin{cases} Y_t = \xi_T + \displaystyle\int_t^T f\big(s, y_s, z_s, y_{s+\delta(s)}, z_{s+\zeta(s)}\big)\, ds - \int_t^T Z_s\, dW_s, \\ \qquad\qquad t \in [0, T]; \\ Y_t = \xi_t, \qquad t \in [T, T+K]; \\ Z_t = \eta_t, \qquad t \in [T, T+K]. \end{cases}$$

Define a mapping $h : L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m \times \mathbb{R}^{m\times d}) \longrightarrow L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m \times \mathbb{R}^{m\times d})$ such that $h[(y_\bullet, z_\bullet)] = (Y_\bullet, Z_\bullet)$. Now we prove that h is a contraction mapping under the norm $\|\cdot\|_\beta$. For two arbitrary elements $(y_\bullet, z_\bullet)$ and $(y'_\bullet, z'_\bullet)$ in $L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m \times \mathbb{R}^{m\times d})$, set $(Y_\bullet, Z_\bullet) = h[(y_\bullet, z_\bullet)]$ and $(Y'_\bullet, Z'_\bullet) = h[(y'_\bullet, z'_\bullet)]$. Denote their differences by

$$(\hat{y}_\bullet, \hat{z}_\bullet) = \big((y-y')_\bullet, (z-z')_\bullet\big), \qquad (\hat{Y}_\bullet, \hat{Z}_\bullet) = \big((Y-Y')_\bullet, (Z-Z')_\bullet\big).$$

By basic estimate (7), we have

$$\begin{aligned} &E\left[\int_0^T \left(\frac{\beta}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2\right) e^{\beta s}\, ds\right] \\ &\quad \le \frac{2}{\beta} E\left[\int_0^T \big| f\big(s, y_s, z_s, y_{s+\delta(s)}, z_{s+\zeta(s)}\big) \right. \\ &\qquad\qquad \left. - f\big(s, y'_s, z'_s, y'_{s+\delta(s)}, z'_{s+\zeta(s)}\big)\big|^2 e^{\beta s}\, ds\right]. \end{aligned}$$

Since $\delta(s)$ and $\zeta(s)$ satisfy (ii) and f satisfies (H1), by the Fubini Theorem, we have

$$\begin{aligned} &E\left[\int_0^T \left(\frac{\beta}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2\right) e^{\beta s}\, ds\right] \\ &\quad \le \frac{2C^2}{\beta} E\left[\int_0^T \big(|\hat{y}_s| + |\hat{z}_s| + E^{\mathscr{F}_s}\big[|\hat{y}_{s+\delta(s)}| + |\hat{z}_{s+\zeta(s)}|\big]\big)^2 e^{\beta s}\, ds\right] \end{aligned}$$

$$\leq \frac{6C^2}{\beta} E\bigg[\int_0^T \big(|\hat{y}_s|^2 + |\hat{z}_s|^2 + 2\big|\hat{y}_{s+\delta(s)}\big|^2 + 2\big|\hat{z}_{s+\zeta(s)}\big|^2\big) e^{\beta s}\, ds\bigg]$$

$$\leq \frac{6C^2(2L+1)}{\beta} E\bigg[\int_0^{T+K} (|\hat{y}_s|^2 + |\hat{z}_s|^2) e^{\beta s}\, ds\bigg].$$

Because $\beta = 12C^2(2L+1)+2$, then

$$E\bigg[\int_0^{T+K} (|\hat{Y}_s|^2 + |\hat{Z}_s|^2) e^{\beta s}\, ds\bigg] \leq \frac{1}{2} E\bigg[\int_0^{T+K} (|\hat{y}_s|^2 + |\hat{z}_s|^2) e^{\beta s}\, ds\bigg],$$

or

$$\|(\hat{Y}_\bullet, \hat{Z}_\bullet)\|_\beta \leq \tfrac{1}{\sqrt{2}} \|(\hat{y}_\bullet, \hat{z}_\bullet)\|_\beta.$$

Consequently, h is a strict contraction mapping of $L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m \times \mathbb{R}^{m\times d})$. It follows by the Fixed Point Theorem that (10) has a unique solution $(Y_\bullet, Z_\bullet) \in L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m \times \mathbb{R}^{m\times d})$. Since f satisfies (H1) and (H2) and since δ, ζ satisfy (i) and (ii), we have $f(\cdot, Y_\bullet, Z_\bullet, Y_{\bullet+\delta(\bullet)}, Z_{\bullet+\zeta(\bullet)}) \in L^2_{\mathscr{F}}(0, T; \mathbb{R}^m)$. Thus, by Lemma 3.1, we obtain $Y_\bullet \in S^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m)$. $\square$

The following example shows that a simple case of the anticipated BSDE (10) has a solution.

EXAMPLE 4.3. Consider the following anticipated BSDE:

$$\begin{cases} Y_t = T W_T - \displaystyle\int_t^T \frac{1}{s+\delta} E^{\mathscr{F}_s}[Y_{s+\delta}]\, ds - \int_t^T Z_s\, dW_s, & t \in [0, T]; \\ Y_t = t W_t, & t \in [T, T+\delta], \end{cases}$$

where $\delta \geq 0$ is a given constant. Then $(t W_t, t)_{t\in[0,T+\delta]}$ is its solution.

The following proposition is an estimate of the solution of the anticipated BSDE (10).

PROPOSITION 4.4. *Assume that f satisfies* (H1) *and* (H2), *and also δ and ζ satisfy* (i) *and* (ii). *Then there exists a positive constant C_0 that only depends on C in* (H1), *L in* (ii), *and T such that for each $\xi_\bullet \in S^2_{\mathscr{F}}(T, T+K; \mathbb{R}^m)$ and each $\eta_\bullet \in L^2_{\mathscr{F}}(T, T+K; \mathbb{R}^{m\times d})$, the solution $(Y_\bullet, Z_\bullet)$ of the anticipated BSDE* (10) *satisfies*

$$\begin{aligned} &E^{\mathscr{F}_t}\bigg[\sup_{t\leq s\leq T} |Y_s|^2 + \int_t^T |Z_s|^2\, ds\bigg] \\ &\quad\leq C_0 E^{\mathscr{F}_t}\bigg[|\xi_T|^2 + \int_T^{T+K} (|\xi_s|^2 + |\eta_s|^2)\, ds + \bigg(\int_t^T |f(s,0,0,0,0)|\, ds\bigg)^2\bigg], \end{aligned} \tag{11}$$

for each $t \in [0, T]$.

PROOF. For $s \in [0, T]$, applying Itô's formula to $e^{\beta s}|Y_s|^2$, we obtain

$$\begin{aligned} e^{\beta s}|Y_s|^2 &+ \int_s^T e^{\beta r}(\beta|Y_r|^2 + |Z_r|^2)\,dr \\ &= e^{\beta T}|\xi_T|^2 - 2\int_s^T e^{\beta r}(Y_r, Z_r\,dW_r) \\ &\quad + 2\int_s^T e^{\beta r}\bigl(f\bigl(r, Y_r, Z_r, Y_{r+\delta(r)}, Z_{r+\zeta(r)}\bigr), Y_r\bigr)\,dr. \end{aligned}$$

Since

$$\begin{aligned} 2\bigl(f&\bigl(r, Y_r, Z_r, Y_{r+\delta(r)}, Z_{r+\zeta(r)}\bigr), Y_r\bigr) \\ &= 2\bigl(f\bigl(r, Y_r, Z_r, Y_{r+\delta(r)}, Z_{r+\zeta(r)}\bigr) - f\bigl(r, Y_r, Z_r, Y_{r+\delta(r)}, 0\bigr), Y_r\bigr) \\ &\quad + 2\bigl(f\bigl(r, Y_r, Z_r, Y_{r+\delta(r)}, 0\bigr) - f(r, Y_r, Z_r, 0, 0), Y_r\bigr) \\ &\quad + 2\bigl(f(r, Y_r, Z_r, 0, 0) - f(r, Y_r, 0, 0, 0), Y_r\bigr) \\ &\quad + 2\bigl(f(r, Y_r, 0, 0, 0) - f(r, 0, 0, 0, 0), Y_r\bigr) + 2(f(r, 0, 0, 0, 0), Y_r) \\ &\le 2CE^{\mathscr{F}_r}\bigl[\bigl|Z_{r+\zeta(r)}\bigr|\bigr]|Y_r| + 2CE^{\mathscr{F}_r}\bigl[\bigl|Y_{r+\delta(r)}\bigr|\bigr]|Y_r| + 2C|Y_r||Z_r| \\ &\quad + 2C|Y_r|^2 + 2(f(r, 0, 0, 0, 0), Y_r) \\ &\le (3LC^2 + 4LTC^2 + 3C^2 + 2C)|Y_r|^2 + \frac{1}{3L}E^{\mathscr{F}_r}\bigl[\bigl|Z_{r+\zeta(r)}\bigr|^2\bigr] \\ &\quad + \frac{1}{4LT}E^{\mathscr{F}_r}\bigl[\bigl|Y_{r+\delta(r)}\bigr|^2\bigr] + \frac{1}{3}|Z_r|^2 + 2(f(r, 0, 0, 0, 0), Y_r), \end{aligned}$$

we get, for $s \in [0, T]$,

$$\begin{aligned} e^{\beta s}|Y_s|^2 &+ \int_s^T e^{\beta r}\biggl[(\beta - 3LC^2 - 4LTC^2 - 3C^2 - 2C)|Y_r|^2 + \frac{2}{3}|Z_r|^2\biggr]dr \\ &\le e^{\beta T}|\xi_T|^2 + 2\int_s^T e^{\beta r}(f(r, 0, 0, 0, 0), Y_r)\,dr - 2\int_s^T e^{\beta r}(Y_r, Z_r\,dW_r) \qquad (12) \\ &\quad + \frac{1}{3L}\int_s^T e^{\beta r}E^{\mathscr{F}_r}\bigl[\bigl|Z_{r+\zeta(r)}\bigr|^2\bigr]dr + \frac{1}{4LT}\int_s^T e^{\beta r}E^{\mathscr{F}_r}\bigl[\bigl|Y_{r+\delta(r)}\bigr|^2\bigr]dr. \end{aligned}$$

Taking conditional expectations under $\mathscr{F}_s$ on both sides of (12), we have

$$\begin{aligned} e^{\beta s}|Y_s|^2 + E^{\mathscr{F}_s}\biggl[\int_s^T e^{\beta r}\biggl[&(\beta - 3LC^2 - 4LTC^2 - 3C^2 - 2C)|Y_r|^2 \\ &+ \frac{2}{3}|Z_r|^2\biggr]dr\biggr] \\ \le E^{\mathscr{F}_s}\biggl[e^{\beta T}|\xi_T|^2 &+ 2\int_s^T e^{\beta r}(f(r, 0, 0, 0, 0), Y_r)\,dr\biggr] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4LT}E^{\mathscr{F}_s}\biggl[\int_s^T e^{\beta r}E^{\mathscr{F}_r}\bigl[\bigl|Y_{r+\delta(r)}\bigr|^2\bigr]\,dr\biggr]\\
&+\frac{1}{3L}E^{\mathscr{F}_s}\biggl[\int_s^T e^{\beta r}E^{\mathscr{F}_r}\bigl[\bigl|Z_{r+\zeta(r)}\bigr|^2\bigr]\,dr\biggr]\\
&\le E^{\mathscr{F}_s}\biggl[e^{\beta T}|\xi_T|^2+2\int_s^T e^{\beta r}(f(r,0,0,0,0),Y_r)\,dr\\
&\qquad\qquad+\frac{1}{4T}\int_s^{T+K}e^{\beta r}|Y_r|^2\,dr\biggr]\\
&+\frac{1}{3}E^{\mathscr{F}_s}\biggl[\int_s^{T+K}e^{\beta r}|Z_r|^2\,dr\biggr].
\end{aligned}
$$

Set $\beta=3LC^2+4LTC^2+3C^2+2C+\frac{1}{4T}$, then

$$
\begin{aligned}
&E^{\mathscr{F}_s}\biggl[\int_s^T e^{\beta r}|Z_r|^2\,dr\biggr]\\
&\qquad\le E^{\mathscr{F}_s}\biggl[3e^{\beta T}|\xi_T|^2+6\int_s^T e^{\beta r}(f(r,0,0,0,0),Y_r)\,dr\biggr]\\
&\qquad\quad+E^{\mathscr{F}_s}\biggl[\int_T^{T+K}e^{\beta r}\Bigl(|\eta_r|^2+\frac{3}{4T}|\xi_r|^2\Bigr)\,dr\biggr].
\end{aligned}
\tag{13}
$$

Since for $t\le s\le T$,

$$
\begin{aligned}
\biggl|\int_s^T e^{\beta r}(Y_r,Z_r\,dW_r)\biggr|&=\biggl|\int_t^T e^{\beta r}(Y_r,Z_r\,dW_r)-\int_t^s e^{\beta r}(Y_r,Z_r\,dW_r)\biggr|\\
&\le\biggl|\int_t^T e^{\beta r}(Y_r,Z_r\,dW_r)\biggr|+\biggl|\int_t^s e^{\beta r}(Y_r,Z_r\,dW_r)\biggr|,
\end{aligned}
$$

by the Burkholder–Davis–Gundy inequality, we have

$$
\begin{aligned}
&E^{\mathscr{F}_t}\biggl[\sup_{t\le s\le T}\biggl|\int_s^T e^{\beta r}(Y_r,Z_r\,dW_r)\biggr|\biggr]\\
&\qquad\le 2E^{\mathscr{F}_t}\biggl[\sup_{t\le s\le T}\biggl|\int_t^s e^{\beta r}(Y_r,Z_r\,dW_r)\biggr|\biggr]\\
&\qquad\le 6E^{\mathscr{F}_t}\biggl[\biggl(\int_t^T e^{2\beta r}|Y_r|^2|Z_r|^2\,dr\biggr)^{1/2}\biggr]\\
&\qquad\le 6E^{\mathscr{F}_t}\biggl[\Bigl(\sup_{t\le r\le T}e^{1/2\beta r}|Y_r|\Bigr)\biggl(\int_t^T e^{\beta r}|Z_r|^2\,dr\biggr)^{1/2}\biggr]\\
&\qquad\le\frac{1}{4}E^{\mathscr{F}_t}\Bigl[\sup_{t\le r\le T}e^{\beta r}|Y_r|^2\Bigr]+36E^{\mathscr{F}_t}\biggl[\int_t^T e^{\beta r}|Z_r|^2\,dr\biggr].
\end{aligned}
\tag{14}
$$

From estimates (12) and (14) we have

$$
\begin{aligned}
&E^{\mathscr{F}_t}\Big[\sup_{t\le s\le T} e^{\beta s}|Y_s|^2\Big]\\
&\quad\le E^{\mathscr{F}_t}\bigg[e^{\beta T}|\xi_T|^2+2\int_t^T e^{\beta r}|f(r,0,0,0,0)||Y_r|\,dr\\
&\qquad\qquad\qquad\qquad +2\sup_{t\le s\le T}\bigg|\int_s^T e^{\beta r}(Y_r, Z_r\,dW_r)\bigg|\bigg]\\
&\qquad + E^{\mathscr{F}_t}\bigg[\frac{1}{3L}\int_t^T e^{\beta r}E^{\mathscr{F}_r}\big[\big|Z_{r+\zeta(r)}\big|^2\big]\,dr\\
&\qquad\qquad +\frac{1}{4LT}\int_t^T e^{\beta r}E^{\mathscr{F}_r}\big[\big|Y_{r+\delta(r)}\big|^2\big]\,dr\bigg]\\
&\quad\le E^{\mathscr{F}_t}[e^{\beta T}|\xi_T|^2]+\frac{1}{2}E^{\mathscr{F}_t}\Big[\sup_{t\le r\le T} e^{\beta r}|Y_r|^2\Big]\\
&\qquad +72E^{\mathscr{F}_t}\bigg[\int_t^T e^{\beta r}|Z_r|^2\,dr\bigg]\\
&\qquad + E^{\mathscr{F}_t}\bigg[\frac{1}{3L}\int_t^T e^{\beta r}\big|Z_{r+\zeta(r)}\big|^2\,dr+\frac{1}{4LT}\int_t^T e^{\beta r}\big|Y_{r+\delta(r)}\big|^2\,dr\bigg]\\
&\qquad +2E^{\mathscr{F}_t}\bigg[\int_t^T e^{\beta r}|f(r,0,0,0,0)||Y_r|\,dr\bigg]\\
&\quad\le E^{\mathscr{F}_t}[e^{\beta T}|\xi_T|^2]+\frac{1}{2}E^{\mathscr{F}_t}\Big[\sup_{t\le r\le T} e^{\beta r}|Y_r|^2\Big]+72E^{\mathscr{F}_t}\bigg[\int_t^T e^{\beta r}|Z_r|^2\,dr\bigg]\\
&\qquad + E^{\mathscr{F}_t}\bigg[\frac{1}{3}\int_t^{T+K} e^{\beta r}|Z_r|^2\,dr+\frac{1}{4T}\int_t^{T+K} e^{\beta r}|Y_r|^2\,dr\bigg]\\
&\qquad +2E^{\mathscr{F}_t}\bigg[\int_t^T e^{\beta r}|f(r,0,0,0,0)||Y_r|\,dr\bigg]\\
&\quad\le E^{\mathscr{F}_t}[e^{\beta T}|\xi_T|^2]+\frac{3}{4}E^{\mathscr{F}_t}\Big[\sup_{t\le r\le T} e^{\beta r}|Y_r|^2\Big]\\
&\qquad +\Big(72+\frac{1}{3}\Big)E^{\mathscr{F}_t}\bigg[\int_t^T e^{\beta r}|Z_r|^2\,dr\bigg]\\
&\qquad + E^{\mathscr{F}_t}\bigg[\int_T^{T+K} e^{\beta r}\Big(\frac{1}{3}|\eta_r|^2+\frac{1}{4T}|\xi_r|^2\Big)\,dr\bigg]\\
&\qquad +2E^{\mathscr{F}_t}\bigg[\int_t^T e^{\beta r}|f(r,0,0,0,0)||Y_r|\,dr\bigg].
\end{aligned}
$$

Denote by $C_0 > 0$ a constant that depends only on T, L and C, which we allow to change from line to line. From the estimate above and estimate (13),

$$
\begin{aligned}
&\frac{1}{4}E^{\mathscr{F}_t}\Big[\sup_{t\le s\le T} e^{\beta s}|Y_s|^2\Big]\\
&\quad\le C_0E^{\mathscr{F}_t}\Big[e^{\beta T}|\xi_T|^2+\int_T^{T+K} e^{\beta r}(|\eta_r|^2+|\xi_r|^2)\,dr\Big]\\
&\qquad + C_0E^{\mathscr{F}_t}\Big[\Big(\sup_{t\le r\le T} e^{1/2\beta r}|Y_r|\Big)\Big(\int_t^T e^{1/2\beta r}|f(r,0,0,0,0)|\,dr\Big)\Big]\\
&\quad\le C_0E^{\mathscr{F}_t}\Big[e^{\beta T}|\xi_T|^2+\int_T^{T+K} e^{\beta r}(|\eta_r|^2+|\xi_r|^2)\,dr\Big]\\
&\qquad + \frac{1}{8}E^{\mathscr{F}_t}\Big[\sup_{t\le r\le T} e^{\beta r}|Y_r|^2\Big]+2C_0^2E^{\mathscr{F}_t}\Big[\Big(\int_t^T e^{1/2\beta r}|f(r,0,0,0,0)|\,dr\Big)^2\Big]\\
&\quad\le C_0E^{\mathscr{F}_t}\Big[|\xi_T|^2+\int_T^{T+K} (|\eta_r|^2+|\xi_r|^2)\,dr\Big]\\
&\qquad + \frac{1}{8}E^{\mathscr{F}_t}\Big[\sup_{t\le r\le T} e^{\beta r}|Y_r|^2\Big]+2C_0^2E^{\mathscr{F}_t}\Big[\Big(\int_t^T |f(r,0,0,0,0)|\,dr\Big)^2\Big].
\end{aligned}
$$

Then

$$
\begin{aligned}
&E^{\mathscr{F}_t}\Big[\sup_{t\le s\le T}|Y_s|^2\Big]+E^{\mathscr{F}_t}\Big[\int_t^T |Z_s|^2\,ds\Big]\\
&\quad\le C_0E^{\mathscr{F}_t}\Big[|\xi_T|^2+\int_T^{T+K}(|\xi_s|^2+|\eta_s|^2)\,ds+\Big(\int_t^T |f(s,0,0,0,0)|\,ds\Big)^2\Big].
\end{aligned}
$$

□

The following proposition shows the importance of the effect of anticipated time on the solution to anticipated BSDEs.

PROPOSITION 4.5. *Let* $(Y_\cdot^{(1)}, Z_\cdot^{(1)})$ *and* $(Y_\cdot^{(2)}, Z_\cdot^{(2)})$ *be respectively solutions of the following two anticipated BSDEs*:

$$
\begin{cases}
-dY_t^{(j)} = f\big(t, Y_t^{(j)}, Z_t^{(j)}, Y_{t+\delta_j(t)}^{(j)}\big)\,dt - Z_t^{(j)}\,dW_t, & t\in[0,T];\\
Y_t^{(j)} = \xi_t, & t\in[T,T+K],
\end{cases}
$$

where $j = 1, 2$. *Assume* $\xi_\cdot \in S^2_{\mathscr{F}}(T, T+K; \mathbb{R}^m)$, δ_1 *and* δ_2 *satisfy* (i) *and* (ii), f *satisfies* (H2), *and there exists a constant* $\bar{C} > 0$, *such that for all* $s \in [0,T]$, $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^{m\times d}$, $\theta, \theta' \in L^2_{\mathscr{F}}(s, T+K; \mathbb{R}^m)$ *and* $r \in [s, T+K]$,

$$
|f(s,y,z,\theta_r)-f(s,y',z',\theta'_r)|\le \bar{C}(|y-y'|+|z-z'|+|E^{\mathscr{F}_s}[\theta_r-\theta'_r]|).
$$

If for any $t \in [0, T]$, $\delta_1(t) \le \delta_2(t)$, *then there exists a constant* $\tilde{M} > 0$ *only depending on* $\bar{C}$, L *and* T *such that*

$$\big|Y_t^{(1)} - Y_t^{(2)}\big|^2 \le \tilde{M} \int_t^T \big(\delta_2(s) - \delta_1(s)\big)\, ds \times E^{\mathscr{F}_t}\bigg[|\xi_T|^2 + \int_T^{T+K} |\xi_s|^2\, ds + \int_t^T |f(s,0,0,0)|^2\, ds\bigg].$$

PROOF. Setting $y_{\bullet} = Y_{\bullet}^{(1)} - Y_{\bullet}^{(2)}$, $z_{\bullet} = Z_{\bullet}^{(1)} - Z_{\bullet}^{(2)}$, then by estimate (6), we obtain, for all $t \in [0, T]$,

$$\begin{aligned}
&|y_t|^2 + E^{\mathscr{F}_t}\bigg[\int_t^T \Big(\frac{\beta}{2}|y_s|^2 + |z_s|^2\Big) e^{\beta(s-t)}\, ds\bigg] \\
&\quad \le \frac{2}{\beta} E^{\mathscr{F}_t}\bigg[\int_t^T \big|f\big(s, Y_s^{(1)}, Z_s^{(1)}, Y_{s+\delta_1(s)}^{(1)}\big) - f\big(s, Y_s^{(2)}, Z_s^{(2)}, Y_{s+\delta_2(s)}^{(2)}\big)\big|^2 \times e^{\beta(s-t)}\, ds\bigg] \\
&\quad \le \frac{2\bar{C}^2}{\beta} E^{\mathscr{F}_t}\bigg[\int_t^T \big(|y_s| + |z_s| + \big|E^{\mathscr{F}_s}[Y_{s+\delta_1(s)}^{(1)} - Y_{s+\delta_2(s)}^{(2)}]\big|\big)^2 e^{\beta(s-t)}\, ds\bigg] \\
&\quad \le \frac{6\bar{C}^2}{\beta} E^{\mathscr{F}_t}\bigg[\int_t^T \big(|y_s|^2 + |z_s|^2 + \big|E^{\mathscr{F}_s}[y_{s+\delta_1(s)}] + E^{\mathscr{F}_s}[Y_{s+\delta_1(s)}^{(2)} - Y_{s+\delta_2(s)}^{(2)}]\big|^2\big) e^{\beta(s-t)}\, ds\bigg] \\
&\quad \le \frac{6\bar{C}^2 + 12\bar{C}^2 L}{\beta} E^{\mathscr{F}_t}\bigg[\int_t^T |y_s|^2 e^{\beta(s-t)}\, ds\bigg] + \frac{6\bar{C}^2}{\beta} E^{\mathscr{F}_t}\bigg[\int_t^T |z_s|^2 e^{\beta(s-t)}\, ds\bigg] \\
&\qquad + \frac{12\bar{C}^2}{\beta} E^{\mathscr{F}_t}\bigg[\int_t^T \big|E^{\mathscr{F}_s}[Y_{s+\delta_1(s)}^{(2)} - Y_{s+\delta_2(s)}^{(2)}]\big|^2 e^{\beta(s-t)}\, ds\bigg].
\end{aligned}$$

But

$$E^{\mathscr{F}_s}[Y_{s+\delta_1(s)}^{(2)} - Y_{s+\delta_2(s)}^{(2)}] = E^{\mathscr{F}_s}\bigg[\int_{s+\delta_1(s)}^{s+\delta_2(s)} f\big(r, Y_r^{(2)}, Z_r^{(2)}, Y_{r+\delta_2(r)}^{(2)}\big)\, dr\bigg],$$

and set $\beta = 6\bar{C}^2$, hence,

$$\begin{aligned}
|y_t|^2 &\le (1 + 2L) E^{\mathscr{F}_t}\bigg[\int_t^T |y_s|^2 e^{\beta(s-t)}\, ds\bigg] \\
&\quad + 2E^{\mathscr{F}_t}\bigg[\int_t^T E^{\mathscr{F}_s}\bigg[\bigg|\int_{s+\delta_1(s)}^{s+\delta_2(s)} f\big(r, Y_r^{(2)}, Z_r^{(2)}, Y_{r+\delta_2(r)}^{(2)}\big)\, dr\bigg|^2\bigg] e^{\beta(s-t)}\, ds\bigg]
\end{aligned}$$

$$
\begin{aligned}
&\le (1+2L)e^{\beta(T-t)} E^{\mathscr{F}_t}\left[\int_t^T |y_s|^2\,ds\right] \\
&\quad + 2E^{\mathscr{F}_t}\left[\int_t^T \big(\delta_2(s)-\delta_1(s)\big)\int_{s+\delta_1(s)}^{s+\delta_2(s)} \big|f\big(r, Y_r^{(2)}, Z_r^{(2)}, Y_{r+\delta_2(r)}^{(2)}\big)\big|^2\,dr \right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\times e^{\beta(s-t)}\,ds\right] \\
&\le (1+2L)e^{\beta(T-t)} E^{\mathscr{F}_t}\left[\int_t^T |y_s|^2\,ds\right] \\
&\quad + 8E^{\mathscr{F}_t}\left[\int_t^T \big(\delta_2(s)-\delta_1(s)\big)e^{\beta(s-t)}\,ds\right.\\
&\qquad\qquad \times\left\{\int_t^T \big(\bar{C}^2|Y_r^{(2)}|^2+\bar{C}^2|Z_r^{(2)}|^2+\bar{C}^2|Y_{r+\delta_2(r)}^{(2)}|^2\right.\\
&\qquad\qquad\qquad\qquad\qquad \left.\left. +|f(r,0,0,0)|^2\big)\,dr\right\}\right] \\
&\le (1+2L)e^{\beta(T-t)} E^{\mathscr{F}_t}\left[\int_t^T |y_s|^2\,ds\right] \\
&\quad + 8\int_t^T \big(\delta_2(s)-\delta_1(s)\big)e^{\beta(s-t)}\,ds \\
&\quad \times E^{\mathscr{F}_t}\left[\int_t^T \big((1+L)\bar{C}^2|Y_r^{(2)}|^2+\bar{C}^2|Z_r^{(2)}|^2+|f(r,0,0,0)|^2\big)\,dr\right.\\
&\qquad\qquad\qquad\qquad \left. + \int_T^{T+K} L\bar{C}^2|\xi_r|^2\,dr\right].
\end{aligned}
$$

From estimate (11), we can find a constant $\bar{M}>0$ depending only on $\bar{C}$, L and T such that

$$
\begin{aligned}
|y_t|^2 &\le \bar{M} E^{\mathscr{F}_t}\left[\int_t^T |y_s|^2\,ds\right] \\
&\quad + \bar{M}\int_t^T \big(\delta_2(s)-\delta_1(s)\big)\,ds \\
&\qquad \times E^{\mathscr{F}_t}\left[|\xi_T|^2+\int_T^{T+K}|\xi_s|^2\,ds+\int_t^T |f(s,0,0,0)|^2\,ds\right].
\end{aligned}
$$

Thus, by Gronwall's inequality,

$$
\begin{aligned}
|y_t|^2 &\le \bar{M}\int_t^T \big(\delta_2(s)-\delta_1(s)\big)\,ds \\
&\qquad \times E^{\mathscr{F}_t}\left[|\xi_T|^2+\int_T^{T+K}|\xi_s|^2\,ds+\int_t^T |f(s,0,0,0)|^2\,ds\right]e^{\bar{M}t}.
\end{aligned}
$$

Fix $\tilde{M} = \bar{M}e^{\bar{M}T}$, therefore,

$$|y_t|^2 \le \tilde{M} \int_t^T \big(\delta_2(s) - \delta_1(s)\big)\, ds \\ \times E^{\mathscr{F}_t}\bigg[|\xi_T|^2 + \int_T^{T+K} |\xi_s|^2\, ds + \int_t^T |f(s,0,0,0)|^2\, ds\bigg]. \qquad \square$$

5. Comparison theorem for 1-dimensional anticipated BSDEs. Lemma 3.3 is a typical version of a comparison theorem. It is a fundamentally important result in BSDE theory. Some further developments in this direction are Cao and Yan [3], Lin [10], Liu and Ren [11], Zhang [16] and Situ [15], without mentioning many other widely circulated papers listed in [13]. Recently Hu and Peng [8] gave a comparison theorem for multidimensional BSDEs. Comparison theorems for BSDEs have received a lot of attention because of their importance. For example, the punishment method in reflected BSDEs is based on a comparison theorem (see [4, 6, 9] and [14]). Moreover, research on properties of g-expectations (see Peng [13]) and the proof of a monotonic limit theorem for BSDEs (see Peng [12]) both depend on comparison theorems.

It is well known that 1-dimensional BSDEs have comparison theorems (see Lemmas 3.3 and 3.4) when their generators satisfy the conditions of existence and uniqueness theorems for BSDEs. It is very important to notice that the conditions on f needed for the comparison theorem for anticipated BSDEs are stronger than those needed for the existence and uniqueness theorem. Using the comparison theorem for anticipated BSDEs, we will solve a stochastic control problem in Section 6.

Let $(Y_\cdot^{(1)}, Z_\cdot^{(1)})$, $(Y_\cdot^{(2)}, Z_\cdot^{(2)})$ be respectively solutions of the following two 1-dimensional anticipated BSDEs:

$$\begin{cases} -dY_t^{(j)} = f_j\big(t, Y_t^{(j)}, Z_t^{(j)}, Y_{t+\delta(t)}^{(j)}\big)\, dt - Z_t^{(j)}\, dW_t, & 0 \le t \le T; \\ Y_t^{(j)} = \xi_t^{(j)}, & T \le t \le T+K, \end{cases}$$

where $j = 1, 2$.

THEOREM 5.1. *Assume that f_1, f_2 satisfies* (H1) *and* (H2), $\xi_\cdot^{(1)}, \xi_\cdot^{(2)} \in S_{\mathscr{F}}^2(T, T+K)$, *$\delta$ satisfies* (i), (ii), *and for all $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f_2(t, y, z, \cdot)$ is increasing, that is, $f_2(t,y,z,\theta_r) \ge f_2(t,y,z,\theta_r')$, if $\theta_r \ge \theta_r'$, $\theta, \theta' \in L_{\mathscr{F}}^2(t, T+K)$, $r \in [t, T+K]$. If $\xi_s^{(1)} \ge \xi_s^{(2)}$, $s \in [T, T+K]$ and $f_1(t,y,z,\theta_r) \ge f_2(t,y,z,\theta_r)$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\theta \in L_{\mathscr{F}}^2(t, T+K)$, $r \in [t, T+K]$, then*

$$Y_t^{(1)} \ge Y_t^{(2)}, \qquad \text{a.e., a.s.}$$

PROOF. Set

$$\begin{cases} Y_t^{(3)} = \xi_T^{(2)} + \int_t^T f_2(s, Y_s^{(3)}, Z_s^{(3)}, Y_{s+\delta(s)}^{(1)})\, ds - \int_t^T Z_s^{(3)}\, dW_s, \\ \qquad\qquad\qquad t \in [0, T]; \\ Y_t^{(3)} = \xi_t^{(2)}, \qquad t \in [T, T+K]. \end{cases}$$

By Lemma 3.2, we know there exists a unique pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot^{(3)}, Z_\cdot^{(3)}) \in S_{\mathscr{F}}^2(0,T) \times L_{\mathscr{F}}^2(0,T;\mathbb{R}^d)$ that satisfies the above BSDE. Since $f_1(s, y, z, Y_{s+\delta(s)}^{(1)}) \geq f_2(s, y, z, Y_{s+\delta(s)}^{(1)})$, $s \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, by Lemma 3.4, we obtain

$$Y_t^{(1)} \geq Y_t^{(3)}, \qquad \text{a.e., a.s.}$$

Set

$$\begin{cases} Y_t^{(4)} = \xi_T^{(2)} + \int_t^T f_2(s, Y_s^{(4)}, Z_s^{(4)}, Y_{s+\delta(s)}^{(3)})\, ds - \int_t^T Z_s^{(4)}\, dW_s, \\ \qquad\qquad\qquad t \in [0, T]; \\ Y_t^{(4)} = \xi_t^{(2)}, \qquad t \in [T, T+K]. \end{cases}$$

Since for all $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f_2(t, y, z, \cdot)$ is increasing and $Y_t^{(1)} \geq Y_t^{(3)}$, a.e., a.s., by Lemma 3.4, we know

$$Y_t^{(3)} \geq Y_t^{(4)}, \qquad \text{a.e., a.s.}$$

For $n = 5, 6, \ldots,$ we consider the following classical BSDE:

$$\begin{cases} Y_t^{(n)} = \xi_T^{(2)} + \int_t^T f_2(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n-1)})\, ds - \int_t^T Z_s^{(n)}\, dW_s, \\ \qquad\qquad\qquad t \in [0, T]; \\ Y_t^{(n)} = \xi_t^{(2)}, \qquad t \in [T, T+K]. \end{cases}$$

Similarly, we have $Y_t^{(4)} \geq Y_t^{(5)} \geq \cdots \geq Y_t^{(n)} \geq \cdots$, a.e., a.s. We use $\|\nu(\cdot)\|_\beta$ in the proof of Theorem 4.2 as the norm in the Banach space $L_{\mathscr{F}}^2(0, T+K; \mathbb{R}) \times L_{\mathscr{F}}^2(0,T;\mathbb{R}^d)$. Set $\hat{Y}_s^{(n)} = Y_s^{(n)} - Y_s^{(n-1)}$, $\hat{Z}_s^{(n)} = Z_s^{(n)} - Z_s^{(n-1)}$, $n \geq 4$. Then, by (7), we have

$$\begin{aligned} &E\left[\int_0^T \left(\frac{\beta}{2}|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2\right) e^{\beta s}\, ds\right] \\ &\quad \leq \frac{2}{\beta} E\left[\int_0^T |f_2(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n-1)}) \right. \\ &\qquad\qquad \left. - f_2(s, Y_s^{(n-1)}, Z_s^{(n-1)}, Y_{s+\delta(s)}^{(n-2)})|^2 e^{\beta s}\, ds\right] \\ &\quad \leq \frac{6C^2}{\beta} E\left[\int_0^T (|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2) e^{\beta s}\, ds\right] + \frac{6C^2 L}{\beta} E\left[\int_0^T |\hat{Y}_s^{(n-1)}|^2 e^{\beta s}\, ds\right]. \end{aligned}$$

Set $\beta = 18C^2L + 18C^2 + 3$. Then

$$\frac{2}{3}E\bigg[\int_0^T (|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2)e^{\beta s}\,ds\bigg] \leq \frac{1}{3}E\bigg[\int_0^T |\hat{Y}_s^{(n-1)}|^2 e^{\beta s}\,ds\bigg] \leq \frac{1}{3}E\bigg[\int_0^T (|\hat{Y}_s^{(n-1)}|^2 + |\hat{Z}_s^{(n-1)}|^2)e^{\beta s}\,ds\bigg].$$

Hence,

$$E\bigg[\int_0^T (|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2)e^{\beta s}\,ds\bigg] \leq \left(\frac{1}{2}\right)^{n-4} E\bigg[\int_0^T (|\hat{Y}_s^{(4)}|^2 + |\hat{Z}_s^{(4)}|^2)e^{\beta s}\,ds\bigg].$$

It follows that $(Y_\bullet^{(n)})_{n\geq 4}$ and $(Z_\bullet^{(n)})_{n\geq 4}$ are respectively Cauchy sequences in $L^2_{\mathscr{F}}(0, T+K)$ and in $L^2_{\mathscr{F}}(0, T; \mathbb{R}^d)$. Denote their limits by $Y_\bullet$ and $Z_\bullet$, respectively. Since $L^2_{\mathscr{F}}(0, T+K)$ and $L^2_{\mathscr{F}}(0, T; \mathbb{R}^d)$ are both Banach spaces, we obtain $(Y_\bullet, Z_\bullet) \in L^2_{\mathscr{F}}(0, T+K) \times L^2_{\mathscr{F}}(0, T; \mathbb{R}^d)$. Note for all $t \in [0, T]$,

$$E\bigg[\int_t^T \big|f_2(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n-1)}) - f_2(s, Y_s, Z_s, Y_{s+\delta(s)})\big|^2 e^{\beta s}\,ds\bigg] \leq 3C^2 E\bigg[\int_t^T (|Y_s^{(n)} - Y_s|^2 + |Z_s^{(n)} - Z_s|^2 + L|Y_s^{(n-1)} - Y_s|^2)e^{\beta s}\,ds\bigg] \to 0,$$

when $n \to \infty$. Therefore, $(Y_\bullet, Z_\bullet)$ satisfies the following anticipated BSDE:

$$\begin{cases} Y_t = \xi_T^{(2)} + \int_t^T f_2(s, Y_s, Z_s, Y_{s+\delta(s)})\,ds - \int_t^T Z_s\,dW_s, & 0 \leq t \leq T; \\ Y_t = \xi_t^{(2)}, & T \leq t \leq T+K. \end{cases}$$

By Theorem 4.2, we know

$$Y_t = Y_t^{(2)}, \qquad \text{a.e., a.s.}$$

Since $Y_t^{(1)} \geq Y_t^{(3)} \geq Y_t^{(4)} \geq Y_t$, it holds immediately

$$Y_t^{(1)} \geq Y_t^{(2)}, \qquad \text{a.e., a.s.} \qquad \square$$

If f_2 is nonincreasing in the anticipated term of $Y_\bullet$, Theorem 5.1 does not hold. The following example shows this.

EXAMPLE 5.2. Given $T > \delta > 0$, consider the following two anticipated BSDEs:

$$\begin{cases} Y_t = c + \int_t^T aE^{\mathscr{F}_s}[Y_{s+\delta}]\,ds - \int_t^T Z_s\,dW_s, & t \in [0, T]; \\ Y_t = c, & t \in [T, T+\delta], \end{cases} \tag{15}$$

and

$$(16)\quad \begin{cases} Y'_t = \int_t^T aE^{\mathscr{F}_s}[I_{[Y'_{s+\delta}<0]}Y'_{s+\delta}]\,ds - \int_t^T Z'_s\,dW_s, & t\in[0,T];\\ Y'_t = 0, & t\in[T,T+\delta], \end{cases}$$

where $a=-\frac{2}{\delta}$, $c<0$ are given constants. Obviously the solution to equation (16) is $(Y'_\cdot, Z'_\cdot)\equiv(0,0)$. When $t\in[T-\delta,T]$, equation (15) becomes

$$Y_t = c + \int_t^T ac\,ds - \int_t^T Z_s\,dW_s.$$

It is easy to see that $Y_t = c + ac(T-t),\ Z_t\equiv 0$ is the solution of equation (15) when $t\in[T-\delta,T]$. But $Y_t>0$ when $t\in[T-\delta,T-\delta/2)$.

If f_2 contains the anticipated term of $Z_\cdot$, Theorem 5.1 does not hold. This is shown in the following example.

EXAMPLE 5.3. Given $T>\delta>0$, consider the two anticipated BSDEs

$$(17)\quad \begin{cases} Y_t = W_T^2 - T - \int_t^T \sqrt{\frac{\pi}{2\delta}}E^{\mathscr{F}_s}[|Z_{s+\delta}-Z_s|]\,ds - \int_t^T Z_s\,dW_s,\\ \qquad\qquad t\in[0,T];\\ Y_t = W_t^2-(T-t), \qquad t\in[T,T+\delta];\\ Z_t = 2W_t, \qquad t\in[T,T+\delta], \end{cases}$$

and

$$(18)\quad \begin{cases} Y'_t = 4W_T^2 - \int_t^T \sqrt{\frac{\pi}{2\delta}}E^{\mathscr{F}_s}[|Z'_{s+\delta}-Z'_s|]\,ds - \int_t^T Z'_s\,dW_s,\\ \qquad\qquad t\in[0,T];\\ Y'_t = 4W_t^2-4(T-t), \qquad t\in[T,T+\delta];\\ Z'_t = 8W_t, \qquad t\in[T,T+\delta]. \end{cases}$$

We can check that the solution of (17) is $(Y_t,Z_t) = (W_t^2 - T-(T-t), 2W_t)$ and that the solution of (18) is $(Y'_t,Z'_t)=(4W_t^2-4(T-t),8W_t)$. We have $Y_T<Y'_T$ but $Y_0>Y'_0$.

THEOREM 5.4. *Under the assumptions of Theorem 5.1, if* $\xi_s^{(1)}\ge\xi_s^{(2)}$, $s\in[T,T+K]$ *and* $f_1(t,Y_t^{(1)},Z_t^{(1)},Y_{t+\delta(t)}^{(1)})\ge f_2(t,Y_t^{(1)},Z_t^{(1)},Y_{t+\delta(t)}^{(1)})$, $t\in[0,T]$, *then*

$$Y_t^{(1)}\ge Y_t^{(2)},\qquad a.e.,\ a.s.$$

We also have a strict comparison. Given the assumptions of Theorem 5.1, suppose $[T,T+K]\subset\{t+\delta(t), t\in[0,T]\}$ *and* f_2 *is strictly increasing in* θ. *Then*

$$Y_0^{(1)}=Y_0^{(2)} \iff \begin{cases} f_1(t,Y_t^{(1)},Z_t^{(1)},Y_{t+\delta(t)}^{(1)}) = f_2(t,Y_t^{(1)},Z_t^{(1)},Y_{t+\delta(t)}^{(1)}),\\ \qquad\qquad t\in[0,T],\\ \xi_s^{(1)}=\xi_s^{(2)}, \qquad s\in[T,T+K]. \end{cases}$$

PROOF. Set

$$\begin{cases} Y_t^{(3)} = \xi_T^{(2)} + \int_t^T f_2\big(s, Y_s^{(3)}, Z_s^{(3)}, Y_{s+\delta(s)}^{(1)}\big)\,ds - \int_t^T Z_s^{(3)}\,dW_s, \\ \qquad\qquad t \in [0,T]; \\ Y_t^{(3)} = \xi_t^{(2)}, \qquad t \in [T, T+K]. \end{cases}$$

Set $\tilde{f}_t = f_1(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}) - f_2(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)})$ and $y_\bullet = Y_\bullet^{(1)} - Y_\bullet^{(3)}$, $z_\bullet = Z_\bullet^{(1)} - Z_\bullet^{(3)}$, $\tilde{\xi}_\bullet = \xi_\bullet^{(1)} - \xi_\bullet^{(2)}$. Then the pair $(y_\bullet, z_\bullet)$ can be regarded as the solution to the linear BSDE

$$\begin{cases} y_t = \tilde{\xi}_T + \int_t^T (a_s y_s + b_s z_s + \tilde{f}_s)\,ds - \int_t^T z_s\,dW_s, \\ \qquad\qquad t \in [0,T]; \\ y_t = \tilde{\xi}_t, \qquad t \in [T, T+K], \end{cases}$$

where

$$a_s = \begin{cases} \dfrac{f_2(s, Y_s^{(1)}, Z_s^{(1)}, Y_{s+\delta(s)}^{(1)}) - f_2(s, Y_s^{(3)}, Z_s^{(1)}, Y_{s+\delta(s)}^{(1)})}{Y_s^{(1)} - Y_s^{(3)}}, & \text{if } Y_s^{(1)} \neq Y_s^{(3)}; \\ 0, & \text{if } Y_s^{(1)} = Y_s^{(3)}, \end{cases}$$

$$b_s = \begin{cases} \dfrac{f_2(s, Y_s^{(3)}, Z_s^{(1)}, Y_{s+\delta(s)}^{(1)}) - f_2(s, Y_s^{(3)}, Z_s^{(3)}, Y_{s+\delta(s)}^{(1)})}{Z_s^{(1)} - Z_s^{(3)}}, & \text{if } Z_s^{(1)} \neq Z_s^{(3)}; \\ 0, & \text{if } Z_s^{(1)} = Z_s^{(3)}. \end{cases}$$

Since f_2 satisfies (H1), $|a_s| \le C$ and $|b_s| \le C$. Set

$$X_t := \exp\biggl[\int_0^t b_s\,dW_s - \frac{1}{2}\int_0^t |b_s|^2\,ds + \int_0^t a_s\,ds\biggr] \ge 0.$$

We apply Itô's formula to $X_s y_s$ on $[t, T]$ and take conditional expectations on both sides:

$$y_t = E^{\mathscr{F}_t}\biggl[\tilde{\xi}_T X_T + \int_t^T \tilde{f}_s X_s\,ds\biggr].$$

Since $\tilde{\xi}_T \ge 0$, $\tilde{f}_t \ge 0$, a.e., a.s., we get $Y_t^{(1)} \ge Y_t^{(3)}$, a.e., a.s.

Then similarly to the proof of Theorem 5.1, we obtain

$$Y_t^{(1)} \ge Y_t^{(2)}, \qquad \text{a.e., a.s.}$$

Now we only need to prove the strict comparison theorem.

($\Longrightarrow$) Suppose $Y_0^{(1)} = Y_0^{(2)}$, by Lemma 3.3, we get

$$f_1\big(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}\big) = f_2\big(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(2)}\big), \qquad t \in [0,T].$$

Since $Y_0^{(1)} \geq Y_0^{(3)} \geq Y_0^{(2)}$, we know $Y_0^{(1)} = Y_0^{(3)}$. Also by Lemma 3.3, we get

$$f_1\big(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}\big) = f_2\big(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}\big), \qquad t \in [0, T].$$

Therefore,

$$f_2\big(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}\big) = f_2\big(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(2)}\big), \qquad t \in [0, T].$$

Note that for all $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f_2(t, y, z, \cdot)$ is strictly increasing, hence, $Y_{t+\delta(t)}^{(1)} = Y_{t+\delta(t)}^{(2)}$, $t \in [0, T]$. In particular, $\xi_t^{(1)} = \xi_t^{(2)}$, $t \in [T, T+K]$.

($\Longleftarrow$) Suppose $f_1(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}) = f_2(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)})$, $t \in [0, T]$ and $\xi_s^{(1)} = \xi_s^{(2)}$, $s \in [T, T+K]$. Then

$$y_t = Y_t^{(1)} - Y_t^{(3)} = E^{\mathscr{F}_t}\left[\tilde{\xi}_T X_T + \int_t^T \tilde{f}_s X_s\, ds\right] \equiv 0.$$

Therefore,

$$\begin{cases} Y_t^{(1)} = \xi_T^{(2)} + \displaystyle\int_t^T f_2\big(s, Y_s^{(1)}, Z_s^{(3)}, Y_{s+\delta(s)}^{(1)}\big)\, ds - \int_t^T Z_s^{(3)}\, dW_s, \\ \qquad\qquad t \in [0, T]; \\ Y_t^{(1)} = \xi_t^{(2)}, \qquad t \in [T, T+K]. \end{cases}$$

By Theorem 4.2, $Y_t^{(1)} = Y_t^{(2)}$, a.e., a.s., in particular, $Y_0^{(1)} = Y_0^{(2)}$. $\square$

COROLLARY 5.5. *Let $(Y_\cdot^{(1)}, Z_\cdot^{(1)})$ and $(Y_\cdot^{(2)}, Z_\cdot^{(2)})$ be respectively the solutions for the following two* 1*-dimensional anticipated BSDEs*:

$$\begin{cases} -dY_t^{(j)} = f\big(t, Y_t^{(j)}, Z_t^{(j)}, Y_{t+\delta_j(t)}^{(j)}\big)\, dt - Z_t^{(j)}\, dW_t, & 0 \leq t \leq T; \\ Y_t^{(j)} = \xi_t, & T \leq t \leq T+K, \end{cases}$$

where $j = 1, 2$. Suppose $\xi_\cdot$ is in $S_{\mathscr{F}}^2(T, T+K)$, f satisfies (H1) *and* (H2), *for all $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f(t, y, z, \cdot)$ is increasing, and δ_1, δ_2 satisfy* (i) *and* (ii). *If $Y_{t+\delta_1(t)}^{(1)} \geq Y_{t+\delta_2(t)}^{(1)}$, a.e., a.s., then*

$$Y_t^{(1)} \geq Y_t^{(2)}, \qquad a.e., \ a.s.$$

PROOF. Set

$$\begin{cases} Y_t^{(3)} = \xi_T + \displaystyle\int_t^T f\big(s, Y_s^{(3)}, Z_s^{(3)}, Y_{s+\delta_2(s)}^{(1)}\big)\, ds - \int_t^T Z_s^{(3)}\, dW_s, \\ \qquad\qquad t \in [0, T]; \\ Y_t^{(3)} = \xi_t, \qquad t \in [T, T+K]. \end{cases}$$

From Lemma 3.2, there exists a unique pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot^{(3)}, Z_\cdot^{(3)}) \in S_{\mathscr{F}}^2(0, T) \times L_{\mathscr{F}}^2(0, T; \mathbb{R}^{1\times d})$ that satisfies the above BSDE. Since $f(s, y, z, Y_{s+\delta_1(s)}^{(1)}) \geq f(s, y, z, Y_{s+\delta_2(s)}^{(1)})$, by Lemma 3.4, we know

$$Y_t^{(1)} \geq Y_t^{(3)}, \qquad \text{a.e., a.s.}$$

The remaining proof is similar to Theorem 5.1, we omit it. $\square$

6. Stochastic control problems. El Karoui, Peng and Quenez [7] applied the duality between SDEs and BSDEs to stochastic control problems. Now we consider if it is feasible to use the duality between SDDEs and anticipated BSDEs to solve these problems. Let $\theta > 0$ be a given constant. Now we consider the following stochastic control problem: the laws of the controlled process belong to a family of equivalent measures whose densities are

$$\begin{cases} dX_s^u = (\alpha(s,u_s)X_s^u + b(s-\theta,u_{s-\theta})X_{s-\theta}^u)\,ds + X_s^u\sigma^T(s,u_s)\,dW_s, \\ \qquad\qquad s\in[t,T+\theta]; \\ X_t^u = 1, \\ X_s^u = 0, \qquad s\in[t-\theta,t), \end{cases}$$

where the coefficients $\alpha(s,u):\mathbb{R}\times\mathbb{R}^k\longrightarrow\mathbb{R}, b(s,u):\mathbb{R}\times\mathbb{R}^k\longrightarrow\mathbb{R}^+$ and $\sigma(s,u):\mathbb{R}\times\mathbb{R}^k\longrightarrow\mathbb{R}^{d\times1}$ are adapted processes uniformly continuous with respect to (s,u). A feasible control $(u_s, s\in[-\theta,T+\theta])$ is a continuous adapted process valued in a compact subset U in $\mathbb{R}^k$. The set of feasible controls is denoted by $\mathcal{U}$. The problem is to maximize over all feasible control processes u the objective function

$$J(u) = E\biggl[X_T^uQ(T) + \int_T^{T+\theta} X_{s-\theta}^uQ(s)b(s-\theta,u_{s-\theta})\,ds + \int_0^T X_s^u l(s,u_s)\,ds\biggr],$$

where $Q(\cdot)\in S^2_{\mathscr{F}}(T,T+\theta)$ is the terminal condition, $(l(\omega,s,u_s), s\in[0,T])$ is the running cost associated with the control process u and $l(s,u)$ is an adapted process uniformly continuous with respect to (s,u). Assume $\alpha(s,u), b(s,u)$, $|\sigma(s,u)|$ and $l(s,u)$ are uniformly bounded by μ. Notice that, by Theorem 2.1, $J(u)=Y_0^u$, where $(Y_\cdot^u, Z_\cdot^u)$ is the solution to the following linear anticipated BSDE:

$$\begin{cases} -dY_t^u = f^u(t,Y_t^u,Z_t^u,Y_{t+\theta}^u)\,dt - Z_t^u\,dW_t, & t\in[0,T]; \\ Y_t^u = Q(t), & t\in[T,T+\theta], \end{cases}$$

where $f^u(t,y,z,\eta_r) = \alpha(t,u_t)y + z\sigma(t,u_t) + b(t,u_t)E^{\mathscr{F}_t}[\eta_r] + l(t,u_t), t\in[0,T], y\in\mathbb{R}, z\in\mathbb{R}^d, \eta_\cdot\in L^2_{\mathscr{F}}(t,T+\theta), r\in[t,T+\theta]$ and

$$Y_t^u = E^{\mathscr{F}_t}\biggl[X_T^uQ(T) + \int_T^{T+\theta} X_{s-\theta}^uQ(s)b(s-\theta,u_{s-\theta})\,ds + \int_t^T X_s^u l(s,u_s)\,ds\biggr].$$

THEOREM 6.1. *Set* $f(t,y,z,\eta_r) = \operatorname{esssup}\{f^u(t,y,z,\eta_r), u\in\mathcal{U}\}, t\in[0,T]$, $y\in\mathbb{R}, z\in\mathbb{R}^d, \eta_\cdot\in L^2_{\mathscr{F}}(t,T+\theta), r\in[t,T+\theta]$. *Then anticipated BSDE*

$$(19)\qquad \begin{cases} -dY_t = f(t,Y_t,Z_t,Y_{t+\theta})\,dt - Z_t\,dW_t, & t\in[0,T]; \\ Y_t = Q(t), & t\in[T,T+\theta], \end{cases}$$

has a unique solution $(Y_\bullet, Z_\bullet)$. *Moreover*, $Y_\bullet$ *is the value function* $Y^*_\bullet$ *of the control problem*, *that is*, *for each* $t \in [0, T]$,

$$Y_t = Y_t^* = \operatorname{esssup}\{Y_t^u, u \in \mathcal{U}\}.$$

PROOF. On one hand, since $\alpha, b, |\sigma|$ and l are uniformly bounded by μ, for all $t \in [0, T]$, $s \in [T, T+\theta]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $\eta_\bullet, \eta'_\bullet \in L^2_{\mathscr{F}}(t, T+\theta)$, and $r \in [t, T+\theta]$,

$$\begin{aligned} f(t, y, z, \eta_r) - f(t, y', z', \eta'_r) &\le \operatorname{esssup}\{\alpha(t, u_t)(y - y') + (z - z')\sigma(t, u_t) \\ &\qquad + b(t, u_t)E^{\mathscr{F}_t}[\eta_r - \eta'_r], u \in \mathcal{U}\} \\ &\le \mu(|y - y'| + |z - z'| + E^{\mathscr{F}_t}[|\eta_r - \eta'_r|]). \end{aligned}$$

Notice $E[\int_0^T |f(t, 0, 0, 0)|^2\, dt] \le \mu^2 T$, then by Theorem 4.2, the anticipated BSDE (19) has a unique solution $(Y.,Z.) \in S^2_{\mathscr{F}}(0, T+K) \times L^2_{\mathscr{F}}(0, T; \mathbb{R}^d)$.

Since for all $u \in \mathcal{U}$, $f^u(t, y, z, \eta) \le f(t, y, z, \eta)$ and $f^u(t, y, z, \eta)$ is increasing in η, by Theorem 5.1, we get $Y_t \ge Y_t^u$, a.e., a.s. Thus, $Y_t \ge Y_t^*$, a.e., a.s.

On the other hand, by the definition of f, we know for all $\varepsilon > 0$, for each $(\omega, t) \in \Omega \times [0, T)$,

$$\begin{aligned} \{u \in \mathcal{U};\ & f(\omega, t, Y_t(\omega), Z_t(\omega), Y_{t+\theta}(\omega)) \\ &\le \alpha(t, u)Y_t(\omega) + Z_t(\omega)\sigma(t, u) \\ &\quad + b(t, u)E^{\mathscr{F}_t}[Y_{t+\theta}(\omega)] + l(\omega, t, u) + \varepsilon\} \ne \varnothing. \end{aligned}$$

Then by a Measurable Selection Theorem, for example, that can be found in Dellacherie [5] or in Beneš [1, 2], there exists a $u^\varepsilon \in \mathcal{U}$ such that

$$f(t, Y_t, Z_t, Y_{t+\theta}) \le f^{u^\varepsilon}(t, Y_t, Z_t, Y_{t+\theta}) + \varepsilon, \qquad \text{a.e., a.s.}$$

Denote the solution to the anticipated BSDE corresponding to $(f^{u^\varepsilon}, Q_\bullet)$ by $(Y^{u^\varepsilon}_\bullet, Z^{u^\varepsilon}_\bullet)$.

First, consider the case when $t \in [T-\theta, T]$. Thus, $t + \theta \in [T, T+\theta]$, $Y_{t+\theta} = Y^{u^\varepsilon}_{t+\theta}$ and

$$\begin{aligned} f^{u^\varepsilon}&(t, Y^{u^\varepsilon}_t, Z^{u^\varepsilon}_t, Y^{u^\varepsilon}_{t+\theta}) - f(t, Y_t, Z_t, Y_{t+\theta}) \\ &\ge f^{u^\varepsilon}(t, Y^{u^\varepsilon}_t, Z^{u^\varepsilon}_t, Y^{u^\varepsilon}_{t+\theta}) - f^{u^\varepsilon}(t, Y_t, Z_t, Y_{t+\theta}) - \varepsilon \\ &= f^{u^\varepsilon}(t, Y^{u^\varepsilon}_t, Z^{u^\varepsilon}_t, Y^{u^\varepsilon}_{t+\theta}) - f^{u^\varepsilon}(t, Y_t, Z_t, Y^{u^\varepsilon}_{t+\theta}) - \varepsilon \\ &= g^{(1)}_t(Y^{u^\varepsilon}_t - Y_t) + g^{(2)}_t(Z^{u^\varepsilon}_t - Z_t) - \varepsilon, \end{aligned}$$

where, for $t \in [T-\theta, T]$,

$$g_t^{(1)} = \begin{cases} \dfrac{f^{u^\varepsilon}(t, Y_t^{u^\varepsilon}, Z_t^{u^\varepsilon}, Y_{t+\theta}^{u^\varepsilon}) - f^{u^\varepsilon}(t, Y_t, Z_t^{u^\varepsilon}, Y_{t+\theta}^{u^\varepsilon})}{Y_t^{u^\varepsilon} - Y_t}, & \text{if } Y_t^{u^\varepsilon} \neq Y_t; \\ 0, & \text{if } Y_t^{u^\varepsilon} = Y_t, \end{cases}$$

$$g_t^{(2)} = \begin{cases} \dfrac{f^{u^\varepsilon}(t, Y_t, Z_t^{u^\varepsilon}, Y_{t+\theta}^{u^\varepsilon}) - f^{u^\varepsilon}(t, Y_t, Z_t, Y_{t+\theta}^{u^\varepsilon})}{Z_t^{u^\varepsilon} - Z_t}, & \text{if } Z_t^{u^\varepsilon} \neq Z_t; \\ 0, & \text{if } Z_t^{u^\varepsilon} = Z_t. \end{cases}$$

That is, for $t \in [T-\theta, T]$,

$$\begin{aligned} Y_t^{u^\varepsilon} - Y_t \geq & \int_t^T \big(g_s^{(1)}(Y_s^{u^\varepsilon} - Y_s) + g_s^{(2)}(Z_s^{u^\varepsilon} - Z_s) - \varepsilon\big)\, ds \\ & - \int_t^T (Z_s^{u^\varepsilon} - Z_s)\, dW_s. \end{aligned}$$

Hence, $Y_t^{u^\varepsilon} - Y_t \geq \tilde{Y}_t^{(1)}$, where $\tilde{Y}_t^{(1)}$ is the solution of BSDE:

$$\tilde{Y}_t^{(1)} = \int_t^T \big(g_s^{(1)} \tilde{Y}_s^{(1)} + g_s^{(2)} \tilde{Z}_s^{(1)} - \varepsilon\big)\, ds - \int_t^T \tilde{Z}_s^{(1)}\, dW_s, \qquad t \in [T-\theta, T].$$

Since $|g_t^{(1)}| \leq \mu$, $|g_t^{(2)}| \leq \mu$, we get

$$\tilde{Y}_t^{(1)} = -\varepsilon E^{\mathscr{F}_t}\left[\int_t^T \tilde{X}_s^{(1)}\, ds\right], \qquad t \in [T-\theta, T],$$

where

$$\tilde{X}_t^{(1)} = \exp\left[\int_0^t g_s^{(2)}\, dW_s - \frac{1}{2}\int_0^t |g_s^{(2)}|^2\, ds + \int_0^t g_s^{(1)}\, ds\right] \geq 0.$$

Therefore, there exists a constant $\rho_1 > 0$ depending only on μ, θ and T such that

$$Y_t^{u^\varepsilon} - Y_t \geq \tilde{Y}_t^{(1)} \geq -\rho_1 \varepsilon, \qquad t \in [T-\theta, T].$$

Second, consider the case when $t \in [T-2\theta, T-\theta]$. Then $t+\theta \in [T-\theta, T]$, $Y_{t+\theta} \leq Y_{t+\theta}^{u^\varepsilon} + \rho_1\varepsilon$. Since for all $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f^u(t, y, z, \cdot)$ is a increasing and linear function, we have

$$\begin{aligned} & f^{u^\varepsilon}(t, Y_t^{u^\varepsilon}, Z_t^{u^\varepsilon}, Y_{t+\theta}^{u^\varepsilon}) - f(t, Y_t, Z_t, Y_{t+\theta}) \\ & \quad \geq f^{u^\varepsilon}(t, Y_t^{u^\varepsilon}, Z_t^{u^\varepsilon}, Y_{t+\theta}^{u^\varepsilon}) - f^{u^\varepsilon}(t, Y_t, Z_t, Y_{t+\theta}) - \varepsilon \\ & \quad \geq f^{u^\varepsilon}(t, Y_t^{u^\varepsilon}, Z_t^{u^\varepsilon}, Y_{t+\theta}^{u^\varepsilon}) - f^{u^\varepsilon}(t, Y_t, Z_t, Y_{t+\theta}^{u^\varepsilon} + \rho_1\varepsilon) - \varepsilon \\ & \quad \geq f^{u^\varepsilon}(t, Y_t^{u^\varepsilon}, Z_t^{u^\varepsilon}, Y_{t+\theta}^{u^\varepsilon}) - f^{u^\varepsilon}(t, Y_t, Z_t, Y_{t+\theta}^{u^\varepsilon}) - \mu\rho_1\varepsilon - \varepsilon \\ & \quad = g_t^{(1)}(Y_t^{u^\varepsilon} - Y_t) + g_t^{(2)}(Z_t^{u^\varepsilon} - Z_t) - (\mu\rho_1 + 1)\varepsilon, \end{aligned}$$

where, for $t \in [T-2\theta, T-\theta]$,

$$g_t^{(1)} = \begin{cases} \dfrac{f^{u^\varepsilon}(t, Y_t^{u^\varepsilon}, Z_t^{u^\varepsilon}, Y_{t+\theta}^{u^\varepsilon}) - f^{u^\varepsilon}(t, Y_t, Z_t^{u^\varepsilon}, Y_{t+\theta}^{u^\varepsilon})}{Y_t^{u^\varepsilon} - Y_t}, & \text{if } Y_t^{u^\varepsilon} \neq Y_t; \\ 0, & \text{if } Y_t^{u^\varepsilon} = Y_t, \end{cases}$$

$$g_t^{(2)} = \begin{cases} \dfrac{f^{u^\varepsilon}(t, Y_t, Z_t^{u^\varepsilon}, Y_{t+\theta}^{u^\varepsilon}) - f^{u^\varepsilon}(t, Y_t, Z_t, Y_{t+\theta}^{u^\varepsilon})}{Z_t^{u^\varepsilon} - Z_t}, & \text{if } Z_t^{u^\varepsilon} \neq Z_t; \\ 0, & \text{if } Z_t^{u^\varepsilon} = Z_t. \end{cases}$$

Therefore, for $t \in [T-2\theta, T-\theta]$,

$$\begin{aligned} Y_t^{u^\varepsilon} - Y_t \geq Y_{T-\theta}^{u^\varepsilon} - Y_{T-\theta} - \int_t^{T-\theta} (Z_s^{u^\varepsilon} - Z_s)\, dW_s \\ + \int_t^{T-\theta} \big(g_s^{(1)}(Y_s^{u^\varepsilon} - Y_s) + g_s^{(2)}(Z_s^{u^\varepsilon} - Z_s) - (\mu\rho_1 + 1)\varepsilon\big)\, ds. \end{aligned}$$

Hence, $Y_t^{u^\varepsilon} - Y_t \geq \tilde{Y}_t^{(2)}$, where $\tilde{Y}_t^{(2)}$ is the solution to the following BSDE: For $t \in [T-2\theta, T-\theta]$,

$$\begin{aligned} \tilde{Y}_t^{(2)} &= Y_{T-\theta}^{u^\varepsilon} - Y_{T-\theta} \\ &\quad + \int_t^{T-\theta} \big(g_s^{(1)} \tilde{Y}_s^{(2)} + g_s^{(2)} \tilde{Z}_s^{(2)} - (\mu\rho_1 + 1)\varepsilon\big)\, ds - \int_t^{T-\theta} \tilde{Z}_s^{(2)}\, dW_s. \end{aligned}$$

Note $|g_t^{(1)}| \leq \mu$ and $|g_t^{(2)}| \leq \mu$. We have, for all $t \in [T-2\theta, T-\theta]$,

$$\tilde{Y}_t^{(2)} = E^{\mathscr{F}_t}\left[(Y_{T-\theta}^{u^\varepsilon} - Y_{T-\theta})\tilde{X}_{T-\theta}^{(2)} - \int_t^{T-\theta} (\mu\rho_1 + 1)\varepsilon \tilde{X}_s^{(2)}\, ds\right],$$

where

$$\tilde{X}_t^{(2)} = \exp\left[\int_0^t g_s^{(2)}\, dW_s - \frac{1}{2}\int_0^t |g_s^{(2)}|^2\, ds + \int_0^t g_s^{(1)}\, ds\right] \geq 0.$$

Since $Y_{T-\theta}^{u^\varepsilon} - Y_{T-\theta} \geq -\rho_1\varepsilon$, there exists a constant $\rho_2 > 0$ depending only on μ, θ and T such that

$$Y_t^{u^\varepsilon} - Y_t \geq \tilde{Y}_t^{(2)} \geq -\rho_2\varepsilon, \qquad t \in [T-2\theta, T-\theta].$$

Similarly, we get constants $\rho_3, \rho_4, \dots, \rho_{[\frac{T}{\theta}]+1} > 0$ such that

$$\begin{aligned} &Y_t^{u^\varepsilon} - Y_t \geq -\rho_n\varepsilon, \qquad t \in [T - n\theta, T - (n-1)\theta], n = 3, 4, \dots, \left[\frac{T}{\theta}\right]; \\ &Y_t^{u^\varepsilon} - Y_t \geq -\rho_{[T/\theta]+1}\varepsilon, \qquad t \in \left[0, T - \left[\frac{T}{\theta}\right]\theta\right]. \end{aligned}$$

Setting $\rho = \max\{\rho_2, \rho_3, \dots, \rho_{[T/\theta]+1}\}$, we obtain

$$Y_t^{u^\varepsilon} - Y_t \geq -\rho\varepsilon, \qquad t \in [0, T].$$

Since $Y_t^{u^\varepsilon} \le Y_t$, a.e., a.s., setting $\varepsilon \to 0$, we get $Y_t^{u^\varepsilon} \to Y_t$, a.e., a.s. Thus,

$$Y_t = Y_t^*, \qquad \text{a.e., a.s.} \qquad \square$$

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