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ORIGINAL RESEARCH

Anticipative information in a Brownian−Poisson market

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Abstract
The anticipative information refers to some information about future events that may be disclosed in advance. This information may regard, for example, financial assets and their future trends. In our paper, we assume the existence of some anticipative information in a market whose risky asset dynamics evolve according to a Brownian motion and a Poisson process. Using Malliavin calculus and filtration enlargement techniques, we derive the information drift of the mentioned processes and, both in the pure jump case and in the mixed one, we compute the additional expected logarithmic utility. Many examples are shown, where the anticipative information is related to some conditions that the constituent processes or their running maximum may verify, in particular, we show new examples considering Bernoulli random variables.

Keywords Optimal portfolio · Malliavin calculus · Clark–Ocone formula · Insider information · Value of the information

Mathematics Subject Classification 91G10 · 60H7 · 93E12

1 Introduction

In this paper we focus on the study of the presence of some anticipative information in a market composed of a bank bond and a risky asset, the latter one driven by a Brownian motion $W = (W_t,\ 0 \le t \le T)$ and a compensated Poisson process $\tilde{N} = (N_t - \int_0^t \lambda_s ds,\ 0 \le t \le T)$ with positive intensity process $\lambda = (\lambda_t,\ 0 \le t \le T)$. In the optimal portfolio problem, a non-informed agent is looking for maximizing her expected logarithmic gains at the end of a

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trading period $T > 0$, while playing with the natural information flow $\mathbb{F} := \{\mathcal{F}_t\}_{0 \le t \le T}$ with $\mathcal{F}_t := \sigma(W_s, N_s : 0 \le s \le t)$. She will be referred to as the $\mathbb{F}$-agent. In addition, we assume that there exists an agent who is informed about a random variable $G \in \mathcal{F}_T$ containing some anticipative information about the path of W and/or N. The anticipative filtration will be the initial enlargement $\mathbb{G} := \mathbb{F} \vee \sigma(G)$ and the agent playing with it will be referred to as the $\mathbb{G}$-agent.

Filtration enlargement is a stochastic calculus technique that allows modeling the incorporation of additional non-adapted information. It has multiple applications, including insider trading or, more general, asymmetric information. We refer to the Bibliographic Notes in Chapter VI of Protter (2005) for a detailed overview on this subject. In the seminal paper Amendinger et al. (1998), it is shown that if the dynamics of the risky asset do not include the discontinuous part N, then the additional gain under logarithmic utility is given by the entropy of the random variable G when it is purely atomic. After that, much progress has been made in the analysis of the additional information in the Brownian case, see Grorud and Pontier (1998) and Amendinger et al. (2003) for the main references. The research on the Poisson process in the initial enlargement framework started with Ankirchner (2008) in which the existence of a compensator is analyzed. Although the entropy is considered, the additional gain of an informed $\mathbb{G}$-agent in the optimal portfolio problem has not been studied. In Di Nunno et al. (2006), a similar framework is studied, however they mainly focused on enlargements that disclose the exact value of the terminal driving processes and their objective is to compute the optimal portfolio without discussing the additional expected logarithmic utility.

Our main motivation is to get an expression for the investor's additional gains in a market whose risky asset dynamics depend also on a Poisson process, in the same spirit of the analysis done in Theorem 4.1 of Amendinger et al. (1998). This is achieved in Theorem 3.11 for the pure jump case and in Theorem 4.6 for the mixed Brownian–Poisson market. Another novelty of our paper is in the kind of examples that we choose to model the additional information. We consider both $G = \tau \wedge T$, being τ the time of the first jump of N, and $G = \mathbb{1}_{\{b_1 \le N_T \le b_2\}}$, for some constants $b_1, b_2 \in \mathbb{N}$. We work also with $G = \mathbb{1}_{\{a_1 \le W_T \le a_2\} \times \{b_1 \le N_T \le b_2\}}$, where the $\mathbb{G}$-agent knows if the pair (W_T, N_T) falls within a certain rectangle or not. However, as the main example we consider $G = \mathbb{1}_{\{a_1 \le M_T \le a_2\} \times \{b_1 \le J_T \le b_2\}}$ being $M_T := \sup_{0 \le s \le T} W_s$ and $J_T := \sup_{0 \le s \le T} \tilde{N}_s$ in which the $\mathbb{G}$-agent knows whether the running maximum processes will be in a certain region or not. Finally, we present an example in which the process λ is truly related with W, $\lambda_t = 1 + \mathbb{1}_{\{W_T \ge 0\}}$, $\forall t \le T$.

For the majority of our computations, we use Malliavin calculus techniques, we suggest Di Nunno et al. (2009) or Nualart (2006) for a general overview on this variational calculus. Malliavin Calculus was applied to the optimal portfolio problem in Ocone and Karatzas (1991) via Clark–Ocone formula. With respect to the enlargement of filtration theory, we highlight Imkeller et al. (2001) for a methodology to compute the information drift in initial enlargements and Brownian setting. In particular, some Malliavin regularity assumptions on the conditional densities are assumed in order to drop the so-called *Jacod hypothesis*. In addition, Corcuera et al. (2004) analyzed the case in which the information is becoming more precise via progressive enlargements. Nualart and Vives (1990) applied for the first time the Malliavin approach to the Poisson process while Mensi and Privault (2003) mimicked the Malliavin methodology of Imkeller et al. (2001) for the Poisson process in the context of initial enlargement of filtrations. Finally Wright et al. (2018) weakened some assumptions of the latter paper by assuming the Malliavin derivative were in $L^2(\mathrm{d}t \times \mathrm{d}\boldsymbol{P})$. Nowadays, the optimal portfolio problem with non-continuous assets is still an active topic of research, for example in Chau et al. (2018), they analyse additional gains generated by an initial enlargement via super-hedging, or Bellalah et al. (2020), that deals with an example related to the Covid-19 crisis. See also Colaneri et al. (2021), where the value of the market price of risk is compared under different information flows.

The paper is organised as follows. In Sect. 2 we describe the framework and we introduce the notation. In Sect. 3 we consider the purely jump market and we get the explicit expression of the compensator of the Poisson process for $\mathcal{F}_T^N$-measurable random variables. The main examples of this section are about the time of the first jump of N and the terminal value of the Poisson process, that is, $G = \mathbb{1}_{\{N_T \in B\}}$ being B an interval. In Sect. 3.1, we state Theorem 3.11 in which we get a nice expression for the additional gain of an agent who plays with an initial enlarged filtration. In Sect. 4 we work in a Brownian–Poisson market, in which the additional expected logarithmic utility is computed in Corollary 4.7. The main examples considered in this subsection are $G = \mathbb{1}_{\{W_T \in A\} \times \{N_T \in B\}}$ and $G = \mathbb{1}_{\{M_T \in A\} \times \{J_T \in B\}}$ for A, B some given intervals. In Sect. 5 we construct a model where the Poisson intensity λ is an anticipative function of W.

2 Model and notation

Let $(\Omega, \mathcal{F}_T, \boldsymbol{P}, \mathbb{F})$ be a filtered probability space, where the filtration $\mathbb{F} := \{\mathcal{F}_t\}_{0 \le t \le T}$ is assumed to be complete and right-continuous. We assume that the agent is going to invest in a market composed by two assets in a finite horizon time $T > 0$. The first one is a risk-less bond $D = (D_t,\ 0 \le t \le T)$ and the second one is a risky stock $S = (S_t,\ 0 \le t \le T)$. The dynamics of both are given by the following SDEs,

$$\frac{dD_t}{D_t} = \rho_t dt, \quad D_0 = 1 \tag{1a}$$

$$\frac{dS_t}{S_{t-}} = \mu_t dt + \sigma_t dW_t + \theta_t d\tilde{N}_t, \quad S_0 = s_0 > 0, \tag{1b}$$

where $W = (W_t,\ 0 \le t \le T)$ is a Brownian motion and $N = (N_t,\ 0 \le t \le T)$ is a Poisson process with strictly positive intensity $\lambda = (\lambda_t,\ 0 \le t \le T)$, while the compensated version of the Poisson process is defined as $\tilde{N}_t := N_t - \int_0^t \lambda_s ds$ with $\int_0^T \lambda_s ds < +\infty$, d$\boldsymbol{P}$-a.s. We assume that W and N are independent and, to ensure that $S = (S_t,\ 0 \le t \le T)$ is well-defined,

$$-1 < \theta_t\ , \quad \mathrm{d}t \times \mathrm{d}\boldsymbol{P} - \text{a.s.} \tag{2}$$

The measurability of λ will be specified in each section. The natural filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$ is generated by the Brownian motion and the Poisson process, and it is augmented by the zero $\boldsymbol{P}$-measure sets, $\mathcal{N}$:

$$\mathcal{F}_t := \sigma(W_s,\ N_s\, :\ 0 \le s \le t) \vee \mathcal{N}.$$

We use the notation $\mathbb{F}^W := \{\mathcal{F}_t^W\}_{0 \le t \le T}$ and $\mathbb{F}^N := \{\mathcal{F}_t^N\}_{0 \le t \le T}$ to refer to the natural filtration of W and N, respectively. About the market coefficients, in (1a) and (1b) we assume that they are $\mathbb{F}$-predictable processes that satisfy the following integrability condition

$$\boldsymbol{E}\left[\int_0^T \left(|\rho_s| + |\mu_s| + \sigma_s^2 + \theta_s^2\right) ds\right] < +\infty. \tag{3}$$

By $\boldsymbol{E}$ and $\boldsymbol{V}$ we refer to the expectation and the variance operators of a given random variable under the measure $\boldsymbol{P}$. Given a σ-algebra $\mathcal{F}$, by $\boldsymbol{E}[\cdot|\mathcal{F}]$ and $\boldsymbol{V}[\cdot|\mathcal{F}]$ we denote the conditional expectation and the conditional variance. We introduce the filtration $\mathbb{H}$, that will be the filtration used to define the allowed strategies, and that depending on the agent will coincide with the filtrations $\mathbb{F}$ and $\mathbb{G}$. We define $L^2(\Omega, \mathcal{H}_T, \mathrm{d}t \times \mathrm{d}\boldsymbol{P}, \mathbb{H})$, or simply $L^2(\mathrm{d}t \times \mathrm{d}\boldsymbol{P})$ when the reference filtration is clear, as the space of all $\mathbb{H}$-adapted processes X such that $\int_0^T \boldsymbol{E}\left[X_s^2\right] ds < +\infty$.

Using the previous set-up, it is assumed that an agent can control her portfolio by a *self-financing* process $\pi = (\pi_t,\ 0 \leq t \leq T)$, with the aim to maximize her expected logarithmic gains at the finite horizon time. It represents the fraction of the monetary unit amount invested in the risky asset. We denote by $X^\pi = (X_t^\pi,\ 0 \leq t \leq T)$ a positive process to model the wealth of the portfolio of the investor under the strategy π. The dynamics of the wealth process are given by the following SDE, for $0 \leq t \leq T$,

$$\frac{dX_t^\pi}{X_{t-}^\pi} = (1-\pi_t)\frac{dD_t}{D_t} + \pi_t \frac{dS_t}{S_{t-}}, \quad X_0^\pi = x_0 > 0, \tag{4}$$

and by using the evolution of both assets given in (1) we get

$$\frac{dX_t^\pi}{X_{t-}^\pi} = (1-\pi_t)\rho_t dt + \pi_t \left(\mu_t dt + \sigma_t dW_t + \theta_t d\tilde{N}_t\right), \quad X_0^\pi = x_0,$$

where the SDE is well-defined on the probability space $(\Omega, \mathcal{F}_T, \boldsymbol{P}, \mathbb{F})$. Before giving a proper definition of the set of processes π that we consider, we look for the natural conditions they should satisfy. Applying the Itô formula to the dynamics of the risky asset given by (1b), we get an explicit solution as follows,

$$\ln \frac{S_t}{s_0} = \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2 + \lambda_s(\ln(1+\theta_s) - \theta_s)\right) ds + \int_0^t \sigma_s dW_s + \int_0^t \ln(1+\theta_s) d\tilde{N}_s.$$

If we apply the Itô formula to the wealth process we get,

$$\ln \frac{X_t^\pi}{x_0} = \int_0^t \left(\rho_s + \pi_s(\mu_s - \rho_s) - \frac{1}{2}\pi_s^2\sigma_s^2 + \lambda_s(\ln(1+\pi_s\theta_s) - \pi_s\theta_s)\right) ds \tag{5}$$

provided that these integrals are well-defined. To ensure this, we assume the following integrability condition,

$$\boldsymbol{E}\left[\int_0^T \left(|\pi_s||\mu_s - \rho_s| + \pi_s^2\sigma_s^2 + \pi_s^2\theta_s^2\right) ds\right] < +\infty. \tag{6}$$

In order to guarantee that X^π is well-defined, we impose that

$$1 + \pi_t\theta_t > 0, \quad \mathrm{d}t \times \mathrm{d}\boldsymbol{P} - \text{a.s.} \tag{7}$$

Now, we can properly define the optimization problem as the supremum of the expected logarithmic gains of the agent's wealth at the finite horizon time T.

$$\mathbb{V}_T^{\mathbb{H}} := \sup_{\pi \in \mathcal{A}(\mathbb{H})} \boldsymbol{E}\left[\ln X_T^\pi | X_0^\pi = x_0\right], \quad \mathbb{H} \supseteq \mathbb{F}. \tag{8}$$

Finally we give the definition of the set $\mathcal{A}(\mathbb{H})$ made of all admissible strategies for the $\mathbb{H}$-agent, that is the one playing with information flow $\mathbb{H} \supseteq \mathbb{F}$.

Definition 2.1 In the financial market (1a)–(1b), we define the set of admissible strategies $\mathcal{A}(\mathbb{H})$ as the set of portfolio processes π predictable with respect to the filtration $\mathbb{H}$ which satisfy the conditions (6) and (7).

Definition 2.2 The additional expected logarithmic utility of a filtration $\mathbb{H} \supseteq \mathbb{F}$ is given by

$$\Delta\mathbb{V}_T^{\mathbb{H}} = \mathbb{V}_T^{\mathbb{H}} - \mathbb{V}_T^{\mathbb{F}},$$

where the quantities on the right-hand side are defined in (8).

In the following statement we summarize the results about the optimal portfolios in markets with Brownian noise, Poisson noise or both. It can be found in Corollary 17 of Di Nunno et al. (2006).

Proposition 2.3 *The optimal strategy π^* for the problem (8) with the information flow $\mathbb{F}$ satisfies*

$$\mu_t - \rho_t - \sigma_t^2\pi_t^* - \lambda\frac{\theta_t^2\pi_t^*}{1+\pi_t^*\theta_t} = 0. \tag{9}$$

In particular, if we have that $\sigma_t > 0$ and $\theta_t = 0$ $\mathrm{d}t \times \mathrm{d}\boldsymbol{P}$-a.s., then we recover the classic Merton problem with the optimal strategy satisfying the relation

$$\pi_t^* = \frac{\mu_t - \rho_t}{\sigma_t^2}. \tag{10}$$

Finally, if we have that $\sigma_t = 0$, $\theta_t \neq 0$ and $\mathrm{d}t \times \mathrm{d}\boldsymbol{P}$-a.s., then the optimal strategy is given by

$$\pi_t^* = \frac{\mu_t - \rho_t}{\lambda\theta_t^2 - \theta_t(\mu_t - \rho_t)}. \tag{11}$$

Let $G \in \mathcal{F}_T$ be a real valued random variable modeling some additional information. We introduce the filtration $\mathbb{G} := \{\mathcal{G}_t\}_{0\le t\le T}$ under which the privileged information is accessible since the beginning time $t = 0$, that is

$$\mathcal{G}_t = \bigcap_{s>t}(\mathcal{F}_s \vee \sigma(G)), \tag{12}$$

We denote with $\boldsymbol{P}^G$ the distribution of G, i.e., $\boldsymbol{P}^G(\cdot) = \boldsymbol{P}(G \in \cdot)$ on $\sigma(G)$ and by $\boldsymbol{P}^G(\cdot|\mathcal{F}) = \boldsymbol{P}(G \in \cdot|\mathcal{F})$ the corresponding conditional probability with respect to a given σ-algebra $\mathcal{F}$. The crucial point is to ensure that any $\mathbb{F}$-semimartingale is also a $\mathbb{G}$-semimartingale, that is known in the literature as the $(\mathcal{H}')$ *hypothesis*, see Jeulin and Yor (1978). An approach widely used in the literature to achieve this is known as the *Jacod hypothesis* and assumes that the conditional distributions $\boldsymbol{P}^G(\cdot|\mathcal{F}_t)$ is absolutely continuous with respect to $\boldsymbol{P}^G$ for $t \in [0, T)$ a.s. In Theorem 2.5 of Jacod (1985) it is proven that the Jacod hypothesis implies the ($\mathcal{H}$') hypothesis. However, this assumption sometimes results too restrictive. Therefore, we use the following proposition valid in the framework of Poisson process to compute the information drift under weaker assumptions. We refer to Proposition 3 in Mensi and Privault (2003) for a proof, for a similar result in the Brownian motion framework see Theorem 2.1 in Imkeller et al. (2001). We present a slightly different statement adapted to finite deterministic time T.

Proposition 2.4 *Let $G \in \mathcal{F}_T^N$, consider $\mathbb{G} = \mathbb{F}^N \vee \sigma(G)$ and $P_t(\omega, dg) := \boldsymbol{P}\left(G \in dg|\mathcal{F}_t^N\right)(\omega)$, with $0 \le t \le T$, denote a version of the conditional law of G given $\mathcal{F}_t^N$ and assume it has the following representation $P_t(\cdot, dg) = P_0(\cdot, dg) + \int_0^t \phi_s(\cdot, dg)d\tilde{N}_s$, $0 \le t \le T$, and there exists a measurable h such that $\phi_s(\cdot, dg) = h_s(\cdot, g)P_s(\cdot, dg)$, $0 \le s < T$, then $\tilde{N}_\cdot - \int_0^\cdot h_s(\cdot, G)ds$ is a $\mathbb{G}$-local martingale, with $\mathbb{G} = \mathbb{F}^N \vee \sigma(G)$.*

The process $h(\cdot, G)$ is usually called the *information drift* and it plays a crucial role in our computations.

To conclude this section, we state the Clark–Ocone formula valid for the case the intensity λ is a deterministic process. The operators D_t and $D_{t,1}$ refer to the Malliavin derivative in the Brownian and Poisson cases respectively and we consider their generalizations to $L^2(\boldsymbol{P})$. We refer to Theorem 13.28 in Di Nunno et al. (2009) for the details and a general background.

Proposition 2.5 *Let λ be a deterministic process and $G \in L^2(\boldsymbol{P})$ be an $\mathcal{F}_T$-measurable random variable, then the following representation holds,*

$$G = \boldsymbol{E}[G] + \int_0^T \boldsymbol{E}[D_t G|\mathcal{F}_t]dW_t + \int_0^T \boldsymbol{E}[D_{t,1}G|\mathcal{F}_t]d\tilde{N}_t. \tag{13}$$

3 Initial enlargements in a pure Poisson market

In this section, we deal with a market in which the source of the noise of the risky asset is only the Poisson process, i.e., we assume that $\sigma_t = 0$, $\mathrm{d}t \times \mathrm{d}\boldsymbol{P}$-a.s. and the intensity process λ is deterministic. Using the predictable representation property (PRP) enjoyed by the compensated Poisson process, we know that for every $\mathcal{F}_T^N$-measurable real valued random variable $G \in L^2(\boldsymbol{P})$, there exists an $\mathbb{F}^N$-predictable process $\varphi \in L^2(\mathrm{d}t \times \mathrm{d}\boldsymbol{P})$ such that

$$G = \boldsymbol{E}[G] + \int_0^T \varphi_s d\tilde{N}_s, \tag{14}$$

usually called the *non-anticipative derivative* of G, see Di Nunno (2007). Let $B \in \mathcal{B}(\mathbb{R})$ be a subset in the Borel σ-algebra on $\mathbb{R}$ and consider the following PRP,

$$\mathbb{1}_{\{G \in B\}} = \boldsymbol{P}^G(B) + \int_0^T \varphi_s(B)d\tilde{N}_s, \tag{15}$$

where by the predictable process $\varphi(B) = (\varphi_t(B) : 0 \leq t \leq T)$ we denote the unique one within the Hilbert space $L^2(\mathrm{d}t \times \mathrm{d}\boldsymbol{P})$ that satisfies (15) for a fixed B.

In the next lemma we prove that $\varphi(\cdot)$ is a vector measure, we refer to Diestel and Uhl (1977) for a general background on the vector measure theory. We make the following assumption, for more details we refer to Definition 4 of Diestel and Uhl (1977).

Assumption 3.1 The vector measure φ is of bounded variation.

Remark 3.2 Assumption 3.1 is trivially satisfied for a bounded discrete random variable as the variation of the vector measure φ is bounded above by a constant times the number of values assumed by the random variable.

In the following lemma we state the Radon-Nikodym derivative for the Hilbert valued random measure φ.

Lemma 3.3 *The set function $B \longrightarrow \varphi(B)$, with $B \in \mathcal{B}(\mathbb{R})$, is a countably additive $L^2(\mathrm{d}t \times \mathrm{d}\boldsymbol{P})$-valued vector measure and there exists a set of processes $\psi = (\psi^g,\ g \in \mathrm{Supp}(G))$ such that $\psi^g = (\psi_t^g,\ 0 \leq t \leq T) \in L^1(\boldsymbol{P}^G)$ satisfying*

$$\varphi_t(B) = \int_B \psi_t^g \boldsymbol{P}^G(dg), \quad B \subset \mathbb{R}.$$

Proof Let $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}(\mathbb{R})$ be a disjoint sequence of subsets satisfying $B = \cup_{i=1}^{\infty} B_i$. Then $\mathbb{1}_{\{G\in B\}} = \sum_{i=1}^{\infty} \mathbb{1}_{\{G\in B_i\}}$ $\boldsymbol{P}$-a.s. in $L^2(\mathrm{d}\boldsymbol{P})$ (see Example 3 in Diestel and Uhl (1977)). Then, using the PRP we get

$$\mathbb{1}_{\{G\in B\}} = \sum_{i=1}^{\infty} \mathbb{1}_{\{G\in B_i\}} = \boldsymbol{P}(G \in B) + \sum_{i=1}^{\infty} \int_0^T \varphi_t(B_i) d\tilde{N}_t = \int_0^T \sum_{i=1}^{\infty} \varphi_t(B_i) d\tilde{N}_t,$$

In the second equality we applied the σ-additivity of the probability measure $\boldsymbol{P}$ while in the third one, thanks to $\varphi(B) \in L^2(\mathrm{d}t \times \mathrm{d}\boldsymbol{P})$ for any $B \in \mathcal{B}(\mathbb{R})$, we used the stochastic Fubini theorem, see Lemma A.1.1 in Mandrekar and Rüdiger (2015). By the uniqueness of the representation we deduce that $\varphi(B) = \sum_{i=1}^{\infty} \varphi(B_i)$ and we conclude that φ is a $L^2(\mathrm{d}t \times \mathrm{d}\boldsymbol{P})$-vector measure. As $\varphi \ll \boldsymbol{P}^G$ on $\sigma(G)$, the last claim of the lemma follows by Proposition 2.1 of Kakihara (2011). $\square$

Therefore, Lemma 3.3 guarantees that there exists a set of processes $\psi^g = (\psi_t^g,\ 0 \le t \le T)$ with $g \in \mathrm{Supp}(G)$ such that the PRP given in (15) can be written as follows

$$\mathbb{1}_{\{G\in B\}} = \boldsymbol{P}^G(B) + \int_0^T \int_B \psi_s^g \boldsymbol{P}^G(dg) d\tilde{N}_s. \tag{16}$$

In particular, by assuming measurability on ψ in all variables, the following representation

$$\mathbb{1}_{\{G\in dg\}} = \boldsymbol{P}^G(dg) + \int_0^T \psi_s^g d\tilde{N}_s\, \boldsymbol{P}^G(dg) \tag{17}$$

holds true, where Fubini's theorem has been applied. When G is purely atomic, the PRP (17) reduces to

$$\mathbb{1}_{\{G=g\}} = \boldsymbol{P}(G = g) + \int_0^T \psi_s^g d\tilde{N}_s\, \boldsymbol{P}(G = g). \tag{18}$$

The following result provides the information drift in terms of the process ψ.

Lemma 3.4 *Let $G \in L^2(\boldsymbol{P})$ be an $\mathcal{F}_T^N$-measurable random variable satisfying Assumption 3.1, then the process $\gamma^G = (\gamma_t^G,\ 0 \le t < T)$ defined as*

$$\gamma_t^g := \frac{\psi_t^g \boldsymbol{P}^G(dg)}{\boldsymbol{P}^G(dg|\mathcal{F}_t^N)}, \quad g \in Supp(G) \tag{19}$$

satisfies that $\tilde{N}_\cdot - \int_0^\cdot \lambda_s \gamma_s^G ds$ is a $\mathbb{G}$-local martingale, provided it is well-defined. γ^G is referred to as the information drift.

Proof Once we got (16), we proceed similarly to Proposition 2.4 but in the case of λ deterministic. Let $A \in \mathcal{F}_s$ and $B \in \sigma(G)$, then

$$\begin{aligned}
\boldsymbol{E}[\mathbb{1}_A \mathbb{1}_{\{G\in B\}}(\tilde{N}_t - \tilde{N}_s)] &= \boldsymbol{E}\left[\mathbb{1}_A \left(\boldsymbol{P}^G(B) + \int_0^T \int_B \psi_s^g \boldsymbol{P}^G(dg) d\tilde{N}_s\right)(\tilde{N}_t - \tilde{N}_s)\right] \\
&= \boldsymbol{E}\left[\mathbb{1}_A \int_s^t \int_B \psi_u^g \boldsymbol{P}^G(dg) dN_u\right] = \boldsymbol{E}\left[\mathbb{1}_A \int_s^t \int_B \lambda_u \gamma_u^g \boldsymbol{P}^G(dg|\mathcal{F}_u) du\right] \\
&= \boldsymbol{E}\left[\mathbb{1}_A \int_s^t \boldsymbol{E}[\lambda_u \gamma_u^G \mathbb{1}_{\{G\in B\}}|\mathcal{F}_u] du\right] = \boldsymbol{E}\left[\mathbb{1}_A \mathbb{1}_{\{G\in B\}} \int_s^t \lambda_u \gamma_u^G du\right].
\end{aligned}$$

As the computation holds true for any $A \in \mathcal{F}_s,\ B \in \sigma(G)$ we conclude that

$$\boldsymbol{E}[\tilde{N}_t - \tilde{N}_s|\mathcal{G}_s] = \boldsymbol{E}\left[\int_s^t \lambda_u \gamma_u^G du|\mathcal{G}_s\right]$$

and the result holds true. □

In order to assure that the optimal strategy in $\mathbb{G}$ is well defined, in the sense of equation (7), we are going to assume some conditions on the information drift γ^G. In particular,

$$\gamma_s^G > -1, \quad \mathrm{d}t \times \mathrm{d}\boldsymbol{P} - a.s. \tag{20}$$

A condition similar to (20) appears in Section 4.1 of Grorud (2000) where they study Poisson processes in enlarged filtrations.

Corollary 3.5 *If* $\mathrm{Supp}(G) = \{0, 1, \ldots, n\}$*, then*

$$\gamma_t^g = \varphi_t^g \frac{\mathbb{1}_{\{G=g\}} - \boldsymbol{E}[\mathbb{1}_{\{G=g\}}|\mathcal{F}_t]}{\boldsymbol{V}[\mathbb{1}_{\{G=g\}}|\mathcal{F}_t]}, \quad \varphi_t^g = \psi_t^g \boldsymbol{P}(G = g), \quad g \in \mathrm{Supp}(G).$$

In addition, if $\mathrm{Supp}(G) = \{0, 1\}$*, then*

$$\gamma_t^G = (-1)^{G+1} \varphi_t \frac{G - \boldsymbol{E}[G|\mathcal{F}_t^N]}{\boldsymbol{V}[G|\mathcal{F}_t^N]}. \tag{21}$$

Proof The former statement comes directly from (19) while for the latter we use the fact that $G = \mathbb{1}_{\{G=1\}}$ and, by the uniqueness of the representation, we conclude that $\varphi = \varphi^1$ and $\varphi^1 = -\varphi^0$, where by φ^0 and φ^1 we refer to the non-anticipative derivative of $\mathbb{1}_{\{G=0\}}$ and $\mathbb{1}_{\{G=1\}}$ respectively. Using that

$$\boldsymbol{E}[G|\mathcal{F}_t^N] = \boldsymbol{P}(G = 1|\mathcal{F}_t^N), \quad \boldsymbol{V}[G|\mathcal{F}_t^N] = \boldsymbol{P}(G = 1|\mathcal{F}_t^N)(1 - \boldsymbol{P}(G = 1|\mathcal{F}_t^N)),$$

the result follows by applying (14) and (18). □

Remark 3.6 Note that if $\mathrm{Supp}(G) = \{0, 1\}$ we can express

$$\gamma_t^g = \frac{(-1)^g \varphi_t}{\boldsymbol{P}(G = 0|\mathcal{F}_t^N) - g}, \quad g \in \{0, 1\}, \tag{22}$$

where φ is the non-anticipative derivative of the Bernoulli random variable G.

By using the Clark–Ocone formula we can deduce that $\varphi_t = \boldsymbol{E}[D_{t,1}G|\mathcal{F}_t^N]$ $\mathrm{d}t \times \mathrm{d}\boldsymbol{P} - a.s.$, which allows to compute some interesting examples. Following Solé et al. (2007), we introduce the following operator

$$\Psi_{t,1}G := G(\omega_{(t,1)}) - G(\omega), \tag{23}$$

being $\omega_{(t,1)}$ the modification of the trajectory ω by adding a new jump of size 1 at time t. In Proposition 5.4 of Solé et al. (2007) it is proved that if $\Psi_{t,1}G \in L^2(\mathrm{d}t \times \mathrm{d}\boldsymbol{P})$, then this operator coincides with the usual Malliavin derivative, in the case of the Poisson process we have that $\Psi_{t,1}G = D_{t,1}G$ $\mathrm{d}t \times \mathrm{d}\boldsymbol{P} - a.s.$

In the next example, we apply Lemma 3.4 to an enlargement with a continuous random variable. In particular, we consider that the informed agent knows when the Poisson process jumps for the first time. Ernst and Rogers (2020, Problem 2) solved a similar problem via the HJB equation.

Example 3.7 We consider the random time $\tau = \inf\{t \geq 0 : N_t = 1\}$ which represents the time of the first jump of the point process N and we define $G = \tau \wedge T$. In this example, we study the enlargement of filtration $\mathbb{G} \supset \mathbb{F}$ induced by G and we compute the information drift given by Lemma 3.4 via the PRP (17). We fix $g \in [0, T]$ and consider the following Clark–Ocone formula

$$\mathbb{1}_{\{G \leq g\}} = \boldsymbol{P}(G \in dg) + \int_0^T \boldsymbol{E}[D_{t,1}\mathbb{1}_{\{G \leq g\}}|\mathcal{F}_t]d\tilde{N}_t.$$

We compute the integrand $\boldsymbol{E}[D_{t,1}\mathbb{1}_{\{G \leq g\}}|\mathcal{F}_t]$ by using the operator Ψ for any $(t, g) \in [0, T]^2$, defined in (23). Note that, when $g < t$, the perturbation is not modifying the indicator so $\Psi_{t,1}\mathbb{1}_{\{G \leq g\}} = 0$. When $g \geq t$, only on the set $\{\tau(\omega) > g\}$ the operator is not null. In general, it holds the following equation

$$\Psi_{t,1}\mathbb{1}_{\{G \leq g\}} = \mathbb{1}_{\{t \leq g\}}\mathbb{1}_{\{\tau > g\}}.$$

Since the Clark–Ocone formula is linear, a limiting argument implies that

$$D_{t,1}\mathbb{1}_{\{G \in dg\}} = -\mathbb{1}_{\{t<g\}}\mathbb{1}_{\{\tau \in dg\}} + \mathbb{1}_{\{t \in dg\}}\mathbb{1}_{\{\tau > g\}}.$$

By (19), the information drift is

$$\gamma_t^g = -\mathbb{1}_{\{t<g\}}\frac{\boldsymbol{P}(\tau \in dg|\mathcal{F}_t^N)}{\boldsymbol{P}(G \in dg|\mathcal{F}_t^N)} + \mathbb{1}_{\{t \in dg\}}\frac{\boldsymbol{P}(\tau > g|\mathcal{F}_t^N)}{\boldsymbol{P}(G \in dg|\mathcal{F}_t^N)}, \quad g \in [0, T], \tag{24}$$

where the first component considers the additional information before the jump and second one the information at the time of the jump. Note that, by the first term appearing in (24), we conclude that γ_t^G does not satisfy assumption (20) as with positive probability $\gamma_t^G = -1$. This is a clear evidence that there is a strong arbitrage in this enlargement. In general, this is an indication that the anticipative information may generate arbitrage opportunities.

Example 3.8 Let $G = \mathbb{1}_{\{N_T \in B\}}$ with $B = [b_1, b_2]$, and $b_1, b_2 \in \mathbb{N}$, $b_2 > b_1 > 0$. We consider the initial enlargement $\mathbb{G} \supset \mathbb{F}$. We compute the process φ as follows, $\Psi_{t,1}\mathbb{1}_{\{N_T \in B\}} = \mathbb{1}_{\{N_T+1 \in B\}} - \mathbb{1}_{\{N_T \in B\}}$, which obviously satisfies the integrability condition and therefore

$$D_{t,1}\mathbb{1}_{\{N_T \in B\}} = \mathbb{1}_{\{N_T = b_1 - 1\}} - \mathbb{1}_{\{N_T = b_2\}},$$

and computing the conditional expectation we obtain the Clark–Ocone formula

$$\varphi_t = \boldsymbol{E}\left[D_{t,1}\mathbb{1}_{\{N_T \in B\}}|\mathcal{F}_t^N\right] = \boldsymbol{P}\left(N_T = b_1 - 1|\mathcal{F}_t^N\right) - \boldsymbol{P}\left(N_T = b_2|\mathcal{F}_t^N\right)$$

and the following PRP holds,

$$\mathbb{1}_{\{N_T \in B\}} = \boldsymbol{P}(N_T \in B) - \int_0^T \left(\boldsymbol{P}\left(N_T = b_2|\mathcal{F}_t^N\right) - \boldsymbol{P}\left(N_T = b_1 - 1|\mathcal{F}_t^N\right)\right) d\tilde{N}_t, \tag{25}$$

giving the following formula for compensator

$$\gamma_t^G = (-1)^G \frac{\boldsymbol{P}\left(N_T = b_1 - 1|\mathcal{F}_t^N\right) - \boldsymbol{P}\left(N_T = b_2|\mathcal{F}_t^N\right)}{\boldsymbol{P}(N_T \in B^c|\mathcal{F}_t^N) - G}. \tag{26}$$

A direct computation shows that $\gamma_t^1 > -1$, while for the case of $G = 0$, some simulations are applied. In the particular case of $\lambda = T = 1$, the computation is direct.

As λ is deterministic, we can compute the probabilities as follows

$$\boldsymbol{P}(N_t - N_s = n|\mathcal{F}_s^N) = e^{-\Lambda(s,t)}\frac{(\Lambda(s,t))^n}{n!}, \quad \Lambda(s,t) := \int_s^t \lambda_u du, \quad n \in \mathbb{N},$$

and the PRP simplifies to

$$\mathbb{1}_{\{N_T \in B\}} = \boldsymbol{P}(N_T \in B) - \int_0^T e^{-\Lambda(t,T)} \left(\frac{(\Lambda(t,T))^{b_2 - N_t}}{(b_2 - N_t)!} \mathbb{1}_{\{N_t \leq b_2\}} \right. \\ \left. - \frac{(\Lambda(t,T))^{b_1 - N_t - 1}}{(b_1 - N_t - 1)!} \mathbb{1}_{\{N_t < b_1\}} \right) d\tilde{N}_t$$

and the information drift is

$$\gamma_t^G = (-1)^G e^{-\Lambda(t,T)} \frac{\frac{(\Lambda(t,T))^{b_1 - N_t - 1}}{(b_1 - N_t - 1)!} \mathbb{1}_{\{N_t < b_1\}} - \frac{(\Lambda(t,T))^{b_2 - N_t}}{(b_2 - N_t)!} \mathbb{1}_{\{N_t \leq b_2\}}}{\boldsymbol{P}(N_T \in B^c | \mathcal{F}_t^N) - G}.$$

Note that, in the simplest case of time-homogeneous Poisson process with constant intensity $\lambda > 0$, we have $\Lambda(t, T) = \lambda(T - t)$.

3.1 Additional expected logarithmic utility

Throughout this subsection, to assure that $\mathbb{V}_T^{\mathbb{F}} < \infty$, we assume that the market coefficients satisfy the following relation

$$0 < \lambda_t - \frac{\mu_t - \rho_t}{\theta_t}, \quad \mathrm{d}t \times \mathrm{d}\boldsymbol{P} - \text{a.s.} \tag{27}$$

Working in the filtration $\mathbb{F}$, if we take expectation in (5), then

$$\boldsymbol{E}\left[\ln \frac{X_T^\pi}{x_0}\right] = \boldsymbol{E}\left[\int_0^T \rho_s + \pi_s(\mu_s - \rho_s) + \lambda_s(\ln(1 + \pi_s\theta_s) - \pi_s\theta_s)\, ds\right],$$

with $\pi \in \mathcal{A}(\mathbb{F})$. Using that the maximum is attained in the strategy given by (11), the solution of the optimal control problem is

$$\mathbb{V}_T^{\mathbb{F}} = \int_0^T \boldsymbol{E}\left[\rho_s - \frac{\mu_s - \rho_s}{\theta_s} + \lambda_s \ln\left(\frac{\lambda_s}{\lambda_s - (\mu_s - \rho_s)/\theta_s}\right)\right] ds, \tag{28}$$

and by using (27) all the terms are well-defined. If $\pi \in \mathcal{A}(\mathbb{G})$, the Itô integral with respect to $\tilde{N}$ is not necessary well-defined, but by Lemma 3.4 we can still use it by taking advantage of the $\mathbb{G}$-semimartingale decomposition. We define the process

$$\widehat{N}_t := \tilde{N}_t - \int_0^t \lambda_s \gamma_s^G ds\,, \quad 0 \leq t \leq T,$$

which is a $\mathbb{G}$-local martingale by Lemma 3.4. Then the dynamics of the wealth process satisfy the following SDE,

$$\frac{dX_t^\pi}{X_{t-}^\pi} = \left((1 - \pi_t)\rho_t + \pi_t\mu_t + \pi_t\theta_t\lambda_t\gamma_t^G\right) dt + \pi_t\theta_t d\widehat{N}_t, \quad X_0 = x_0, \tag{29}$$

and we have the following explicit solution

$$\ln \frac{X_t^\pi}{x_0} = \int_0^t \left(\rho_s + \pi_s(\mu_s - \rho_s) + \lambda_s(1 + \gamma_s^G)\ln(1 + \pi_s\theta_s) - \lambda_s\pi_s\theta_s\right) ds \\ + \int_0^t \ln(1 + \pi_s\theta_s) d\widehat{N}_s.$$

As it is argued in Amendinger et al. (1998), by using the integrability condition of $\ln(1+\pi_t\theta_t)$, the stochastic integral satisfies

$$\boldsymbol{E}\left[\int_0^T \ln(1+\pi_s\theta_s)d\widehat{N}_s\right] = 0.$$

In addition, for any $\mathbb{F}^N$-predictable $\beta = (\beta_t,\ 0 \le t \le T)$,

$$0 = \boldsymbol{E}\left[\int_0^t \beta_s d\tilde{N}_s - \int_0^t \beta_s d\widehat{N}_s\right] = \boldsymbol{E}\left[\int_0^t \beta_s\lambda_s\gamma_s^G ds\right] \tag{30}$$

provided that $\boldsymbol{E}\left[\int_0^t |\beta_s|\lambda_s ds\right] < +\infty$. Then,

$$\boldsymbol{E}\left[\ln\frac{X_T^\pi}{x_0}\right] = \int_0^T \boldsymbol{E}\left[\rho_s + \pi_s(\mu_s-\rho_s) + \lambda_s(1+\gamma_s^G)\ln(1+\pi_s\theta_s) - \lambda_s\pi_s\theta_s\right]ds. \tag{31}$$

In the next proposition we compute the optimal strategy for a $\mathbb{G}$-agent.

Proposition 3.9 *Let $G \in L^2(\boldsymbol{P})$ be an $\mathcal{F}_T^N$-measurable random variable satisfying Assumption 3.1 with information drift γ^G verifying (20) and assume $\sigma_t = 0$, $\theta_t \neq 0$ $\mathrm{d}t \times \mathrm{d}\boldsymbol{P}$ a.s. Then, the optimal portfolio π^G solving the problem (8) with information flow $\mathbb{G}$ is given by*

$$\pi_t^G = \frac{\mu_t-\rho_t}{\lambda_t\theta_t^2 - \theta_t(\mu_t-\rho_t)} + \frac{\lambda_t\gamma_t^G}{\lambda_t\theta_t - (\mu_t-\rho_t)}. \tag{32}$$

Proof We refer to the "Appendix A" for the details of the proof. □

Remark 3.10 Note that the strategy (32) satisfies the admissibility condition (7) thanks to the assumptions (20) and (27). In particular,

$$\begin{aligned}1+\pi_t^G\theta_t &= 1 + \frac{\mu_t-\rho_t}{\lambda_t\theta_t-(\mu_t-\rho_t)} + \frac{\lambda_t\theta_t\gamma_t^G}{\lambda_t\theta_t-(\mu_t-\rho_t)}\\ &= \frac{\lambda_t\theta_t(1+\gamma_t^G)}{\lambda_t\theta_t-(\mu_t-\rho_t)} = \frac{\lambda_t(1+\gamma_t^G)}{\lambda_t-(\mu_t-\rho_t)/\theta_t} > 0 \quad \mathrm{d}t\times\mathrm{d}\boldsymbol{P}-a.s.\end{aligned} \tag{33}$$

The next theorem is one of the main results as it gives the difference of the expected gains under logarithmic utility in a pure jump market of the additional information $G \in \mathcal{F}_T^N$.

Theorem 3.11 *Let $\mathbb{G} \supset \mathbb{F}$ be the initial enlargement with $G \in L^2(\boldsymbol{P})$ an $\mathcal{F}_T^N$-measurable random variable satisfying Assumption 3.1 with information drift γ^G verifying (20), then*

$$\Delta\mathbb{V}_T^{\mathbb{G}} = \int_0^T \boldsymbol{E}\left[\lambda_s(1+\gamma_s^G)\ln\Big(\lambda_s(1+\gamma_s^G)\Big) - \lambda_s\ln\lambda_s\right]ds \ge 0. \tag{34}$$

Proof We refer to the "Appendix A" for the details of the proof. □

4 Mixed Brownian–Poisson market

We extend the results of Sect. 3 to the case of Brownian–Poisson market, i.e., the dynamics of the risky asset are driven by a Brownian motion and a compensated Poisson process and the market coefficients satisfy $\sigma_t, \theta_t \neq 0$ $\mathrm{d}t \times \mathrm{d}\boldsymbol{P}$-a.s. We also maintain the assumption that

λ is deterministic in this section. Using the Clark–Ocone formula stated in (13), we can apply the same representation formula for every $\mathcal{F}_T$-measurable random variable $G \in L^2(\boldsymbol{P})$,

$$G = \boldsymbol{E}[G] + \int_0^T \phi_s dW_s + \int_0^T \varphi_s d\tilde{N}_s,$$

where the processes ϕ and φ are related to the Malliavin derivative. Let $B \in \mathcal{B}(\mathbb{R})$ be a subset and we consider the following PRP

$$\mathbb{1}_{\{G \in B\}} = \boldsymbol{P}^G(B) + \int_0^T \phi_s(B) dW_s + \int_0^T \varphi_s(B) d\tilde{N}_s, \tag{35}$$

given in Theorem 13.28 of Di Nunno et al. (2009). As before, we shall make the following assumptions.

Assumption 4.1 The vector measures ϕ, φ are of bounded variation.

If we reason as in Remark 3.2, we can prove that the previous Assumption 4.1 is satisfied for discrete random variables.

We fix g in the support of G, by reasoning as in the beginning of Sect. 3, see Lemma 3.3, we conclude that there exist two $\mathbb{F}$-adapted processes ζ^g, ψ^g with $g \in \mathrm{Supp}(G)$ such that the following PRP

$$\mathbb{1}_{\{G \in B\}} = \boldsymbol{P}^G(B) + \int_0^T \int_B \zeta_s^g \boldsymbol{P}^G(dg) dW_s + \int_0^T \int_B \psi_s^g \boldsymbol{P}^G(dg) d\tilde{N}_s.$$

Provided the measurability of ζ and ψ in all variables, we can write the following representation

$$\mathbb{1}_{\{G \in dg\}} = \boldsymbol{P}^G(dg) + \int_0^T \zeta_s^g dW_s\, \boldsymbol{P}^G(dg) + \int_0^T \psi_s^g d\tilde{N}_s\, \boldsymbol{P}^G(dg), \tag{36}$$

with $\phi_t^g = \zeta_t^g \boldsymbol{P}(G = g)$ and $\varphi_t^g = \psi_t^g \boldsymbol{P}(G = g)$ in (36). when G is purely atomic. Note that, by using our standing assumptions, we have achieved a representation usually assumed in the literature, see Proposition 4 of Wright et al. (2018) for the Poisson case.

Lemma 4.2 *Let $G \in L^2(\boldsymbol{P})$ be an $\mathcal{F}_T$-measurable random variable satisfying the Assumption 4.1, then the processes α^G and γ^G defined as*

$$\alpha_t^g := \frac{\zeta_t^g \boldsymbol{P}^G(dg)}{\boldsymbol{P}^G(dg|\mathcal{F}_t)}, \quad \gamma_t^g := \frac{\psi_t^g \boldsymbol{P}^G(dg)}{\boldsymbol{P}^G(dg|\mathcal{F}_t)}, \quad 0 \le t < T, \tag{37}$$

satisfy that $W_\cdot - \int_0^\cdot \alpha_s^G ds$ and $\tilde{N}_\cdot - \int_0^\cdot \lambda_s \gamma_s^G ds$ are $\mathbb{G}$-local martingales, provided they are well-defined.

Proof Taking into account that $[W, \tilde{N}]_t^{\mathbb{F}} = 0$, the proof of the following lemma follows from the same lines as the proof of Lemma 3.4. Let $A \in \mathcal{F}_s$ and $B \in \sigma(G)$, then

$$\begin{aligned}
&\boldsymbol{E}[\mathbb{1}_A \mathbb{1}_{\{G \in B\}} (\tilde{N}_t - \tilde{N}_s)] \\
&\quad = \boldsymbol{E}\left[\mathbb{1}_A \left(\boldsymbol{P}^G(B) + \int_0^T \int_B \zeta_s^g \boldsymbol{P}^G(dg) dW_s + \int_0^T \int_B \psi_s^g \boldsymbol{P}^G(dg) d\tilde{N}_s\right)(\tilde{N}_t - \tilde{N}_s)\right] \\
&\quad = \boldsymbol{E}\left[\mathbb{1}_A \int_s^t \int_B \psi_u^g \boldsymbol{P}^G(dg) dN_u\right] = \boldsymbol{E}\left[\mathbb{1}_A \int_s^t \int_B \lambda_u \gamma_u^g \boldsymbol{P}^G(dg|\mathcal{F}_u) du\right] \\
&\quad = \boldsymbol{E}\left[\mathbb{1}_A \mathbb{1}_{\{G \in B\}} \int_s^t \lambda_u \gamma_u^G du\right],
\end{aligned}$$

and the result follows true by mimicking the computation for the Brownian motion W. □

As in the previous section, we keep assuming
$\gamma_s^G > -1 \, \mathrm{d}t \times \mathrm{d}\boldsymbol{P}\text{-}a.s.$
in order to maintain that the stochastic $\mathbb{G}$-intensity $\lambda(1+\gamma^G)$ is positive.

Corollary 4.3 *If* $\mathrm{Supp}(G) = \{0, 1, \ldots, n\}$, *then*

$$\alpha_t^g = \phi_t^g \frac{\mathbb{1}_{\{G=g\}} - \boldsymbol{E}[\mathbb{1}_{\{G=g\}}|\mathcal{F}_t]}{\boldsymbol{V}[\mathbb{1}_{\{G=g\}}|\mathcal{F}_t]}, \quad \gamma_t^g = \varphi_t^g \frac{\mathbb{1}_{\{G=g\}} - \boldsymbol{E}[\mathbb{1}_{\{G=g\}}|\mathcal{F}_t]}{\boldsymbol{V}[\mathbb{1}_{\{G=g\}}|\mathcal{F}_t]},$$

where we consider $\phi_t^g = \zeta_t^g \boldsymbol{P}(G=g)$ *and* $\varphi_t^g = \psi_t^g \boldsymbol{P}(G=g)$ *with* $g \in \mathrm{Supp}(G)$. *In addition, if* $\mathrm{Supp}(G) = \{0, 1\}$, *then*

$$\alpha_t^G = (-1)^{G+1} \phi_t \frac{G - \boldsymbol{E}[G|\mathcal{F}_t]}{\boldsymbol{V}[G|\mathcal{F}_t]}, \quad \gamma_t^G = (-1)^{G+1} \varphi_t \frac{G - \boldsymbol{E}[G|\mathcal{F}_t]}{\boldsymbol{V}[G|\mathcal{F}_t]}. \tag{38}$$

When we consider a $\mathbb{G}$-agent playing with $\pi \in \mathcal{A}(\mathbb{G})$, the Itô integral fails for both the Brownian motion and the Poisson process. Using Lemma 4.2 we can define the following $\mathbb{G}$-local martingales

$$\widehat{W}_t := W_t - \int_0^t \alpha_s^G ds, \quad \widehat{N}_t := \tilde{N}_t - \int_0^t \lambda_s \gamma_s^G ds, \quad 0 \le t \le T \,. \tag{39}$$

The dynamics of the wealth process satisfy the following SDE,

$$\frac{dX_t^\pi}{X_{t-}^\pi} = \left((1-\pi_t)\rho_t + \pi_t\mu_t + \pi_t\sigma_t\alpha_t^G + \pi_t\theta_t\lambda_t\gamma_t^G\right) dt + \pi_t\sigma_t d\widehat{W}_t + \pi_t\theta_t d\widehat{N}_t. \tag{40}$$

To short the notation, we define the following terms

$$\begin{aligned} d_s^G &:= \mu_s - \rho_s + \alpha_s^G\sigma_s - \lambda_s\theta_s, & c_s^G &:= \lambda_s(1+\gamma_s^G), \\ d_s &:= \mu_s - \rho_s - \lambda_s\theta_s, & c_s &:= \lambda_s. \end{aligned}$$

Finally, using the integrability conditions, we can compute the expectation of the stochastic integrals and we get,

$$\boldsymbol{E}\left[\ln \frac{X_T^\pi}{x_0}\right] = \int_0^T \boldsymbol{E}\left[\rho_s + \pi_s d_s^G - \frac{1}{2}\pi_s^2\sigma_s^2 + c_s^G \ln(1+\pi_s\theta_s) - \lambda_s\pi_s\theta_s\right] ds.$$

Proposition 4.4 *The optimal strategy of the problem* (8) *with* $G \in L^2(\boldsymbol{P})$ *being an* $\mathcal{F}_T$-*measurable random variable satisfying Assumption* 4.1 *such that the information drift* γ^G *verifies* (20), *information flow* $\mathbb{G}$ *and both Brownian and Poisson noises is given by,*

$$\pi_s = \frac{1}{2}\left(\frac{d_s^G}{\sigma_s^2} - \frac{1}{\theta_s}\right) + \mathrm{sgn}(\theta_s)\frac{1}{2}\sqrt{\left(\frac{d_s^G}{\sigma_s^2} + \frac{1}{\theta_s}\right)^2 + 4\frac{c_s^G}{\sigma_s^2}}, \quad 0 \le s \le T. \tag{41}$$

Proof We refer to the "Appendix A" for the details of the proof. □

Remark 4.5 If we denote by $\pi_s^W := (\mu_s - \rho_s + \alpha_s^G\sigma_s)/\sigma_s^2$ the optimal strategy for the only Brownian market with additional information then we can write the general strategy for the Poisson-Brownian mixed market as

$$\pi_s = \frac{1}{2}\left(\pi_s^W - \lambda_s\frac{\theta_s}{\sigma_s^2} - \frac{1}{\theta_s}\right) + \mathrm{sgn}(\theta_s)\frac{1}{2}\sqrt{\left(\pi_s^W - \lambda_s\frac{\theta_s}{\sigma_s^2} + \frac{1}{\theta_s}\right)^2 + 4\lambda_s\frac{1+\gamma_s^G}{\sigma_s^2}}.$$

The expected logarithmic gains under $\mathbb{G}$ are computed in the following theorem.

Theorem 4.6 *Let $\mathbb{G} \supset \mathbb{F}$ be the initial enlargement with $G \in L^2(\boldsymbol{P})$ being an $\mathcal{F}_T$-measurable random variable satisfying Assumption 4.1 such that the information drift γ^G verifies (20), then*

$$\mathbb{V}_T^{\mathbb{G}} = \int_0^T \boldsymbol{E}\Bigg[\frac{1}{2}\left(\frac{d_s^G}{\sigma_s}\right)^2 + \frac{1}{2}\frac{d_s^G}{\sigma_s}\operatorname{sgn}(\theta_s)\sqrt{\left(\frac{d_s^G}{\sigma_s} + \frac{\sigma_s}{\theta_s}\right)^2 + 4c_s^G} - \frac{1}{2}\frac{d_s}{\theta_s} + c_s^G \ln c_s^G$$
$$+ \lambda_s \ln\left(2\frac{\theta_s^2}{\sigma_s^2}\right) - c_s^G \ln\left(\sqrt{\left(\frac{d_s^G\theta_s}{\sigma_s^2} + 1\right)^2 + 4\frac{\theta_s^2}{\sigma_s^2}c_s^G} - \left(\frac{d_s^G\theta_s}{\sigma_s^2} + 1\right)\right)\Bigg]ds. \quad (42)$$

Proof We refer to the "Appendix A" for the details of the proof. □

Corollary 4.7 *Under the set-up of Theorem 4.6,*

$$\Delta\mathbb{V}_T^{\mathbb{G}} = \int_0^T \boldsymbol{E}\left[\frac{1}{2}\left(\alpha_s^G\right)^2 + c_s^G \ln c_s^G - \lambda_s \ln \lambda_s + f_s^G - f_s + h_s - h_s^G\right] ds \quad (43)$$

where we define the following terms,

$$f_s^{(G)} := \frac{1}{2}\left(\frac{d_s^{(G)}}{\sigma_s}\right)\operatorname{sgn}(\theta_s)\sqrt{\left(\frac{d_s^{(G)}}{\sigma_s} + \frac{\sigma_s}{\theta_s}\right)^2 + 4c_s^{(G)}}$$
$$h_s^{(G)} := c_s^{(G)} \ln\left(\sqrt{\left(\frac{d_s^{(G)}\theta_s}{\sigma_s^2} + 1\right)^2 + 4\frac{\theta_s^2}{\sigma_s^2}c_s^{(G)}} - \left(\frac{d_s^{(G)}\theta_s}{\sigma_s^2} + 1\right)\right).$$

Remark 4.8 Corollary 4.7 can be seen as a generalization of the additional expected logarithmic utility in Amendinger et al. (1998) for the case of market with jumps. If we assume that the information is only related to the Brownian motion, i.e., $\gamma^G = 0$, the additional gains can be easily computed. For the sake of simplicity, we also choose the market coefficients as $\rho = \mu$, $\lambda = \sigma = 1$ and $\theta_s \in \{-a, a\}$, for any $a \in (0, 1)$, and we obtain

$$\mathbb{V}_T^{\mathbb{G}} - \mathbb{V}_T^{\mathbb{F}} = \int_0^T \boldsymbol{E}\Bigg[\frac{1}{2}\left(\alpha_s^G\right)^2 + \frac{a}{2}\sqrt{\left(1 - \frac{1}{a}\right)^2 + 4}$$
$$+\frac{1}{2}\left(\frac{\alpha_s^G\theta_s - a^2}{a}\right)\sqrt{\left(\frac{\alpha_s^G\theta_s - a^2}{a} + \frac{1}{\theta_s}\right)^2 + 4} + \ln 2a^2$$
$$- \ln\left(\sqrt{(\alpha_s^G\theta_s + 1 - a^2)^2 + 4a^2} - \alpha_s^G\theta_s - 1 + a^2\right)\Bigg] ds. \quad (44)$$

Accordingly, the additional expected logarithmic utility in the mixed-market when the information is only related to the Poisson process, i.e. $\alpha^G = 0$, is

$$\begin{aligned}\mathbb{V}_T^{\mathbb{G}} - \mathbb{V}_T^{\mathbb{F}} = \int_0^T \boldsymbol{E}\Bigg[&(1+\gamma_s^G)\ln\left(1+\gamma_s^G\right) + \frac{a}{2}\sqrt{\left(1-\frac{1}{a}\right)^2 + 4}\\ &-\frac{a}{2}\sqrt{\left(-a+\frac{1}{\theta_s}\right)^2 + 4(1+\gamma_s^G)} + \ln 2a^2\\ &-(1+\gamma_s^G)\ln\left(\sqrt{(1+a^2)^2 + 4a^2\gamma_s^G} - 1 + a^2\right)\Bigg] ds, \end{aligned} \tag{45}$$

under the same simplification of the market coefficients.

Example 4.9 Let $A = (-\infty, a]$ and $B = (-\infty, b]$ be two half-bounded intervals. We define the Bernoulli random variable as the following product indicator,

$$G = \mathbb{1}_{\{W_T \le a\} \times \{N_T \le b\}} = \mathbb{1}_{\{W_T \le a\}} \mathbb{1}_{\{N_T \le b\}}.$$

According to León et al. (2002), thanks to the independence of the Brownian motion and the Poisson process, the Malliavin derivatives in each direction can be easily computed as follows,

$$D_t G = \mathbb{1}_{\{N_T \le b\}} D_t \mathbb{1}_{\{W_T \le a\}}, \quad D_{t,1} G = \mathbb{1}_{\{W_T \le a\}} D_{t,1} \mathbb{1}_{\{N_T \le b\}}$$

so the conditional expectations are calculated as follows,

$$\boldsymbol{E}\left[D_t G | \mathcal{F}_t\right] = \boldsymbol{E}\left[\mathbb{1}_{\{N_T \le b\}} D_t \mathbb{1}_{\{W_T \le a\}} | \mathcal{F}_t\right] = -\boldsymbol{E}\left[\mathbb{1}_{\{N_T \le b\}} \delta_a(W_T) | \mathcal{F}_t\right],$$

where we refer to Section 5 of Bermin (2002) for the generalized Malliavin derivative of the indicator function and the definition of the Dirac delta $\delta_a(\cdot)$. In order to compute the conditional expectation, we consider the following conditional distribution function,

$$F(x, y) := \boldsymbol{P}(W_T \le x, N_T \le y | \mathcal{F}_t) = \int_{-\infty}^{x} \sum_{k=0}^{y} f_{W_T|W_t}(u) p_{N_T|N_t}(k) du,$$

where $f_{W_T|W_t}(u)$ denotes the density function of $(W_T|W_t)$ and $p_{N_T|N_t}(k)$ the probability function of $(N_T|N_t)$. Both of them are well-known, then,

$$\begin{aligned}\boldsymbol{E}\left[D_t G | \mathcal{F}_t\right] &= -\int_{-\infty}^{\infty} \sum_{k=0}^{\infty} f_{W_T|W_t}(u) p_{N_T|N_t}(k) \mathbb{1}_{\{k \in B\}} \delta_a(u) du\\ &= -f_{W_T|W_t}(a) \sum_{k=0}^{b} p_{N_T|N_t}(k)\\ &= -\frac{\exp\left(-\frac{(a-W_t)^2}{2(T-t)}\right)}{\sqrt{2\pi(T-t)}} \sum_{k=0}^{b-N_t} e^{-\Lambda(t,T)} \frac{(\Lambda(t,T))^k}{k!} \mathbb{1}_{\{N_t \le b\}}.\end{aligned}$$

With respect to the Poisson component, we compute the conditional expectation of $D_{t,1}G$ as follows,

$$\begin{aligned}\boldsymbol{E}\left[D_{t,1}G|\mathcal{F}_t\right] &= \boldsymbol{E}\left[\mathbb{1}_{\{W_T\le a\}}D_{t,1}\mathbb{1}_{\{N_T\le b\}}|\mathcal{F}_t\right] = -\boldsymbol{E}\left[\mathbb{1}_{\{W_T\le a\}}\mathbb{1}_{\{N_T=b\}}|\mathcal{F}_t\right]\\ &= -\boldsymbol{E}\left[\mathbb{1}_{\{W_T\le a\}\times\{N_T=b\}}|\mathcal{F}_t\right] = -\boldsymbol{P}\left(W_T\le a,\ N_T=b|\mathcal{F}_t\right)\\ &= -\left(\int_{-\infty}^{a}\frac{\exp\left(-\frac{(x-W_t)^2}{2(T-t)}\right)}{\sqrt{2\pi(T-t)}}dx\right)\left(e^{-\Lambda(t,T)}\frac{(\Lambda(t,T))^{b-N_t}}{(b-N_t)!}\mathbb{1}_{\{N_t\le b\}}\right)\end{aligned}$$

where we have used our Example 3.8, with $b_1=0,\ b_2=b$, in order to compute the Malliavin derivative. Then, the processes α^G and γ^G appearing in (38) are determined. Finally, we deduce the PRP via Clark–Ocone formula,

$$\begin{aligned}&\mathbb{1}_{\{W_T\le a\}\times\{N_T\le b\}} = \boldsymbol{P}(W_T\le a)\boldsymbol{P}(N_T\le b)\\ &-\int_0^T\left(\frac{\exp\left(-\frac{(a-W_t)^2}{2(T-t)}\right)}{\sqrt{2\pi(T-t)}}\right)\left(\sum_{k=0}^{b-N_t}e^{-\Lambda(t,T)}\frac{(\Lambda(t,T))^k}{k!}\right)dW_t\\ &-\int_0^T\left(\int_{-\infty}^{a-W_t}\frac{\exp\left(-\frac{(x-a+W_t)^2}{2(T-t)}\right)}{\sqrt{2\pi(T-t)}}dx\right)\left(e^{-\Lambda(t,T)}\frac{(\Lambda(t,T))^{b-N_t}}{(b-N_t)!}\mathbb{1}_{\{N_t\le b\}}\right)d\tilde{N}_t.\end{aligned}$$

Example 4.10 Let's define

$$M_{s,t} := \sup_{s\le u\le t} W_u,\quad J_{s,t} := \sup_{s\le u\le t}\tilde{N}_u, \tag{46}$$

and $M_t := M_{0,t}$ and $J_t := J_{0,t}$. To short the notation, we define the intervals $A := (a_1, a_2]$ and $B = (b_1, b_2]$. We consider the following example

$$G = \mathbb{1}_{\{M_T\in A\}\times\{J_T\in B\}} = \mathbb{1}_{\{M_T\in A\}}\mathbb{1}_{\{J_T\in B\}} \tag{47}$$

and we proceed as before.

$$\begin{aligned}D_tG &= \mathbb{1}_{\{J_T\in B\}}D_t\mathbb{1}_{\{M_T\in A\}} = \mathbb{1}_{\{J_T\in B\}}D_t\left(\mathbb{1}_{\{M_T\le a_2\}}-\mathbb{1}_{\{M_T\le a_1\}}\right)\\ &= \mathbb{1}_{\{J_T\in B\}}\mathbb{1}_{\{M_t\le M_{t,T}\}}\left(-\delta_{a_2}(M_T)+\delta_{a_1}(M_T)\right),\end{aligned}$$

we refer to Bermin (2002) for a detailed explanation of the Malliavin derivative of the running maximum M_T. Thanks to the independence, we restrict our analysis to the following term

$$\begin{aligned}\boldsymbol{E}\left[D_t\mathbb{1}_{\{M_T\in A\}}|\mathcal{F}_t\right] &= \boldsymbol{E}\left[\mathbb{1}_{\{M_t\le M_{t,T}\}}\left(-\delta_{a_2}(M_{t,T})+\delta_{a_1}(M_{t,T})\right)|\mathcal{F}_t\right]\\ &= \int_0^{+\infty}\mathbb{1}_{\{M_t\le m\}}\left(-\delta_{a_2}(m)+\delta_{a_1}(m)\right)f_t^M(m)dm\\ &= \mathbb{1}_{\{M_t\le a_1\}}f_t^M(a_1)-\mathbb{1}_{\{M_t\le a_2\}}f_t^M(a_2),\end{aligned} \tag{48}$$

being f_t^M the density of the random variable $M_{t,T}$ given $\mathcal{F}_t$, which is equivalent to consider the variable M_{T-t} in the domain $(W_t, +\infty)$, i.e.,

$$f_t^M(m) = \frac{2e^{-\frac{(m-W_t)^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}},\quad m\ge W_t.$$

On the other hand we compute the conditional expectation of the remained Poisson term,

$$\begin{aligned}\boldsymbol{E}\left[\mathbb{1}_{\{J_T\in B\}}|\mathcal{F}_t\right] &= \boldsymbol{P}\left(J_T \in B|\mathcal{F}_t\right) = \boldsymbol{P}\left(\max\{J_t, J_{t,T}\} \in B|\mathcal{F}_t\right)\\ &= \boldsymbol{P}\left(J_t + (J_{t,T} - J_t)^+ \in B|\mathcal{F}_t\right)\\ &= \boldsymbol{P}\left((J_{T-t} - b_t)^+ \in (b_1 - j, b_2 - j]\right)|_{j=J_t},\end{aligned}$$

where $b_t := j - \tilde{N}_t$ and using that $J_{t,T} - \tilde{N}_t$ is independent of $\mathcal{F}_t$. We aim to compute

$$\begin{aligned}\boldsymbol{E}\left[\mathbb{1}_{\{J_T\in B\}}|\mathcal{F}_t\right] = \boldsymbol{P}&\left((J_{T-t} - b_t)^+ > b_1 - j\right)|_{j=J_t}\\ &- \boldsymbol{P}\left((J_{T-t} - b_t)^+ > b_2 - j\right)|_{j=J_t}.\end{aligned} \tag{49}$$

Each one of the probabilities can be computed as

$$\begin{aligned}\boldsymbol{P}\left((J_{T-t} - b_t)^+ > b_1 - j\right)|_{j=J_t} &= \mathbb{1}_{\{b_1-J_t<0\}} + \mathbb{1}_{\{b_1-J_t\geq 0\}}\overline{F}^N_{T-t}(b_1 - \tilde{N}_t)\\ &= 1 + \mathbb{1}_{\{b_1-J_t\geq 0\}}\left(\overline{F}^N_{T-t}(b_1 - \tilde{N}_t) - 1\right)\\ \boldsymbol{P}\left((J_{T-t} - b_t)^+ > b_2 - j\right)|_{j=J_t} &= \mathbb{1}_{\{b_2-J_t<0\}} + \mathbb{1}_{\{b_2-J_t\geq 0\}}\overline{F}^N_{T-t}(b_2 - \tilde{N}_t)\\ &= 1 + \mathbb{1}_{\{b_2-J_t\geq 0\}}\left(\overline{F}^N_{T-t}(b_2 - \tilde{N}_t) - 1\right)\end{aligned}$$

where the survival function is defined as $\overline{F}^N_{T-t}(x) = \boldsymbol{P}(J_{T-t} > x)$ for every $x \geq 0$. See Kuznetsov (2010) for an explicit computation of the distribution of the running supremum. In terms of the distribution function F^N, (49) can be simplified as follows,

$$\begin{aligned}\boldsymbol{E}\left[\mathbb{1}_{\{J_T\in B\}}|\mathcal{F}_t\right] &= \mathbb{1}_{\{b_1-J_t\geq 0\}}\left(\overline{F}^N_{T-t}(b_1 - \tilde{N}_t) - 1\right)\\ &\quad - \mathbb{1}_{\{b_2-J_t\geq 0\}}\left(\overline{F}^N_{T-t}(b_2 - \tilde{N}_t) - 1\right)\\ &= \mathbb{1}_{\{b_2-J_t\geq 0\}}F^N_{T-t}(b_2 - \tilde{N}_t) - \mathbb{1}_{\{b_1-J_t\geq 0\}}F^N_{T-t}(b_1 - \tilde{N}_t).\end{aligned} \tag{50}$$

Then, taking into account (48) and (50) the process α^G is fully determined by (38). We proceed in the same way in order to compute $\boldsymbol{E}[D_{t,1}G|\mathcal{F}_t]$. Using the operator Ψ given in (23), we can compute the following Malliavin derivative

$$D_{t,1}\mathbb{1}_{\{J_T\in B\}} = \mathbb{1}_{\{\max\{J_t,1+J_{t,T}\}\in B\}} - \mathbb{1}_{\{J_T\in B\}}$$

where the second term has been calculated in (50). For the first one we have

$$\begin{aligned}\boldsymbol{E}[\mathbb{1}_{\{\max\{J_t,1+J_{t,T}\}\in B\}}|\mathcal{F}_t] &= \boldsymbol{P}(\max\{J_t, 1 + J_{t,T}\} \in B|\mathcal{F}_t)\\ &= \mathbb{1}_{\{b_2-J_t\geq 0\}}F^N_{T-t}(b_2 - \tilde{N}_t - 1)\\ &\quad - \mathbb{1}_{\{b_1-J_t\geq 0\}}F^N_{T-t}(b_1 - \tilde{N}_t - 1)\end{aligned}$$

where we have omitted some steps as they were similar to ones shown before. Finally

$$\boldsymbol{E}\left[\mathbb{1}_{\{M_T\in A\}}|\mathcal{F}_t\right] = \mathbb{1}_{\{a_2-M_t\geq 0\}}F^W_{T-t}(a_2 - W_t) - \mathbb{1}_{\{a_1-M_t\geq 0\}}F^W_{T-t}(a_1 - W_t), \tag{51}$$

where in this case $\overline{F}^W_t(y) = 2(1 - \Phi(y/\sqrt{t}))$ and again the process γ^G is determined by (38).

5 Stochastic intensity in Brownian–Poisson market

In this section, we assume that $\tilde{N}$ is an inhomogeneous Poisson process with stochastic intensity. We introduce $\mathcal{F}_t^\lambda := \sigma(\lambda_s, s \le t)$ and we assume that $\mathcal{F}_T^\lambda \subset \mathcal{F}_T$. We will look at examples where $\mathcal{F}_T^\lambda \subset \mathcal{F}_T^W$, allowing some dependence of λ with respect to W, however more general cases may be considered, we prefer not to include them for easy of presentation. In particular, following Section 8.4.2 of Jeanblanc et al. (2009), we assume that

$$\boldsymbol{E}\left[\int_0^T \phi_s dN_s\right] = \boldsymbol{E}\left[\int_0^T \phi_s \lambda_s ds\right],$$

being ϕ any $\mathbb{F} \vee \mathbb{F}^\lambda$-adapted process and $\tilde{N}_\cdot = N_\cdot - \int_0^\cdot \lambda_s ds$ is an $\mathbb{F} \vee \mathbb{F}^\lambda$-martingale. The standing assumption of the section is that G satisfies the following representation for any $B \in \sigma(G)$.

Assumption 5.1

$$\mathbb{1}_{\{G \in B\}} = \boldsymbol{P}^G(B|\mathcal{F}_T^\lambda) + \int_0^T \int_B \zeta_s^g \boldsymbol{P}^G(dg) dW_s + \int_0^T \int_B \psi_s^g \boldsymbol{P}^G(dg) d\tilde{N}_s, \tag{52}$$

and there exist some $\mathbb{F} \vee \mathcal{F}_T^\lambda$-adapted processes $\bar{\alpha}^\lambda$ and $\bar{\gamma}^\lambda$ such that $W_\cdot - \int_0^\cdot \bar{\alpha}_s^\lambda ds$ and $\tilde{N}_\cdot - \int_0^\cdot \bar{\gamma}_s^\lambda ds$ are $\mathbb{F} \vee \mathcal{F}_T^\lambda$-martingales, provided ζ and ψ are as usual measurable in all variables.

From the Assumption 5.1 we conclude that with $\widehat{\alpha}_t^\lambda := \boldsymbol{E}[\bar{\alpha}_t^\lambda | \mathcal{F}_t \vee \mathcal{F}_t^\lambda]$, $0 \le t \le T$, the process $W_\cdot - \int_0^\cdot \widehat{\alpha}_s^\lambda ds$ is an $\mathbb{F} \vee \mathbb{F}^\lambda$-martingale.

Next proposition gives the equivalent of Lemma 4.2 for the case of λ not adapted to the Poisson filtration. Note that additional terms appear in the information drifts induced by the unpredictability of λ.

Proposition 5.2 *Let $G \in L^2(\boldsymbol{P})$ be an $\mathcal{F}_T$-measurable random variable satisfying Assumption 5.1 and suppose that $\mathbb{F} \vee \mathbb{F}^\lambda \subset \mathbb{G}$ holds. Then, for any $(t, g) \in [0, T) \times \mathrm{Supp}(G)$, we define*

$$\alpha_t^g := \frac{\zeta_t^g \boldsymbol{P}^G(dg)}{\boldsymbol{P}^G(dg|\mathcal{F}_t)}, \quad \widetilde{\alpha}_t^\lambda := \frac{\bar{\alpha}_t^\lambda \boldsymbol{P}^G(dg|\mathcal{F}_T^\lambda)}{\boldsymbol{P}^G(dg|\mathcal{F}_t \vee \mathcal{F}_T^\lambda)},$$
$$\gamma_t^g := \frac{\psi_t^g \boldsymbol{P}^G(dg)}{\boldsymbol{P}^G(dg|\mathcal{F}_t)}, \quad \widetilde{\gamma}_t^\lambda := \frac{\bar{\gamma}_t^\lambda \boldsymbol{P}^G(dg|\mathcal{F}_T^\lambda)}{\boldsymbol{P}^G(dg|\mathcal{F}_t \vee \mathcal{F}_T^\lambda)}, \quad \widehat{\gamma}_t^g := \frac{\zeta_t^g \widehat{\alpha}_t^\lambda \boldsymbol{P}^G(dg)}{\boldsymbol{P}^G(dg|\mathcal{F}_t \vee \mathcal{F}_t^\lambda)},$$

such that the processes

$$W_\cdot - \int_0^\cdot \left(\alpha_s^G + \widetilde{\alpha}_s^\lambda\right) ds,$$
$$N_\cdot - \int_0^\cdot \left(\lambda_v + \widetilde{\gamma}_v^\lambda + \lambda_v \gamma_v^G - \lambda_v \boldsymbol{E}\left[\int_v^T \widehat{\gamma}_u^G du | \mathcal{G}_v\right]\right) dv$$

are $\mathbb{G}$-local martingales, provided the integrands, i.e. the information drifts, are well-defined.

Proof We refer to the "Appendix A" for the details of the proof. □

Example 5.3 We consider a stochastic intensity given by

$$\lambda_t = 1 + \mathbb{1}_{\{W_T \ge 0\}}, \quad 0 \le t \le T.$$

We introduce $G = \mathbb{1}_{\{W_T \geq 0\}}$ and note that $\mathcal{F}_t^\lambda = \mathcal{F}_T^\lambda = \sigma(G)$, for any $0 \leq t \leq T$ and we conclude that $\widehat{\alpha}^\lambda = \bar{\alpha}^\lambda = \widetilde{\alpha}^\lambda$ and $\widetilde{\gamma}^\lambda = \bar{\gamma} = 0$. Let's define the initial enlargement $\mathbb{G} = \mathbb{F} \vee \sigma(G)$. Then the representation (5.1) is reduced to

$$G = \boldsymbol{E}[G|\mathcal{F}_T^\lambda]$$

so we conclude that $\zeta^g = \psi^g = 0$ and then $\gamma^G = \widehat{\gamma}^G = \alpha^G = 0$. We use the well-known PRP

$$\mathbb{1}_{\{W_T \geq 0\}} = \boldsymbol{P}(W_T \geq 0) + \int_0^T \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{W_t^2}{2(T-t)}\right) dW_t,$$

from which we conclude, see Proposition 4.7 in D'Auria and Salmerón (2020) for details, that

$$\widetilde{\alpha}_t^\lambda = \frac{(-1)^\lambda}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{W_t^2}{2(T-t)}\right), \quad \lambda \in \{1, 2\},$$

and the information drifts appearing in Proposition 5.2 are determined. By concatenating the information drifts, the technique can be easily generalized to the case when

$$\lambda_t = 1 + \mathbb{1}_{\{W_{t_k} \geq W_{t_{k-1}}\}}, \quad t_{k-1} \leq t \leq t_k, \quad k \in 1, \ldots, n,$$

$G_k = \mathbb{1}_{\{W_{t_k} \geq W_{t_{k-1}}\}}$ and the enlargement $\mathbb{G} = \mathbb{F} \vee \sigma(G_1, \ldots, G_n)$.

Conclusion

In this paper we show how to incorporate anticipative information in a filtration generated by a Brownian motion and a Poisson process. We compute the compensators in a general framework of additional information (see Lemma 4.2), and then we focus on the purely atomic case to consider more explicit examples (see Corollary 4.3). In particular, we study the case in which a $\mathbb{G}$-agent knows if the final pair of random variables (W_T, N_T) are within a certain rectangular region, as well as the case that considers a similar type of information about the pair of running maximums (M_T, J_T), see Examples 4.9 and 4.10. We also study the case when λ is not $\mathbb{F}$-adapted in Sect. 5 and give an example when it is an anticipative function of W, see Example 5.3.

When the dynamics of the risky asset are driven by the Poisson process only, we give the exact value of the additional information in terms of an entropy similar to the corresponding continuous case, see Theorem 3.11. Finally, we extend that formula for the mixed case in Theorem 4.6 and Corollary 4.7.

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Data availability Not applicable.

Declarations

Conflict of interest The authors declare no conflicts of interest in this paper.

Appendix A

Proof of Proposition 3.9 We proceed by maximizing point-wise the expression given by (31). After defining the integrand

$$I(\pi) := \rho_s + \pi(\mu_s - \rho_s) + \lambda_s(1+\gamma_s^G)\ln(1+\pi\theta_s) - \lambda_s\pi\theta_s$$

we look for each $\omega \in \Omega$, the strategy π_s^G that solves for each time $s \in [0, T]$ the equation $0 = I'(\pi_s^G)$. Therefore

$$0 = \mu_s - \rho_s + \frac{\lambda_s(1+\gamma_s^G)\theta_s}{1+\pi_s^G\theta_s} - \lambda_s\theta_s$$

and solving for π_s^G the result holds true since $I''(\pi_s^G) < 0$. $\square$

Proof of Theorem 3.11 By (31) we have

$$\mathbb{V}_T^{\mathbb{G}} = \int_0^T \boldsymbol{E}\left[\rho_s + \pi_s^G(\mu_s - \rho_s) + \lambda_s(1+\gamma_s^G)\ln(1+\pi_s^G\theta_s) - \lambda_s\pi_s^G\theta_s\right]ds, \quad (53)$$

being π^G the process defined in the Proposition 3.9. We compute the following terms

$$\pi_s^G(\mu_s - \rho_s - \lambda_s\theta_s) = -\lambda_s\gamma_s^G - \frac{\mu_s - \rho_s}{\theta_s}$$
$$\ln\left(1+\pi_s^G\theta_s\right) = \ln\left(\frac{\lambda_s(1+\gamma_s^G)}{\lambda_s - (\mu_s - \rho_s)/\theta_s}\right).$$

By substituting them in (53) we have the following expression,

$$\mathbb{V}_T^{\mathbb{G}} = \int_0^T \boldsymbol{E}\left[\rho_s - \lambda_s\gamma_s^G - \frac{\mu_s - \rho_s}{\theta_s} + \lambda_s(1+\gamma_s^G)\ln\left(\frac{\lambda_s(1+\gamma_s^G)}{\lambda_s - (\mu_s - \rho_s)/\theta_s}\right)\right]ds$$

Finally, we compute the difference as follows

$$\mathbb{V}_T^{\mathbb{G}} - \mathbb{V}_T^{\mathbb{F}} = \int_0^T \boldsymbol{E}\left[-\lambda_s \gamma_s^G + \lambda_s (1+\gamma_s^G) \ln\left(\frac{\lambda_s (1+\gamma_s^G)}{\lambda_s - (\mu_s - \rho_s)/\theta_s}\right)\right.$$
$$\left. -\lambda_s \ln\left(\frac{\lambda_s}{\lambda_s - (\mu_s - \rho_s)/\theta_s}\right)\right] ds$$
$$= \int_0^T \boldsymbol{E}\left[-\lambda_s \gamma_s^G + \lambda_s (1+\gamma_s^G) \ln\left(1+\gamma_s^G\right)\right] ds \geq 0$$

where we have applied (30) in order to simplify the expression as follows

$$\int_0^T \boldsymbol{E}\left[\gamma_s^G \ln\left(\frac{\lambda_s}{\lambda_s - (\mu_s - \rho_s)/\theta_s}\right)\right] ds = 0.$$

It can be checked that the function $h(x, y) = -xy + x(1+y)\ln(1+y)$ is positive when $x > 0$ and $y > -1$ and we get the non-negativity of the additional gains. Finally by adding and subtracting the term $\lambda_s(1+\gamma_s^G)\ln\lambda_s$ and by applying (30) to the term $-\lambda_s\gamma_s^G$ and $\lambda_s\gamma_s^G \ln\lambda_s$ we get the result. □

Proof of Proposition 4.4 We proceed by maximizing point-wise. We define

$$I(\pi) := \rho_s + \pi d_s^G - \frac{1}{2}\pi^2\sigma_s^2 + c_s^G \ln(1+\pi\theta_s)$$

and we consider the first order condition $0 = I'(\pi)$ as follows,

$$0 = d_s^G - \pi\sigma_s^2 + c_s^G \frac{\theta_s}{1+\pi\theta_s}. \tag{54}$$

Then we derive the next equation,

$$0 = \left(\sigma_s^2\theta_s\right)\pi^2 + \left(\sigma_s^2 - \theta_s d_s^G\right)\pi - \left(d_s^G + \theta_s c_s^G\right)$$

with the following solutions,

$$\pi^{\pm} = \frac{\theta_s d_s^G - \sigma_s^2 \pm \sqrt{\left(\theta_s d_s^G - \sigma_s^2\right)^2 + 4\sigma_s^2\theta_s\left(d_s^G + \theta_s c_s^G\right)}}{2\sigma_s^2\theta_s}$$
$$= \frac{1}{2}\left(\frac{d_s^G}{\sigma_s^2} - \frac{1}{\theta_s}\right) \pm \frac{\operatorname{sgn}(\theta_s)}{2}\sqrt{\left(\frac{d_s^G}{\sigma_s^2} + \frac{1}{\theta_s}\right)^2 + 4\frac{c_s^G}{\sigma_s^2}},$$

where in the last step we have used arithmetic computations. It can be verified that $I''(0) < 0$ and the pair of strategies $\pi^{\pm}$ are maximum if and only if they are admissible. Then we need to check if the condition $1 + \pi^{\pm}\theta_s > 0$ is satisfied. We rewrite the pair as follows,

$$1 + \pi^{\pm}\theta_s = \frac{1}{2}\left(\frac{d_s^G\theta_s}{\sigma_s^2} + 1\right) \pm \frac{1}{2}\sqrt{\left(\frac{d_s^G\theta_s}{\sigma_s^2} + 1\right)^2 + 4\theta_s^2\frac{c_s^G}{\sigma_s^2}} \tag{55}$$

and we deduce that the unique optimal solution is $\pi^+ = (\pi_t^+,\, 0 \leq t \leq T)$, then the result holds with the fact that $I''(\pi^+) < 0$. □

Proof of Theorem 4.6 We recall the notation of d_s^G and c_s^G in the previous computation. Using (54),

$$\begin{aligned}
\mathbb{V}_T^{\mathbb{G}} &= \int_0^T \boldsymbol{E}\Big[\rho_s + \pi_s d_s^G - \frac{\pi_s^2\sigma_s^2}{2} + c_s^G \ln(1+\pi_s\theta_s)\Big]ds \\
&= \int_0^T \boldsymbol{E}\Big[\rho_s + \frac{\pi_s^2\sigma_s^2}{2} - c_s^G\left(\frac{\pi_s\theta_s}{1+\pi_s\theta_s} + \ln\left(\frac{1}{1+\pi_s\theta_s}\right)\right)\Big]ds \\
&= \int_0^T \boldsymbol{E}\Big[\rho_s + \frac{\pi_s^2\sigma_s^2}{2} - c_s^G + c_s^G\left(\frac{1}{1+\pi_s\theta_s} - \ln\left(\frac{1}{1+\pi_s\theta_s}\right)\right)\Big]ds
\end{aligned}$$

Note that, using (55)

$$\begin{aligned}
(\pi_s\sigma_s)^2 &= \frac{1}{2}\left(\frac{d_s^G}{\sigma_s}\right)^2 + \frac{1}{2}\left(\frac{\sigma_s}{\theta_s}\right)^2 + c_s^G \\
&\quad + \frac{1}{2}\left(\frac{d_s^G}{\sigma_s} - \frac{\sigma_s}{\theta_s}\right)\operatorname{sgn}(\theta_s)\sqrt{\left(\frac{d_s^G}{\sigma_s} + \frac{\sigma_s}{\theta_s}\right)^2 + 4c_s^G} \\
\frac{c_s^G}{1+\pi_s\theta_s} &= \frac{c_s^G}{2}\,\frac{\sqrt{\left(\frac{d_s^G\theta_s}{\sigma_s^2}+1\right)^2 + 4\frac{\theta_s^2}{\sigma_s^2}c_s^G} - \left(\frac{d_s^G\theta_s}{\sigma_s^2}+1\right)}{\frac{\theta_s^2}{\sigma_s^2}c_s^G} \\
&= \frac{\sigma_s^2}{2\theta_s^2}\left(\sqrt{\left(\frac{d_s^G\theta_s}{\sigma_s^2}+1\right)^2 + 4\frac{\theta_s^2}{\sigma_s^2}c_s^G} - \left(\frac{d_s^G\theta_s}{\sigma_s^2}+1\right)\right) \\
&= \frac{\sigma_s}{2\theta_s}\left(\operatorname{sgn}(\theta_s)\sqrt{\left(\frac{d_s^G}{\sigma_s}+\frac{\sigma_s}{\theta_s}\right)^2 + 4c_s^G} - \left(\frac{d_s^G}{\sigma_s}+\frac{\sigma_s}{\theta_s}\right)\right) \\
-c_s^G\ln\left(\frac{1}{1+\pi_s\theta_s}\right) &= c_s^G\ln c_s^G + c_s^G\ln\left(2\frac{\theta_s^2}{\sigma_s^2}\right) \\
&\quad - c_s^G\ln\left(\sqrt{\left(\frac{d_s^G\theta_s}{\sigma_s^2}+1\right)^2 + 4\frac{\theta_s^2}{\sigma_s^2}c_s^G} - \left(\frac{d_s^G\theta_s}{\sigma_s^2}+1\right)\right)
\end{aligned}$$

By rearranging all the terms we get

$$\begin{aligned}
\mathbb{V}_T^{\mathbb{G}} = \int_0^T \boldsymbol{E}\Bigg[&\frac{1}{2}\left(\frac{d_s^G}{\sigma_s}\right)^2 + \frac{1}{2}\frac{d_s^G}{\sigma_s}\operatorname{sgn}(\theta_s)\sqrt{\left(\frac{d_s^G}{\sigma_s}+\frac{\sigma_s}{\theta_s}\right)^2 + 4c_s^G} - \frac{d_s^G}{2\theta_s} + c_s^G\ln c_s^G \\
&+ c_s^G\ln\left(2\frac{\theta_s^2}{\sigma_s^2}\right) - c_s^G\ln\left(\sqrt{\left(\frac{d_s^G\theta_s}{\sigma_s^2}+1\right)^2 + 4\frac{\theta_s^2}{\sigma_s^2}c_s^G} - \left(\frac{d_s^G\theta_s}{\sigma_s^2}+1\right)\right)\Bigg]ds,
\end{aligned}$$

and by simplifying the terms $-\frac{d_s^G}{2\theta_s}$ and $c_s^G\ln\left(2\frac{\theta_s^2}{\sigma_s^2}\right)$ because of (30), the result follows true. ◻

Proof of Proposition 5.2 Let $A \in \mathcal{F}_s$ and $B \in \sigma(G)$, then by (35) we have

$$\begin{aligned}
&\boldsymbol{E}[\mathbb{1}_A \mathbb{1}_{\{G\in B\}}(\tilde{N}_t - \tilde{N}_s)] \\
&\quad = \boldsymbol{E}\left[\mathbb{1}_A \left(\boldsymbol{P}^G(B|\mathcal{F}_T^\lambda) + \int_0^T \int_B \zeta_s^g \boldsymbol{P}^G(dg)dW_s + \int_0^T \int_B \psi_s^g \boldsymbol{P}^G(dg)d\tilde{N}_s\right)(\tilde{N}_t - \tilde{N}_s)\right].
\end{aligned}$$

For the first term we have

$$\begin{aligned}
\boldsymbol{E}\left[\mathbb{1}_A \boldsymbol{P}^G(B|\mathcal{F}_T^\lambda)(\tilde{N}_t - \tilde{N}_s)\right] &= \boldsymbol{E}\left[\mathbb{1}_A \boldsymbol{P}^G(B|\mathcal{F}_T^\lambda)\boldsymbol{E}[(\tilde{N}_t - \tilde{N}_s)|\mathcal{F}_s \vee \mathcal{F}_T^\lambda]\right] \\
&= \boldsymbol{E}\left[\mathbb{1}_A \boldsymbol{P}^G(B|\mathcal{F}_T^\lambda)\boldsymbol{E}[\int_s^t \bar{\gamma}_u^\lambda \, du|\mathcal{F}_s \vee \mathcal{F}_T^\lambda]\right] \\
&= \boldsymbol{E}\left[\mathbb{1}_A \boldsymbol{P}^G(B|\mathcal{F}_T^\lambda)\int_s^t \bar{\gamma}_u^\lambda \, du\right] \\
&= \boldsymbol{E}\left[\mathbb{1}_A \int_s^t \int_B \bar{\gamma}_u^\lambda \boldsymbol{P}^G(dg|\mathcal{F}_T^\lambda)\, du\right] \\
&= \boldsymbol{E}\left[\mathbb{1}_A \int_s^t \int_B \widetilde{\gamma}_u^\lambda \boldsymbol{P}^G(dg|\mathcal{F}_u \vee \mathcal{F}_T^\lambda)\, du\right] \\
&= \boldsymbol{E}\left[\mathbb{1}_A \int_s^t \boldsymbol{E}[\mathbb{1}_{\{G\in B\}}\widetilde{\gamma}_u^\lambda|\mathcal{F}_u \vee \mathcal{F}_T^\lambda]\, du\right] \\
&= \boldsymbol{E}\left[\mathbb{1}_A \mathbb{1}_{\{G\in B\}} \int_s^t \widetilde{\gamma}_u^\lambda \, du\right].
\end{aligned}$$

With respect to the second term we use the fact that $\widehat{W}_\cdot := W_\cdot - \int_0^\cdot \widehat{\alpha}_u^G ds$ is a $\mathbb{F} \vee \mathbb{F}^\lambda$-martingale, then

$$\begin{aligned}
&\boldsymbol{E}\left[\mathbb{1}_A \int_0^T \int_B \zeta_s^g \boldsymbol{P}^G(dg)dW_s(\tilde{N}_t - \tilde{N}_s)\right] \\
&\quad = \boldsymbol{E}\left[\mathbb{1}_A \left(\int_0^T \int_B \zeta_u^g \boldsymbol{P}^G(dg)d\widehat{W}_u + \int_0^T \int_B \zeta_u^g \boldsymbol{P}^G(dg)\widehat{\alpha}_u^\lambda du\right)(\tilde{N}_t - \tilde{N}_s)\right] \\
&\quad = \boldsymbol{E}\left[\mathbb{1}_A \int_0^T \int_B \zeta_u^g \boldsymbol{P}^G(dg)\widehat{\alpha}_u^\lambda du\,(\tilde{N}_t - \tilde{N}_s)\right],
\end{aligned}$$

where in the second equality we applied the Itô isometry for $\mathbb{F} \vee \mathbb{F}^\lambda$-martingales. The latter integral in $[0, T]$ is going to be split in $[0, s]$, $[s, t]$ and $[t, T]$. To short the notation we introduce the $\mathbb{F}$-adapted process $Y_u^B := \int_B \zeta_u^g \boldsymbol{P}^G(dg)$, $u \in [0, T]$. For the first interval we have,

$$\boldsymbol{E}\left[\mathbb{1}_A \int_0^s Y_u^B \widehat{\alpha}_u^\lambda du\,(\tilde{N}_t - \tilde{N}_s)\right] = \boldsymbol{E}\left[\int_s^t \mathbb{1}_A \int_0^s Y_u^B \widehat{\alpha}_u^\lambda du\, d\tilde{N}_v\right] = 0,$$

where we used that $\tilde{N}$ is a $\mathbb{F} \vee \mathbb{F}^\lambda$ martingale and it is the expectation of a well-define Itô integral. For the interval $[t, T]$,

$$\begin{aligned}
\boldsymbol{E}\left[\mathbb{1}_A \int_t^T Y_u^B \widehat{\alpha}_u^\lambda du\,(\tilde{N}_t - \tilde{N}_s)\right] &= -\boldsymbol{E}\left[\mathbb{1}_A \int_s^t \int_t^T \lambda_v Y_u^B \widehat{\alpha}_u^\lambda \, du dv\right] \\
&= -\boldsymbol{E}\left[\mathbb{1}_A \int_s^t \left(\int_v^T - \int_v^t\right) \lambda_v Y_u^B \widehat{\alpha}_u^\lambda \, du dv\right] \qquad (56)
\end{aligned}$$

and we continue with the first integral.

$$\boldsymbol{E}\left[\mathbb{1}_A\int_s^t\int_v^T \lambda_v Y_u^B \widehat{\alpha}_u^\lambda\, du dv\right] = \boldsymbol{E}\left[\mathbb{1}_A\int_s^t\int_v^T \lambda_v \int_B \widehat{\gamma}_u^g \boldsymbol{P}^G(dg|\mathcal{F}_u,\mathcal{F}_u^\lambda)\, du dv\right]$$
$$= \boldsymbol{E}\left[\mathbb{1}_A\int_s^t\int_v^T \lambda_v \boldsymbol{E}[\mathbb{1}_{\{G\in B\}}\widehat{\gamma}_u^G|\mathcal{F}_u,\mathcal{F}_u^\lambda]\, du dv\right] = \boldsymbol{E}\left[\mathbb{1}_A\mathbb{1}_{\{G\in B\}}\int_s^t\int_v^T \lambda_v\widehat{\gamma}_u^G\, du dv\right].$$

Next we prove that the interval $[s,t]$ simplifies with the second integral in (56). Indeed,

$$\boldsymbol{E}\left[\mathbb{1}_A\int_s^t Y_u^B\widehat{\alpha}_u^\lambda du\,(\tilde{N}_t-\tilde{N}_s)\right] = \boldsymbol{E}\left[\mathbb{1}_A\int_s^t \left(\int_s^v Y_u^B\widehat{\alpha}_u^\lambda du + \int_v^t Y_u^B\widehat{\alpha}_u^\lambda du\right) d\tilde{N}_v\right]$$
$$= \boldsymbol{E}\left[\mathbb{1}_A\int_s^t\int_v^t Y_u^B\widehat{\alpha}_u^\lambda du d\tilde{N}_v\right] = \boldsymbol{E}\left[\mathbb{1}_A\int_s^t (\tilde{N}_u-\tilde{N}_s)Y_u^B\widehat{\alpha}_u^\lambda\, du\right]$$
$$= -\boldsymbol{E}\left[\mathbb{1}_A\int_s^t\int_s^u \lambda_v dv Y_u^B\widehat{\alpha}_u^\lambda\, du\right] = -\boldsymbol{E}\left[\mathbb{1}_A\int_s^t\int_v^t \lambda_v Y_u^B\widehat{\alpha}_u^\lambda\, du dv\right], \tag{57}$$

where we used that $N_u - N_s$, $s\le u$, is $\mathbb{F}$-adapted and for any $\mathbb{F}$-adapted process $X = (X_t,\ 0\le t\le T)$ we have

$$0 = \boldsymbol{E}\left[\int_0^T X_u\, d\widehat{W}_u - \int_0^T X_u\, dW_u\right] = \boldsymbol{E}\left[\int_0^T X_u\widehat{\alpha}_u^\lambda\, du\right].$$

We consider now

$$\boldsymbol{E}\left[\mathbb{1}_A\int_0^T\int_B \psi_s^g\boldsymbol{P}^G(dg)d\tilde{N}_s(\tilde{N}_t-\tilde{N}_s)\right] = \boldsymbol{E}\left[\mathbb{1}_A\int_s^t\int_B \psi_u^g\boldsymbol{P}^G(dg)dN_u\right]$$
$$= \boldsymbol{E}\left[\mathbb{1}_A\int_s^t\int_B \lambda_u\gamma_u^g\boldsymbol{P}^G(dg|\mathcal{F}_u)du\right] = \boldsymbol{E}\left[\mathbb{1}_A\mathbb{1}_{\{G\in B\}}\int_s^t \lambda_u\gamma_u^G du\right],$$

where in the first equality we used the Itô isometry and the fact that $[\tilde{N}_\cdot]^{\mathbb{F}\vee\mathbb{F}^\lambda} = N_\cdot$, in the second one we applied the definition of γ^G while in the third one we used the properties of the conditional expectation operator. By putting together all the terms and taking $\mathcal{G}_t$-conditional expectation, we conclude that

$$N_\cdot - \int_0^\cdot\left(\lambda_v + \widetilde{\gamma}_v^\lambda + \lambda_v\gamma_v^G - \lambda_v\boldsymbol{E}\left[\int_v^T\widehat{\gamma}_u^G du|\mathcal{G}_v\right]\right)dv$$

is a $\mathbb{G}$-local martingale. To finish the proof, we sketch the computation in order to calculate the information drift of W.

$$\boldsymbol{E}[\mathbb{1}_A\mathbb{1}_{\{G\in B\}}(W_t-W_s)]$$
$$= \boldsymbol{E}\left[\mathbb{1}_A\left(\boldsymbol{P}^G(B|\mathcal{F}_T^\lambda) + \int_0^T\int_B \zeta_s^g\boldsymbol{P}^G(dg)dW_s + \int_0^T\int_B \psi_s^g\boldsymbol{P}^G(dg)d\tilde{N}_s\right)(W_t-W_s)\right].$$

For the case of the two first terms we have

$$\boldsymbol{E}\left[\mathbb{1}_A\boldsymbol{P}^G(B|\mathcal{F}_T^\lambda)(W_t-W_s)\right] = \boldsymbol{E}\left[\mathbb{1}_A\boldsymbol{P}^G(B|\mathcal{F}_T^\lambda)\int_s^t\bar{\alpha}_u^\lambda du\right]$$
$$= \boldsymbol{E}\left[\mathbb{1}_A\mathbb{1}_{\{G\in B\}}\int_s^t\widetilde{\alpha}_u^\lambda du\right],$$
$$\boldsymbol{E}\left[\mathbb{1}_A\int_0^T\int_B \zeta_s^g\boldsymbol{P}^G(dg)dW_s(W_t-W_s)\right] = \boldsymbol{E}\left[\mathbb{1}_A\mathbb{1}_{\{G\in B\}}\int_s^t\alpha_u^G du\right].$$

By defining $Z_u^B := \int_B \psi_s^g \boldsymbol{P}^G(dg)$, we proceed entirely as in the Poisson case. We omit some details that can be consulted in the first part of the proof.

$$\boldsymbol{E}\left[\mathbb{1}_A \int_0^T Z_v^B d\tilde{N}_v (W_t - W_s)\right] = \boldsymbol{E}\left[\mathbb{1}_A \int_0^T Z_v^B d\tilde{N}_v \left(\widehat{W}_t - \widehat{W}_s + \int_s^t \widehat{\alpha}_u^\lambda du\right)\right]$$
$$= \boldsymbol{E}\left[\mathbb{1}_A \int_0^T Z_v^B d\tilde{N}_v \int_s^t \widehat{\alpha}_u^\lambda du\right] = \boldsymbol{E}\left[\mathbb{1}_A \left(\int_0^s + \int_s^t\right) Z_v^B d\tilde{N}_v \int_s^t \widehat{\alpha}_u^\lambda du\right].$$

For the first integral we have

$$\boldsymbol{E}\left[\mathbb{1}_A \int_0^s Z_v^B d\tilde{N}_v \int_s^t \widehat{\alpha}_u^\lambda du\right] = -\boldsymbol{E}\left[\mathbb{1}_A \int_s^t \left(\int_0^u - \int_s^u\right) Z_v^B \lambda_v \widehat{\alpha}_u^\lambda \, dv du\right].$$

As in the Poisson case, the second integral is simplified with the $[s, t]$ interval. So we deal with the first one,

$$\boldsymbol{E}\left[\mathbb{1}_A \int_s^t \int_0^u Z_v^B \lambda_v \widehat{\alpha}_u^\lambda \, dv du\right] = \boldsymbol{E}\left[\mathbb{1}_A \int_s^t \int_0^u \boldsymbol{E}[\mathbb{1}_{\{G\in B\}} \lambda_v \gamma_v^G | \mathcal{F}_v] \widehat{\alpha}_u^\lambda \, dv du\right] = 0.$$

So we conclude that the process

$$W_\cdot - \int_0^\cdot \left(\widetilde{\alpha}_v^\lambda + \alpha_v^G\right) dv$$

is a $\mathbb{G}$-martingale and the result holds true. □

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