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Research Article

Antiperiodic Solutions for Liénard-Type Differential Equation with p-Laplacian Operator

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The existence of antiperiodic solutions for Liénard-type and Duffing-type differential equations with p-Laplacian operator has been studied by using degree theory. The results obtained improve and enrich some known works to some extent.

1. Introduction

Antiperiodic problems arise naturally from the mathematical models of various of physical processes (see [1, 2]), and also appear in the study of partial differential equations and abstract differential equations (see [3–5]). For instance, electron beam focusing system in travelling-wave tube's theories is an antiperiodic problem (see [6]).

During the past twenty years, antiperiodic problems have been studied extensively by numerous scholars. For example, for first-order ordinary differential equations, a Massera's type criterion was presented in [7] and the validity of the monotone iterative technique was shown in [8]. Moreover, for higher-order ordinary differential equations, the existence of antiperiodic solutions was considered in [9–12]. Recently, existence results were extended to antiperiodic boundary value problems for impulsive differential equations (see [13]), and antiperiodic wavelets were discussed in [14].

Wang and Li (see [15]) discussed the existence of solutions of the following antiperiodic boundary value problem for second-order conservative system:

$$q'' = u(t,q), \qquad q(0) = -q(T), \qquad q'(0) = -q'(T) \tag{1.1}$$

using of the main assumption as follows:

(A_1) There exist constants $0 \le c < 8$ and $M > 0$, such that

$$\left|u(t,q)\right| \le \frac{c}{T^2}\left|q\right| + M, \quad \forall t \in [0,T],\ q \in \mathbb{R}. \tag{1.2}$$

The turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type of problems, Leibenson (see [16]) introduced the following p-Laplacian equation:

$$\left(\phi_p(x')\right)' = f(t,x,x'), \tag{1.3}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Obviously, the inverse operator of ϕ_p is ϕ_q, where $q > 1$ is a constant such that $1/p + 1/q = 1$.

Notice that, when $p = 2$, the nonlinear operator $(\phi_p(x'))'$ reduces to the linear operator x''.

In the past few decades, many important results relative to (1.3) with certain boundary conditions have been obtained. We refer the reader to [17–20] and the references cited therein. However, to the best of our knowledge, there exist relatively few results for the existence of antiperiodic solutions of (1.3). Moreover, it is well known that the existence of antiperiodic solutions plays a key role in characterizing the behavior of nonlinear differential equations (see [21]). Thus, it is worthwhile to continue to investigate the existence of antiperiodic solutions for (1.3).

A primary purpose of this paper is to study the existence of antiperiodic solutions for the following Liénard-type p-Laplacian equation:

$$\left(\phi_p(x')\right)' + f(x)x' + g(t,x) = e(t) \tag{1.4}$$

and antiperiodic solutions with symmetry for Duffing-type p-Laplacian equation as follows:

$$\left(\phi_p(x')\right)' + g(t,x) = e(t), \tag{1.5}$$

where $f, e \in C(\mathbb{R},\mathbb{R})$, $g \in C(\mathbb{R}^2,\mathbb{R})$ with $f(-x) \equiv f(x)$, $g(t+\pi,-x) \equiv -g(t,x)$, and $e(t+\pi) \equiv -e(t)$. That is, we will prove that (1.4) or (1.5) has at least one solution $x(t)$ satisfying

$$x(t+\pi) = -x(t), \quad \forall t \in \mathbb{R}. \tag{1.6}$$

Note that, $x(t)$ is also a 2π-periodic solution of (1.4) or (1.5) if $x(t)$ is a π-antiperiodic solution of (1.4) or (1.5). Hence, from the arguments in this paper, we can also obtain the existence results of periodic solutions for above equations.

The rest of this paper is organized as follows. Section 2 contains some necessary preliminaries. In Section 3, we establish some sufficient conditions for the existence of antiperiodic solutions of (1.4), basing on Leray-Schauder principle. Then, in Section 4, we obtain two existence results of antiperiodic solutions with symmetry for (1.5). Finally, in Section 5, some explicit examples are given to illustrate the main results. Our results are different from those of bibliographies listed above.

2. Preliminaries

For convenience, we introduce some notations as follows:

$$\begin{aligned}
C^{k,2\pi} &= \left\{x \in C^k(\mathbb{R},\mathbb{R}) : x(t+2\pi) \equiv x(t)\right\},\\
C^{k,\pi} &= \left\{x \in C^{k,2\pi} : x(t+\pi) \equiv -x(t)\right\},\\
C_0^{k,\pi} &= \left\{x \in C^{k,\pi} : x(-t) \equiv x(t)\right\},\\
C_1^{k,\pi} &= \left\{x \in C^{k,\pi} : x(-t) \equiv -x(t)\right\},\\
\|x\|_\infty &= \max_{t\in[0,2\pi]} |x(t)|, \quad x \in C^{0,2\pi},\\
\|x\|_{C^k} &= \max_{i\in\{0,1,\ldots,k\}} \left\{\left\|x^{(i)}\right\|_\infty\right\}, \quad x \in C^{k,2\pi},
\end{aligned} \tag{2.1}$$

and $\|\cdot\|_p$ denotes norm in $L^p([0,2\pi],\mathbb{R})$.

For each $x \in C^{0,\pi}$, there exists the following Fourier series expansion:

$$x(t) = \sum_{i=0}^{\infty}[a_{2i+1}\cos(2i+1)t + b_{2i+1}\sin(2i+1)t], \tag{2.2}$$

where $a_{2i+1}, b_{2i+1} \in \mathbb{R}$. Let us define the mapping $J : C^{0,\pi} \to C^{1,\pi}$ by

$$\begin{aligned}
(Jx)(t) &= \int_0^t x(s)ds - \sum_{i=0}^{\infty}\frac{b_{2i+1}}{2i+1}\\
&= \sum_{i=0}^{\infty}\left[\frac{a_{2i+1}}{2i+1}\sin(2i+1)t - \frac{b_{2i+1}}{2i+1}\cos(2i+1)t\right], \quad \forall t \in \mathbb{R}.
\end{aligned} \tag{2.3}$$

Notice that, $x \in C_0^{0,\pi}$ may be written as Fourier series as follows:

$$x(t) = \sum_{i=0}^{\infty} a_{2i+1}\cos(2i+1)t \tag{2.4}$$

and $x \in C_1^{0,\pi}$ may be written as the following Fourier series:

$$x(t) = \sum_{i=0}^{\infty} b_{2i+1}\sin(2i+1)t. \tag{2.5}$$

We define the mapping $J_0 : C_0^{0,\pi} \to C_1^{1,\pi}$ by

$$(J_0x)(t) = \int_0^t x(s)ds = \sum_{i=0}^{\infty} \frac{a_{2i+1}}{2i+1} \sin(2i+1)t, \quad \forall t \in \mathbb{R} \tag{2.6}$$

and the mapping $J_1 : C_1^{0,\pi} \to C_0^{1,\pi}$ by

$$(J_1x)(t) = \int_0^t x(s)ds - \sum_{i=0}^{\infty} \frac{b_{2i+1}}{2i+1} = -\sum_{i=0}^{\infty} \frac{b_{2i+1}}{2i+1} \cos(2i+1)t, \quad \forall t \in \mathbb{R}. \tag{2.7}$$

It is easy to prove that the mappings J, J_0, J_1 are completely continuous by using Arzelà-Ascoli theorem.

Next, we introduce a Wirtinger inequality (see [22]) and a continuation theorem (see [23, 24]) as follows.

Lemma 2.1 (Wirtinger inequality). *For each $x \in W^{1,p}([0, 2\pi], \mathbb{R})$ such that $x(0) = x(2\pi)$ and $\int_0^{2\pi} |x(t)|^{p-2}x(t)dt = 0$, one has*

$$\lambda_1 \|x\|_p^p \le \|x'\|_p^p, \tag{2.8}$$

where

$$\lambda_1 = \left(\frac{\pi_p}{\pi}\right)^p, \quad \pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}. \tag{2.9}$$

Lemma 2.2 (Continuation theorem). *Let Ω be open-bounded in a linear normal space X. Suppose that f is a completely continuous field on $\overline{\Omega}$. Moreover, assume that the Leray-Schauder degree*

$$\deg(f, \Omega, p) \neq 0, \quad \text{for } p \in X \setminus f(\partial\Omega). \tag{2.10}$$

Then equation $f(x) = p$ has at least one solution in Ω.

3. Antiperiodic Solutions for (1.4)

In this section, an existence result of antiperiodic solutions for (1.4) will be given.

Theorem 3.1. *Assume that*

(H_1) *there exists a nonnegative function $\alpha \in C(\mathbb{R}, \mathbb{R}^+)$ such that*

$$\limsup_{|x| \to +\infty} \frac{|g(t,x)|}{|x|^{p-1}} = \alpha(t), \quad \forall t \in \mathbb{R}, \tag{3.1}$$

where

$$\|\alpha\|_\infty < \left(\frac{\pi_p}{\pi}\right)^p (= \lambda_1). \tag{3.2}$$

Then (1.4) *has at least one antiperiodic solution.*

Remark 3.2. When $p = 2$, λ_1 is equal to 1. It is easy to see that condition (A_1) in [15] is stronger than condition (H_1) of Theorem 3.1.

For making use of Leray-Schauder degree theory to prove the existence of antiperiodic solutions for (1.4), we consider the homotopic equation of (1.4) as follows:

$$(\phi_p(x'))' = -\lambda f(x)x' - \lambda g(t,x) + \lambda e(t), \quad \lambda \in [0,1]. \tag{3.3}$$

Define the operator $L_p : D(L_p) \subset C^{1,\pi} \to L^1([0,2\pi],\mathbb{R})$ by

$$(L_p x)(t) = (\phi_p(x'(t)))', \quad \forall t \in \mathbb{R}, \tag{3.4}$$

where

$$D(L_p) = \left\{x \in C^{1,\pi} : \phi_p(x'(t)) \text{ is absolutely continuous on } \mathbb{R}\right\}. \tag{3.5}$$

Let $N : C^{1,\pi} \to L^1([0,2\pi],\mathbb{R})$ be the Nemytski operator

$$(Nx)(t) = -f(x(t))x'(t) - g(t,x(t)) + e(t), \quad \forall t \in \mathbb{R}. \tag{3.6}$$

Obviously, the operator L_p is invertible and the antiperiodic problem of (3.3) is equivalent to the operator equation

$$L_p x = \lambda N x, \quad x \in D(L_p). \tag{3.7}$$

We begin with some lemmas below.

Lemma 3.3. *Suppose that the assumption* (H_1) *is true. Then the antiperiodic solution* $x(t)$ *of* (3.3) *satisfies*

$$\|x'\|_p \le K_1, \tag{3.8}$$

where K_1 *is a positive constant only dependent of* λ_1 *and* $\|e\|_\infty$.

Proof. Multiplying the both sides of (3.3) with $x(t)$ and integrating it over $[0, 2\pi]$, we get

$$\int_0^{2\pi} (\phi_p(x'(t)))' x(t)dt = -\lambda \int_0^{2\pi} f(x(t))x'(t)x(t)dt - \lambda \int_0^{2\pi} g(t, x(t))x(t)dt + \lambda \int_0^{2\pi} e(t)x(t)dt. \tag{3.9}$$

Noting that

$$\int_0^{2\pi} (\phi_p(x'(t)))' x(t)dt = -\int_0^{2\pi} \phi_p(x'(t))x'(t)dt = -\|x'\|_p^p \tag{3.10}$$

and $\int_0^{2\pi} f(x(t))x(t)x'(t)dt = 0$, we have

$$\|x'\|_p^p \leq \int_0^{2\pi} |g(t, x(t))||x(t)|dt + \int_0^{2\pi} |e(t)||x(t)|dt. \tag{3.11}$$

By hypothesis (H_1), there exists a nonnegative constant β such that

$$|g(t, x)| \leq \alpha(t)|x|^{p-1} + \beta, \quad \forall t, x \in \mathbb{R}. \tag{3.12}$$

Thus, from (3.11), we have

$$\|x'\|_p^p \leq \|\alpha\|_\infty \|x\|_p^p + (\beta + \|e\|_\infty) \int_0^{2\pi} |x(t)|dt. \tag{3.13}$$

That is,

$$\|x'\|_p^p \leq \|\alpha\|_\infty \|x\|_p^p + K_2\|x\|_p, \tag{3.14}$$

where $K_2 = (2\pi)^{1/q}(\beta + \|e\|_\infty)$.

For each $x \in C^{1,\pi}$, we get

$$\int_0^{2\pi} x(t)dt = \int_0^{\pi} x(t)dt + \int_0^{\pi} x(t + \pi)dt = 0. \tag{3.15}$$

Similarly, we obtain that

$$\int_0^{2\pi} x'(t)dt = 0, \tag{3.16}$$

$$\int_0^{2\pi} |x(t)|^{p-2}x(t)dt = 0. \tag{3.17}$$

Basing on Lemma 2.1, it can be shown from (3.17) and (3.14) that

$$\|x'\|_p^p \le \frac{\|\alpha\|_\infty}{\lambda_1}\|x'\|_p^p + \frac{K_2}{\lambda_1^{1/p}}\|x'\|_p. \tag{3.18}$$

Let $K_1 = (K_2\lambda_1^{1/q}/(\lambda_1 - \|\alpha\|_\infty))^{1/(p-1)} > 0$, then

$$\|x'\|_p \le K_1. \tag{3.19}$$

The proof is complete. □

Lemma 3.4. *Suppose that the assumption* (H_1) *is true. Then, for the possible antiperiodic solution* $x(t)$ *of* (3.3), *there exists a prior bounds in* $C^{1,\pi}$, *that is,* $x(t)$ *satisfies*

$$\|x\|_{C^1} \le T_1, \tag{3.20}$$

where T_1 *is a positive constant independent of* λ.

Proof. By (3.15), there exists $t_1 \in [0, 2\pi]$ such that $x(t_1) = 0$. Hence, (3.8) yields that

$$\|x\|_\infty \le \int_0^{2\pi} |x'(t)|dt \le (2\pi)^{1/q}\|x'\|_p \le (2\pi)^{1/q}K_1 := K_3. \tag{3.21}$$

Letting

$$\begin{gathered} K_4 = \max\{|f(x)| : \|x\|_\infty \le K_3\}, \\ K_5 = \max\{|g(t,x)| + |e(t)| : t \in [0,2\pi],\ \|x\|_\infty \le K_3\}. \end{gathered} \tag{3.22}$$

From (3.16), there exists $t_2 \in [0, 2\pi]$ such that $x'(t_2) = 0$, which implies that $\phi_p(x'(t_2)) = 0$. Therefore, integrating the both sides of (3.3) over $[t_2, t]$, we have

$$\phi_p(x'(t)) = -\lambda\int_{t_2}^t f(x(t))x'(t)dt - \lambda\int_{t_2}^t g(t,x(t))dt + \lambda\int_{t_2}^t e(t)dt. \tag{3.23}$$

Thus, we get from (3.8) that

$$\begin{aligned} |\phi_p(x'(t))| &\le K_4\int_0^{2\pi}|x'(t)|dt + 2\pi K_5 \\ &\le K_4(2\pi)^{1/q}\|x'\|_p + 2\pi K_5 \\ &\le (2\pi)^{1/q}K_4K_1 + 2\pi K_5 := K_6^{p-1}, \quad \forall t \in [0,2\pi]. \end{aligned} \tag{3.24}$$

Noting that $|\phi_p(x'(t))| = |x'(t)|^{p-1}$, we obtain that

$$\|x'\|_\infty \le K_6. \tag{3.25}$$

Combining (3.21) with (3.25), we have

$$\|x\|_{C^1} \le T_1, \tag{3.26}$$

where $T_1 = \max\{K_3, K_6\}$. The proof is complete. □

Now we give the proof of Theorem 3.1.

Proof of Theorem 3.1. Setting

$$\Omega = \left\{x \in C^{1,\pi} : \|x\|_{C^1} < T_1 + 1\right\}. \tag{3.27}$$

Obviously, the set Ω is an open-bounded set in $C^{1,\pi}$ and zero element $\theta \in \Omega$.

From the definition of operator N, it is easy to see that

$$(Nx)(t+\pi) \equiv -(Nx)(t), \quad \forall x \in C^{1,\pi}. \tag{3.28}$$

Hence, the operator N sends $C^{1,\pi}$ into $C^{0,\pi}$. Let us define the operator $F_\lambda : \overline{\Omega} \to C^{1,\pi}$ by

$$F_\lambda x = J\phi_q J\lambda N x = \phi_q(\lambda) L_p^{-1} N x, \quad \lambda \in [0,1]. \tag{3.29}$$

Obviously, the operator F_λ is completely continuous in $\overline{\Omega}$ and the fixed points of operator F_1 are the antiperiodic solutions of (1.4).

With this in mind, let us define the completely continuous field $h_\lambda(x) : \overline{\Omega} \times [0,1] \to C^{1,\pi}$ by

$$h_\lambda(x) = x - F_\lambda x. \tag{3.30}$$

By (3.20), we get that zero element $\theta \notin h_\lambda(\partial\Omega)$ for all $\lambda \in [0,1]$. So that, the following Leray-Schauder degrees are well defined and

$$\deg(\mathrm{id} - F_1, \Omega, \theta) = \deg(h_1, \Omega, \theta) = \deg(h_0, \Omega, \theta) = \deg(\mathrm{id}, \Omega, \theta) = 1 \neq 0. \tag{3.31}$$

Consequently, the operator F_1 has at least one fixed point in Ω by using Lemma 2.2. Namely, (1.4) has at least one antiperiodic solution. The proof is complete. □

4. Antiperiodic Solutions with Symmetry for (1.5)

In this section, we will prove the existence of even antiperiodic solutions or odd antiperiodic solutions for (1.5).

Theorem 4.1. *Assume that*

(H_2) *the functions* $g(t,x)$ *and* $e(t)$ *are even in* t*, that is,*

$$g(-t,\cdot) = g(t,\cdot), \quad e(-t) = e(t), \quad \forall t \in \mathbb{R} \tag{4.1}$$

and the assumption (H_1) *is true. Then* (1.5) *has at least one even antiperiodic solution* $x(t)$*, that is,* $x(t)$ *satisfies*

$$x(t+\pi) = -x(t), \quad x(-t) = x(t), \quad \forall t \in \mathbb{R}. \tag{4.2}$$

Proof. We consider the homotopic equation of (1.5) as follows:

$$(\phi_p(x'))' = -\lambda g(t,x) + \lambda e(t), \quad \lambda \in [0,1]. \tag{4.3}$$

Define the operator $N_0 : C_0^{1,\pi} \to L^1([0,2\pi],\mathbb{R})$ by

$$(N_0 x)(t) = -g(t,x(t)) + e(t), \quad \forall t \in \mathbb{R}. \tag{4.4}$$

Obviously, the operator N_0 is continuous.

Basing on the proof of Theorem 3.1, for the possible even antiperiodic solution $x(t)$ of (4.3), there exists *a prior bounds* in $C_0^{1,\pi}$, that is, $x(t)$ satisfies

$$\|x\|_{C^1} \le T_2, \tag{4.5}$$

where T_2 is a positive constant independent of λ. So that, our problem is reduced to construct one completely continuous operator G_λ in $C_0^{1,\pi}$ which sends $C_0^{1,\pi}$ into $C_0^{1,\pi}$, such that the fixed points of operator G_1 in some open-bounded set are the even antiperiodic solutions of (1.5).

With this in mind, let us define the following set:

$$\Omega_0 = \left\{x \in C_0^{1,\pi} : \|x\|_{C^1} < T_2 + 1\right\}. \tag{4.6}$$

Obviously, the set Ω_0 is an open-bounded set in $C_0^{1,\pi}$ and zero element $\theta \in \Omega_0$.

By hypothesis (H_2), it is easy to see that

$$(N_0 x)(-t) \equiv (N_0 x)(t), \quad \forall x \in C_0^{1,\pi}. \tag{4.7}$$

Hence, the operator N_0 sends $C_0^{1,\pi}$ into $C_0^{0,\pi}$. Let us define the completely continuous operator $G_\lambda : \overline{\Omega_0} \to C_0^{1,\pi}$ by

$$G_\lambda x = J_1 \phi_q J_0 \lambda N_0 x = \phi_q(\lambda)\left(L_p|_{D(L_p)\cap C_0^{0,\pi}}\right)^{-1} N_0 x, \quad \lambda \in [0,1]. \tag{4.8}$$

From the similar arguments in the proof of Theorem 3.1, we can prove that there exists at least one fixed point of operator G_1 in Ω_0. Thus, (1.5) has at least one even antiperiodic solution. The proof is complete. □

Theorem 4.2. *Assume that*

(H_3) *the function* $g(t,x)$ *is odd in* t,x *and* $e(t)$ *is odd in* t, *that is,*

$$g(-t,-x) = -g(t,x), \quad e(-t) = -e(t), \quad \forall t,x \in \mathbb{R} \tag{4.9}$$

and the assumption (H_1) *is true. Then* (1.5) *has at least one odd antiperiodic solution* $x(t)$, *that is,* $x(t)$ *satisfies*

$$x(t+\pi) = -x(t), \quad x(-t) = -x(t), \quad \forall t \in \mathbb{R}. \tag{4.10}$$

Proof. We consider the homotopic equation (4.3) of (1.5). Define the operator $N_1 : C_1^{1,\pi} \to L^1([0,2\pi],\mathbb{R})$ by

$$(N_1x)(t) = -g(t,x(t)) + e(t), \quad \forall t \in \mathbb{R}. \tag{4.11}$$

Obviously, the operator N_1 is continuous.

Based on the proof of Theorem 3.1, for the possible odd antiperiodic solutions of (4.3), there exists *a prior bounds* in $C_1^{1,\pi}$. Hence, our problem is reduced to construct one completely continuous operator P_λ in $C_1^{1,\pi}$ which sends $C_1^{1,\pi}$ into $C_1^{1,\pi}$, such that the fixed points of operator P_1 in some open-bounded set are the odd antiperiodic solutions of (1.5).

With this in mind, let us define the set as follows:

$$\Omega_1 = \left\{x \in C_1^{1,\pi} : \|x\|_{C^1} < T_2 + 1\right\}. \tag{4.12}$$

Obviously, the set Ω_1 is an open-bounded set in $C_1^{1,\pi}$ and zero element $\theta \in \Omega_1$.

From the hypothesis (H_3), it is easy to see that

$$(N_1x)(-t) \equiv -(N_1x)(t), \quad \forall x \in C_1^{1,\pi}. \tag{4.13}$$

Thus, the operator N_1 sends $C_1^{1,\pi}$ into $C_1^{0,\pi}$. Let us define the completely continuous operator $P_\lambda : \overline{\Omega_1} \to C_1^{1,\pi}$ by

$$P_\lambda x = J_0\phi_q J_1 \lambda N_1 x = \phi_q(\lambda)\left(L_p|_{D(L_p)\cap C_1^{0,\pi}}\right)^{-1} N_1 x, \quad \lambda \in [0,1]. \tag{4.14}$$

By a similar way as the proof of Theorem 3.1, we can prove that there exists at least one fixed point of operator P_1 in Ω_1. So that, (1.5) has at least one odd antiperiodic solution. The proof is complete. □

5. Examples

In this section, we will give some examples to illustrate our main results.

Consider the following second-order differential equation with p-Laplacian operator:

$$(\phi_4(x'))' + x^2 x' + g(t, x) = e(t). \quad (5.1)$$

Example 5.1. Let

$$g(t, x) = \frac{1}{2}\sin^2 t \cdot x^3, \qquad e(t) = \cos t. \quad (5.2)$$

For $p = 4$, by direct calculation, we can get $\lambda_1 = 3/4$. Choosing $\alpha(t) = (1/2)\sin^2 t$, then (5.1) satisfies the condition of Theorem 3.1. So it has at least one antiperiodic solution.

Moreover, the conditions of Theorem 4.1 are also satisfied. Thus (5.1) has at least one even antiperiodic solution.

Example 5.2. Let

$$g(t, x) = \frac{1}{2}\sin^2 t \cdot x^3, \qquad e(t) = \sin t. \quad (5.3)$$

We choose $\alpha(t) = (1/2)\sin^2 t$. Obviously, (5.1) satisfies all the conditions of Theorem 4.2. Hence it has at least one odd antiperiodic solution.

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