

ANTIPODAL SETS IN ORIENTED REAL GRASSMANN MANIFOLDS

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ABSTRACT. We reduce the problem of classifying all maximal antipodal sets in the oriented real Grassmann manifold $\tilde{G}_k(\mathbb{R}^n)$ to that of classifying all maximal subsets satisfying certain conditions in the set consisting of subsets of cardinality k in $\{1, \dots, n\}$. Using this reduction we classify all maximal antipodal sets in $\tilde{G}_k(\mathbb{R}^n)$ for $k \leq 4$. We construct some maximal antipodal subsets for higher k .

1. INTRODUCTION

Chen and Nagano [1] introduced the notion of antipodal sets in Riemannian symmetric spaces and showed some fundamental properties of antipodal sets. They also explicitly described antipodal sets in many compact Riemannian symmetric spaces, but they did not mention antipodal sets in oriented real Grassmann manifolds. In this paper we describe antipodal sets in oriented real Grassmann manifolds.

Let M be a Riemannian symmetric space and denote by s_x the geodesic symmetry of M at $x \in M$. A subset S in M is called an *antipodal* set if $s_x(y) = y$ for any $x, y \in S$. We define the *2-number* $\#_2 M$ of M as the supremum of $\#S$ for all antipodal sets S in M . It is known that $\#_2 M$ is finite. See Proposition 2.1 in Tanaka-Tasaki [4]. We call an antipodal set S in M *great* if $\#_2 M = \#S$. A great antipodal set is maximal in all antipodal sets with respect to inclusion relation. In [4] we showed that any antipodal set in a symmetric R -space is included in a great antipodal set and any two great antipodal sets are congruent, however, antipodal sets in the oriented real Grassmann manifolds do not have such a nice property.

We denote by $\tilde{G}_k(\mathbb{R}^n)$ the oriented real Grassmann manifold consisting of oriented real vector subspaces of dimension k in \mathbb{R}^n equipped with the standard inner product. We reduce the problem of classifying all maximal antipodal sets in $\tilde{G}_k(\mathbb{R}^n)$ to that of classifying maximal

2010 *Mathematics Subject Classification.* 53C40 (Primary), 53D12 (Secondary).

Key words and phrases. antipodal set, oriented real Grassmann manifold, 2-number, Riemannian symmetric space.

subsets satisfying certain conditions in the set consisting of subsets of cardinality k in $\{1, \dots, n\}$. In the case where $k = 1, 2$ antipodal sets in $\tilde{G}_k(\mathbb{R}^n)$ are simple. In comparison with these we can see that antipodal sets in $\tilde{G}_k(\mathbb{R}^n)$ are not simple in the cases where $k \geq 3$ by using the reduction. In Sections 4, 5, and 6 we classify all of maximal antipodal sets of $\tilde{G}_k(\mathbb{R}^n)$ in the case where $k \leq 4$. In Section 7 we construct some maximal antipodal subsets for higher k .

2. REAL GRASSMANN MANIFOLDS

The orthogonal group $O(n)$ acts transitively on $G_k(\mathbb{R}^n)$. We define an inner product on the Lie algebra $\mathfrak{o}(n)$ of $O(n)$ by

$$\langle X, Y \rangle = \frac{1}{2} \operatorname{tr}({}^tXY) \quad (X, Y \in \mathfrak{o}(n)),$$

which induces a biinvariant Riemannian metric on $O(n)$ and an $O(n)$ -invariant Riemannian metric on $G_k(\mathbb{R}^n)$. With respect to this Riemannian metric $G_k(\mathbb{R}^n)$ is a Riemannian symmetric space. It is known that $G_k(\mathbb{R}^n)$ is a symmetric R -space. For $V \in G_k(\mathbb{R}^n)$ the reflection r_V with respect to V is equal to $1_V - 1_{V^\perp}$, which induces the geodesic symmetry s_V of $G_k(\mathbb{R}^n)$ at V . We denote by $\langle v_1, \dots, v_m \rangle$ the vector subspace of \mathbb{R}^n spanned by $v_1, \dots, v_m \in \mathbb{R}^n$. We denote by $\operatorname{Inc}_k(n)$ the set of all strictly increasing maps from $\{1, \dots, k\}$ to $\{1, \dots, n\}$. For a connected Riemannian manifold M , two subsets X and Y in M are said to be *congruent*, if X is transformed to Y by an element of the identity component of the group of all isometries of M .

Lemma 2.1. *Let e_1, \dots, e_n be an orthonormal basis of \mathbb{R}^n . The subset*

$$A = \{ \langle e_{\alpha(1)}, \dots, e_{\alpha(k)} \rangle \mid \alpha \in \operatorname{Inc}_k(n) \}$$

in $G_k(\mathbb{R}^n)$ is a maximal antipodal set and any maximal antipodal set in $G_k(\mathbb{R}^n)$ is congruent with A . Thus A is a great antipodal set and we have

$$\#_2 G_k(\mathbb{R}^n) = \#\operatorname{Inc}_k(n) = \binom{n}{k}.$$

This is showed in [1] and [4].

3. ORIENTED REAL GRASSMANN MANIFOLDS

We denote by $\tilde{G}_k(\mathbb{R}^n)$ the oriented real Grassmann manifold consisting of oriented vector subspaces of dimension k in \mathbb{R}^n . In this case $O(n)$ also acts transitively on $\tilde{G}_k(\mathbb{R}^n)$. The biinvariant Riemannian metric on $O(n)$ defined in the previous section induces an $O(n)$ -invariant Riemannian metric on $\tilde{G}_k(\mathbb{R}^n)$. With respect to this Riemannian metric

$\tilde{G}_k(\mathbb{R}^n)$ is a Riemannian symmetric space, but it is not true in general that $\tilde{G}_k(\mathbb{R}^n)$ is a symmetric R -space.

We denote by $\bigwedge^k \mathbb{R}^n$ the exterior algebra of \mathbb{R}^n of degree k . The inner product on \mathbb{R}^n naturally induces an inner product on $\bigwedge^k \mathbb{R}^n$. With respect to this inner product

$$\{e_{\alpha(1)} \wedge \cdots \wedge e_{\alpha(k)} \mid \alpha \in \text{Inc}_k(n)\}$$

is an orthonormal basis of $\bigwedge^k \mathbb{R}^n$, if e_1, \dots, e_n is an orthonormal basis of \mathbb{R}^n . We can regard $\tilde{G}_k(\mathbb{R}^n)$ as a Riemannian submanifold of $\bigwedge^k \mathbb{R}^n$ by identifying $\tilde{V} \in \tilde{G}_k(\mathbb{R}^n)$ with $v_1 \wedge \cdots \wedge v_k$, where v_1, \dots, v_k is a positively oriented orthonormal basis of \tilde{V} .

For $\tilde{V} \in \tilde{G}_k(\mathbb{R}^n)$ we denote by V the vector subspace determined by \tilde{V} . The reflection $r_V = 1_V - 1_{V^\perp}$ induces the geodesic symmetry $s_{\tilde{V}}$ of $\tilde{G}_k(\mathbb{R}^n)$ at \tilde{V} defined by

$$s_{\tilde{V}}(v_1 \wedge \cdots \wedge v_k) = r_V(v_1) \wedge \cdots \wedge r_V(v_k).$$

This is equal to $(\bigwedge^k r_V)(v_1 \wedge \cdots \wedge v_k)$. Hence the geodesic symmetry $s_{\tilde{V}}$ is the restriction of the linear transformation $\bigwedge^k r_V$ of $\bigwedge^k \mathbb{R}^n$ to $\tilde{G}_k(\mathbb{R}^n)$. We define a double covering map $p : \tilde{G}_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n)$ by $p(\tilde{V}) = V$. We have $s_{p(x)} \circ p = p \circ s_x$ for $x \in \tilde{G}_k(\mathbb{R}^n)$, because p is an isometric covering map.

Lemma 3.1. *For any antipodal set S in $\tilde{G}_k(\mathbb{R}^n)$ there exists an orthonormal basis v_1, \dots, v_n of \mathbb{R}^n which satisfies*

$$S \subset \{\pm v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)} \mid \alpha \in \text{Inc}_k(n)\},$$

where we regard $\tilde{G}_k(\mathbb{R}^n)$ as a submanifold of $\bigwedge^k \mathbb{R}^n$.

Proof. For any $x, y \in S$ we have

$$s_{p(x)}(p(y)) = p(s_x(y)) = p(y),$$

thus $p(S)$ is an antipodal set in $G_k(\mathbb{R}^n)$. According to Lemma 2.1 there exists an orthonormal basis v_1, \dots, v_n of \mathbb{R}^n which satisfies

$$p(S) \subset \{\langle v_{\alpha(1)}, \dots, v_{\alpha(k)} \rangle \mid \alpha \in \text{Inc}_k(n)\}.$$

Hence we get

$$S \subset \{\pm v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)} \mid \alpha \in \text{Inc}_k(n)\}.$$

□

For a set X we denote by $P_k(X)$ the set of all subsets α in X whose cardinalities $\#\alpha$ are equal to k . We simply denote $P_k(n) = P_k(\{1, \dots, n\})$. The map

$$\text{Inc}_k(n) \rightarrow P_k(n); \alpha \mapsto \{\alpha(1), \dots, \alpha(k)\}$$

is bijective and we identify $\text{Inc}_k(n)$ with $P_k(n)$ by this bijective map.

Lemma 3.2. *For any maximal antipodal set S in $\tilde{G}_k(\mathbb{R}^n)$ there exist an orthonormal basis v_1, \dots, v_n of \mathbb{R}^n and a subset A of $P_k(n)$ which satisfy*

$$S = \{\pm v_{\alpha(1)} \wedge \dots \wedge v_{\alpha(k)} \mid \alpha \in A\}.$$

Proof. By Lemma 3.1, there exists an orthonormal basis v_1, \dots, v_n of \mathbb{R}^n which satisfies

$$S \subset \{\pm v_{\alpha(1)} \wedge \dots \wedge v_{\alpha(k)} \mid \alpha \in \text{Inc}_k(n)\}.$$

We set $\vec{v}_\alpha = v_{\alpha(1)} \wedge \dots \wedge v_{\alpha(k)}$ and

$$A = \{\alpha \in P_k(n) \mid \vec{v}_\alpha \in S \text{ or } -\vec{v}_\alpha \in S\}.$$

This definition of A implies $S \subset \{\pm \vec{v}_\alpha \mid \alpha \in A\}$.

If $x, y \in S$ then $s_x(y) = y$. In this case we have $s_x(-y) = -s_x(y) = -y$, because s_x is the restriction of a linear transformation of $\bigwedge^k \mathbb{R}^n$. By the definition we have $s_{-y} = s_y$ and $S \cup \{-y\}$ is also an antipodal set. Since S is a maximal antipodal set, we have $S = \{\pm \vec{v}_\alpha \mid \alpha \in A\}$. \square

For an orthonormal basis $\mathbf{v} = \{v_1, \dots, v_n\}$ of \mathbb{R}^n and a subset A of $P_k(n)$ we define

$$\mathcal{A}_{\mathbf{v}}(A) = \{\pm \vec{v}_\alpha \mid \alpha \in A\}.$$

We have to determine which subset A of $P_k(n)$ defines a maximal antipodal set $\mathcal{A}_{\mathbf{v}}(A)$ in $\tilde{G}_k(\mathbb{R}^n)$.

Lemma 3.3. *Let $\mathbf{v} = \{v_1, \dots, v_n\}$ be an orthonormal basis of \mathbb{R}^n and A be a subset of $P_k(n)$. $\mathcal{A}_{\mathbf{v}}(A)$ is an antipodal set in $\tilde{G}_k(\mathbb{R}^n)$ if and only if the cardinality $\#(\beta - \alpha)$ is even for any $\alpha, \beta \in A$, where $\beta - \alpha = \{b \in \beta \mid b \notin \alpha\}$.*

Proof. For $\alpha, \beta \in P_k(n)$ we have

$$s_{\vec{v}_\alpha}(\vec{v}_\beta) = r_{p(\vec{v}_\alpha)} v_{\beta(1)} \wedge \dots \wedge r_{p(\vec{v}_\alpha)} v_{\beta(k)} = (-1)^{\#(\beta - \alpha)} \vec{v}_\beta.$$

So $s_{\vec{v}_\alpha}(\vec{v}_\beta) = \vec{v}_\beta$ if and only if $\#(\beta - \alpha)$ is even. Hence $\mathcal{A}_{\mathbf{v}}(A)$ is an antipodal set in $\tilde{G}_k(\mathbb{R}^n)$ if and only if the cardinality $\#(\beta - \alpha)$ is even for any $\alpha, \beta \in A$. \square

By the above lemma we introduce the following definition.

Definition 3.1. $\alpha, \beta \in P_k(n)$ are *antipodal*, if the cardinality $\#(\beta - \alpha)$ is even. A subset A of $P_k(n)$ is said to be *antipodal*, if any $\alpha, \beta \in A$ are antipodal. By the definition $\alpha, \beta \in P_k(n)$ are antipodal if and only if $\#(\alpha \cap \beta) \equiv k \pmod{2}$.

We denote by $\text{Sym}(X)$ the symmetric group on a finite set X . If $X = \{1, \dots, n\}$, we simply write $\text{Sym}(n) = \text{Sym}(X)$. Two subsets X and Y in $P_k(X)$ are said to be *congruent*, if X is transformed to Y by an element of $\text{Sym}(X)$.

If $A \subset B \subset P_k(n)$, then $\mathcal{A}_v(A) \subset \mathcal{A}_v(B)$. This implication and the lemmas mentioned above imply the following theorem.

Theorem 3.1. *If v_1, \dots, v_n is an orthonormal basis of \mathbb{R}^n and A is a maximal antipodal subset of $P_k(n)$, then $\mathcal{A}_v(A)$ is a maximal antipodal set in $\tilde{G}_k(\mathbb{R}^n)$. Conversely for any maximal antipodal set S in $\tilde{G}_k(\mathbb{R}^n)$ there exist an orthonormal basis v_1, \dots, v_n of \mathbb{R}^n and a maximal antipodal subset A of $P_k(n)$ which satisfy $S = \mathcal{A}_v(A)$. Moreover the correspondence above induces a bijection from the set of all congruent classes of maximal antipodal subsets in $P_k(n)$ to the set of all congruent classes of maximal antipodal sets in $\tilde{G}_k(\mathbb{R}^n)$.*

Corollary 3.1.

$$\#_2 \tilde{G}_k(\mathbb{R}^n) = 2 \max\{\#A \mid A \text{ is a maximal antipodal subset of } P_k(n)\},$$

In order to determine maximal antipodal sets in $\tilde{G}_k(\mathbb{R}^n)$ we have to determine maximal antipodal subsets of $P_k(n)$. We investigate maximal antipodal subsets of $P_k(n)$ in the sequel sections.

4. MAXIMAL ANTIPODAL SUBSETS OF $P_k(n)$

In this section we consider a strategy to determine all congruent classes of maximal antipodal subsets in $P_k(n)$.

We denote by α^c the complement of $\alpha \in P_k(n)$ in $\{1, \dots, n\}$. For $\alpha, \beta \in P_k(n)$ we have $\beta - \alpha = \alpha^c - \beta^c$, which implies the following lemma.

Lemma 4.1. *A subset $A \subset P_k(n)$ is antipodal if and only if $A^c = \{\alpha^c \mid \alpha \in A\} \subset P_{n-k}(n)$ is antipodal. Moreover A is a maximal antipodal subset of $P_k(n)$ if and only if A^c is a maximal antipodal subset of $P_{n-k}(n)$.*

We identify $P_k(n)$ with $\text{Inc}_k(n)$ and define the lexicographic order on $P_k(n)$. With respect to this order $\{1, 2, \dots, k\}$ is the minimum element. From now on we describe how to construct representatives of all congruent classes of maximal antipodal subsets of $P_k(n)$. At first we

take $A_1 = \{\{1, 2, \dots, k\}\}$. Next we construct some antipodal subsets of cardinality $i + 1$ from an antipodal subset A_i of cardinality i . We define the stabilizer $S(A_i)$ of A_i by

$$S(A_i) = \{g \in \text{Sym}(n) \mid g(A_i) = A_i\}.$$

We consider the set of elements which are antipodal to A_i :

$$A(A_i) = \{\alpha \in P_k(n) - A_i \mid A_i \cup \{\alpha\} \text{ is antipodal}\}.$$

If $A(A_i)$ is empty, A_i is a maximal antipodal subset of $P_k(n)$ and the procedure in the case of the orbit A_i ends. So we consider the case where $A(A_i)$ is not empty. $S(A_i)$ stabilizes A_i and preserves the antipodal relations, hence $S(A_i)$ stabilizes $A(A_i)$, too. We decompose $A(A_i)$ to a disjoint union of the orbits O_1, \dots, O_j of $S(A_i)$. We take the minimum element α_a of O_a for each $1 \leq a \leq j$, add α_a to A_i and obtain $A_{i+1,a} = A_i \cup \{\alpha_a\}$. Since α_a belongs to $A(A_i)$, each $A_{i+1,a}$ is antipodal. We divide the procedure to the cases by the orbits of $S(A_i)$. We take each $A_{i+1,a}$ as A_{i+1} and repeat the above procedure until reaching a maximal antipodal subset of $P_k(n)$.

Lemma 4.2. *We can obtain representatives of all congruent classes of maximal antipodal subsets of $P_k(n)$ by the procedure mentioned above.*

Proof. Let A be a maximal antipodal subset of $P_k(n)$. Transforming A by an element of $\text{Sym}(n)$ we can suppose that $A_1 \subset A$. Transforming A by an element of $S(A_1)$ we can take the minimum element α of an orbit of $S(A_1)$ in $A(A_1)$ which belongs to A . The second antipodal subset A_2 satisfies $A_2 = A_1 \cup \{\alpha\} \subset A$. We can repeat this procedure until $A_i = A$ for $i = \#A$. \square

Remark 4.1. If there exists a subset B of $A(A_j)$ satisfying that any element of B and any other element of $A(A_j)$ are antipodal in the procedure mentioned above, then we can add B to A_j and obtain $A_{j+\#B} = A_j \cup B$.

It may happen that two of the maximal antipodal subsets obtained by the procedure mentioned above are congruent. It actually happens in the proof of Theorem 6.1, which classify maximal antipodal subsets of $P_4(n)$.

In the case where $k = 1$ any two different elements in $P_1(n)$ are not antipodal, so we get the following proposition.

Proposition 4.1. *$\{\{1\}\}$ is a maximal antipodal subset of $P_1(n)$. Conversely any maximal antipodal subset of $P_1(n)$ is congruent with it.*

The corresponding maximal antipodal set of $\tilde{G}_1(\mathbb{R}^n) = S^{n-1}$ is $\{\pm v\}$ for $v \in S^{n-1}$. This proposition and Corollary 3.1 imply the following corollary.

Corollary 4.1. $\#_2 S^{n-1} = 2$.

Proposition 4.2. *For a natural number l we put*

$$A(2, 2l) = \{\{1, 2\}, \{3, 4\}, \dots, \{2l-1, 2l\}\}.$$

$A(2, 2[n/2])$ is a maximal antipodal subset of $P_2(n)$. Conversely any maximal antipodal subset of $P_2(n)$ is congruent with it.

Proof. Two different elements α, β in $P_2(n)$ are antipodal if and only if $\alpha \cap \beta = \emptyset$. We take $A_1 = \{\{1, 2\}\}$. We have $A(A_1) = P_2(\{3, \dots, n\})$ and it is an orbit of $S(A_1)$. We take the minimum element $\{3, 4\}$ of $A(A_1)$, add it to A_1 and obtain $A_2 = \{\{1, 2\}, \{3, 4\}\}$. We can repeat this procedure until we get

$$\{\{1, 2\}, \{3, 4\}, \dots, \{2[n/2]-1, 2[n/2]\}\}.$$

Hence it is a maximal antipodal subset of $P_2(n)$, and any maximal antipodal subset of $P_2(n)$ is congruent with it. \square

Corollary 4.2. $\#_2 \tilde{G}_2(\mathbb{R}^n) = 2[n/2]$.

Remark 4.2. We can construct a 2-form from a subset A of $P_2(2l)$ and an orthonormal basis $\{e_i\}$ of \mathbb{R}^{2l} as follows:

$$\sum_{\alpha \in A} \bar{e}_\alpha^* = \sum_{\alpha \in A} e_{\alpha(1)}^* \wedge e_{\alpha(2)}^*.$$

The 2-form constructed from the maximal antipodal subset $\{\{1, 2\}, \dots, \{2l-1, 2l\}\}$ of $P_2(2l)$ is the Kähler form

$$e_1^* \wedge e_2^* + \dots + e_{2l-1}^* \wedge e_{2l}^*$$

on $\mathbb{C}^l = \mathbb{R}^{2l}$ with a suitable Hermitian structure.

In comparison with the cases where $k = 1$ and $k = 2$, the procedure of constructing maximal antipodal subsets of $P_k(n)$ is not simple in the cases where k is greater than 2.

5. MAXIMAL ANTIPODAL SUBSETS OF $P_3(n)$

In the case where $k = 3$, two different elements α, β in $P_3(n)$ are antipodal if and only if $\#(\alpha \cap \beta) = 1$. Let

$$A(3, 2l + 1) = \{\{1, 2, 3\}, \{1, 4, 5\}, \dots, \{1, 2l, 2l + 1\}\},$$

$$B(3, 6) = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}\},$$

$$B(3, 7) = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}, \\ \{1, 6, 7\}, \{2, 5, 7\}, \{3, 4, 7\}\}.$$

These are antipodal subsets in $P_3(2l + 1)$, $P_3(6)$ and $P_3(7)$ respectively. These satisfy

$$A(3, 5) \subset B(3, 6) \subset B(3, 7), \quad A(3, 5) \subset A(3, 7) \subset B(3, 7).$$

These antipodal subsets can be visualized by Figure 1.

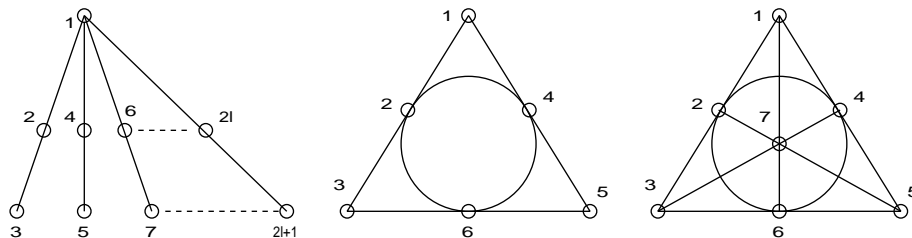


FIGURE 1. $A(3, 2l + 1)$, $B(3, 6)$ and $B(3, 7)$

As stated in Theorem 5.1, $A(3, 2l + 1)$ is a maximal antipodal subset in $P_3(2l + 1)$ and $P_3(2l + 2)$ except for the cases of $A(3, 5) \subset P_3(6)$ and $A(3, 7) \subset P_3(7), P_3(8)$. The relation of inclusions mentioned above shows that these are not maximal.

Remark 5.1. The intersection relation between elements in $B(3, 7)$ is equal to that between projective lines in the projective plane F_2P^2 over the binary field F_2 consisting of 0 and 1.

Theorem 5.1. Let $l = \lfloor (n - 1)/2 \rfloor$. Each in the following table is a maximal antipodal subset of $P_3(n)$:

n	3, 4	5	6	7, 8	more than 8
	$A(3, 3)$	$A(3, 5)$	$B(3, 6)$	$B(3, 7)$	$A(3, 2l + 1), B(3, 7)$

Conversely any maximal antipodal subset of $P_3(n)$ is congruent with one of them.

Before proving the theorem we prepare some notation. For subsets $A, B \subset X$ satisfying $A \cap B = \emptyset$, we denote

$$P_k(A) \times P_l(B) = \{\alpha \cup \beta \mid \alpha \in P_k(A), \beta \in P_l(B)\} \subset P_{k+l}(X).$$

For more than two subsets in X we can similarly define $P_{k_1}(A_1) \times \cdots \times P_{k_l}(A_l)$.

Proof. We take $A_1 = \{\{1, 2, 3\}\}$. We have

$$A(A_1) = P_1(\{1, 2, 3\}) \times P_2(\{4, \dots, n\}).$$

If $n \leq 4$, $A(A_1)$ is empty and $A_1 = A(3, 3)$ is a maximal antipodal subset of $P_3(n)$. Thus we suppose that $n \geq 5$. The stabilizer $S(A_1)$ of A_1 is equal to $\text{Sym}(\{1, 2, 3\}) \times \text{Sym}(\{4, \dots, n\})$. So $A(A_1)$ is an orbit of $S(A_1)$. We take the minimum element $\{1, 4, 5\}$ of $A(A_1)$, add it to A_1 and obtain $A_2 = \{\{1, 2, 3\}, \{1, 4, 5\}\} = A(3, 5)$.

$$\begin{aligned} A(A_2) &= \{\alpha \in A(A_1) - \{\{1, 4, 5\}\} \mid \alpha \text{ and } \{1, 4, 5\} \text{ are antipodal}\} \\ &= P_1(\{2, 3\}) \times P_1(\{4, 5\}) \times P_1(\{6, \dots, n\}) \\ &\quad \cup P_1(\{1\}) \times P_2(\{6, \dots, n\}). \end{aligned}$$

If $n = 5$, $A(A_2)$ is empty and $A_2 = A(3, 5)$ is a maximal antipodal subset of $P_3(5)$. Thus we suppose that $n \geq 6$. The stabilizer $S(A_2)$ fixes 1 and induces permutations on $\{\{2, 3\}, \{4, 5\}\}$, so it acts transitively on $P_1(\{2, 3\}) \times P_1(\{4, 5\}) \times P_1(\{6, \dots, n\})$ and $P_1(\{1\}) \times P_2(\{6, \dots, n\})$. These are two orbits of $S(A_2)$. If $n = 6$, then the second orbit is empty and $A(A_2) = P_1(\{2, 3\}) \times P_1(\{4, 5\}) \times P_1(\{6\})$. We take the minimum element $\{2, 4, 6\}$ of the orbit $A(A_2)$, add it to A_2 and obtain $A_{3,1} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}\}$.

$$\begin{aligned} A(A_{3,1}) &= \{\alpha \in A(A_2) - \{\{2, 4, 6\}\} \mid \alpha \text{ and } \{2, 4, 6\} \text{ are antipodal}\} \\ &= \{\{3, 5, 6\}\}. \end{aligned}$$

We add $\{3, 5, 6\}$ to $A_{3,1}$ and obtain

$$\{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}\} = B(3, 6),$$

which is a maximal antipodal subset of $P_3(6)$. We have obtained representatives of all congruent classes of maximal antipodal subsets in $P_3(n)$ for $n \leq 6$.

We suppose that $n \geq 7$. In this case $A(A_2)$ has two orbits of $S(A_2)$:

$$\begin{aligned} &P_1(\{2, 3\}) \times P_1(\{4, 5\}) \times P_1(\{6, \dots, n\}), \\ &P_1(\{1\}) \times P_2(\{6, \dots, n\}), \end{aligned}$$

which are not empty. Hence we divide the procedure to two cases where we take minimum elements in the two orbits.

(1) We take the minimum element $\{2, 4, 6\}$ of the orbit $P_1(\{2, 3\}) \times P_1(\{4, 5\}) \times P_1(\{6, \dots, n\})$, add it to A_2 and obtain

$$A_{3,1} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}\}.$$

We have

$$\begin{aligned}
A(A_{3,1}) &= \{\alpha \in A(A_2) - \{\{2, 4, 6\}\} \mid \alpha \text{ and } \{2, 4, 6\} \text{ are antipodal}\} \\
&= \{\{3, 5, 6\}\} \cup P_1(\{2\}) \times P_1(\{5\}) \times P_1(\{7, \dots, n\}) \\
&\quad \cup P_1(\{3\}) \times P_1(\{4\}) \times P_1(\{7, \dots, n\}) \\
&\quad \cup P_1(\{1\}) \times P_1(\{6\}) \times P_1(\{7, \dots, n\}) \\
&= \{\{3, 5, 6\}\} \cup \{\{1, 6\}, \{2, 5\}, \{3, 4\}\} \times P_1(\{7, \dots, n\}).
\end{aligned}$$

In $A(A_{3,1})$, $\{3, 5, 6\}$ and other elements are antipodal. We can add it to $A_{3,1}$ and obtain

$$A_{4,1} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}\} = B(3, 6).$$

We have

$$A(A_{4,1}) = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\} \times P_1(\{7, \dots, n\}).$$

The stabilizer $S(A_{4,1})$ induces permutations on

$$\{\{1, 6\}, \{2, 5\}, \{3, 4\}\},$$

so it acts transitively on $A(A_{4,1})$. This is an orbit of $S(A_{4,1})$. We take the minimum element $\{1, 6, 7\}$ of $A(A_{4,1})$, add it to $A_{4,1}$ and obtain

$$A_{5,1} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}, \{1, 6, 7\}\}.$$

We have

$$A(A_{5,1}) = \{\{2, 5, 7\}, \{3, 4, 7\}\}.$$

These two elements are antipodal. We can add these to $A_{5,1}$ and obtain

$$\begin{aligned}
&\{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}, \{1, 6, 7\}, \{2, 5, 7\}, \{3, 4, 7\}\} \\
&= B(3, 7),
\end{aligned}$$

which is a maximal antipodal subset of $P_3(n)$ for $n \geq 7$.

(2) We take the minimum element $\{1, 6, 7\}$ of the orbit $P_1(\{1\}) \times P_2(\{6, \dots, n\})$, add it to A_2 and obtain

$$A_{3,2} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}\}.$$

We have

$$\begin{aligned}
A(A_{3,2}) &= P_1(\{2, 3\}) \times P_1(\{4, 5\}) \times P_1(\{6, 7\}) \\
&\quad \cup P_1(\{1\}) \times P_2(\{8, \dots, n\}).
\end{aligned}$$

The stabilizer $S(A_{3,2})$ fixes 1 and induces permutations on

$$\{\{2, 3\}, \{4, 5\}, \{6, 7\}\},$$

so it acts transitively on $P_1(\{2, 3\}) \times P_1(\{4, 5\}) \times P_1(\{6, 7\})$ and $P_1(\{1\}) \times P_2(\{8, \dots, n\})$. These are two orbits of $S(A_{3,2})$. If $n \leq 8$, then the second orbit is empty and $A(A_{3,2}) = P_1(\{2, 3\}) \times P_1(\{4, 5\}) \times P_1(\{6, 7\})$. We take the minimum element $\{2, 4, 6\}$, add it to $A_{3,2}$ and obtain

$$A_{4,2} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}\}.$$

We have

$$A(A_{4,2}) = \{\{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}.$$

These are antipodal to each other. We can add these to $A_{4,2}$ and obtain $B(3, 7)$. As we have showed in the case (1), $B(3, 7)$ is a maximal antipodal subset of $P_3(n)$ for $n \geq 7$. We have obtained representatives of all congruent classes of maximal antipodal subsets in $P_3(n)$ for $n \leq 8$.

We suppose that $n \geq 9$. In this case $A(A_{3,2})$ has two orbits of $S(A_{3,2})$:

$$\begin{aligned} &P_1(\{2, 3\}) \times P_1(\{4, 5\}) \times P_1(\{6, 7\}), \\ &P_1(\{1\}) \times P_2(\{8, \dots, n\}), \end{aligned}$$

which are not empty. If we take the minimum element $\{2, 4, 6\}$, then we reach $B(3, 7)$ in a way similar to the above argument. Hence we take the minimum element $\{1, 8, 9\}$ of the second orbit $P_1(\{1\}) \times P_2(\{8, \dots, n\})$, add it to $A_{3,2}$ and obtain

$$A_{4,3} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{1, 8, 9\}\}.$$

We have

$$A(A_{4,3}) = P_1(\{1\}) \times P_2(\{10, \dots, n\}).$$

The stabilizer $S(A_{4,3})$ fixes 1 and induces permutations on

$$\{\{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}\},$$

so it acts transitively on $A(A_{4,3})$. We can repeat this procedure until we get $A(3, 2l + 1)$, where $l = \lfloor (n - 1)/2 \rfloor$. Therefore we complete the proof of the theorem. \square

Corollary 5.1. *The 2-number $\#_2 \tilde{G}_3(\mathbb{R}^n)$ is as follows:*

n	4	5	6	7, ..., 16	more than 16
$\#_2 \tilde{G}_3(\mathbb{R}^n)$	2	4	8	14	$2\lfloor (n - 1)/2 \rfloor$

Proof. We have

$$\#A(3, 2l + 1) = l, \quad \#B(3, 6) = 4, \quad \#B(3, 7) = 7$$

and obtain the corollary by Corollary 3.1. \square

Remark 5.2. We can construct 3-forms from maximal antipodal subsets of $P_3(n)$ in a way similar to Remark 4.2. The special Lagrangian 3-form on \mathbb{C}^3 is defined by

$$\begin{aligned} & \operatorname{Re}[(e_1^* + \sqrt{-1}e_2^*) \wedge (e_3^* + \sqrt{-1}e_4^*) \wedge (e_5^* + \sqrt{-1}e_6^*)] \\ &= e_1^* \wedge e_3^* \wedge e_5^* - e_1^* \wedge e_4^* \wedge e_6^* - e_2^* \wedge e_3^* \wedge e_6^* - e_2^* \wedge e_4^* \wedge e_5^*, \end{aligned}$$

which corresponds to $\{\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$. This is a maximal antipodal subset of $P_3(6)$, so it is congruent with $B(3, 6)$ by Theorem 5.1. We can also directly see that the subset transformed by acting the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 5 & 3 & 4 \end{pmatrix}$ on it is equal to $B(3, 6)$.

The 3-form constructed from $B(3, 7)$ is equal to

$$\begin{aligned} & e_1^* \wedge e_2^* \wedge e_3^* + e_1^* \wedge e_4^* \wedge e_5^* + e_2^* \wedge e_4^* \wedge e_6^* + e_3^* \wedge e_5^* \wedge e_6^* + e_1^* \wedge e_6^* \wedge e_7^* \\ &+ e_2^* \wedge e_5^* \wedge e_7^* + e_3^* \wedge e_4^* \wedge e_7^*, \end{aligned}$$

which was found by Harvey-Lawson [2] as a 3-form on $\operatorname{Im}\mathbb{O} = \mathbb{R}^7$ invariant under the action of the exceptional compact simple Lie group G_2 on $\operatorname{Im}\mathbb{O}$.

6. MAXIMAL ANTIPODAL SUBSETS OF $P_4(n)$

We define three antipodal subsets of $P_4(n)$ as follows:

$$\begin{aligned} A(4, 2l) &= \{\alpha \cup \beta \in P_4(2l) \mid \alpha, \beta \in \{\{1, 2\}, \{3, 4\}, \dots, \{2l-1, 2l\}\}\}, \\ B(4, 7) &= B^c(3, 7) = \{\alpha^c \mid \alpha \in B(3, 7)\}, \\ B(4, 8) &= B(4, 7) \cup B(3, 7) \times \{\{8\}\}. \end{aligned}$$

It is easy to see that $A(4, 2l)$ is antipodal. Lemma 4.1 and Theorem 5.1 imply that $B(4, 7)$ is a maximal antipodal subset of $P_4(7)$. Since $B(3, 7)$ is antipodal in $P_3(7)$, so is $B(3, 7) \times \{\{8\}\}$ in $P_4(8)$. For $\alpha, \beta \in B(3, 7)$ we have $\alpha^c \cap (\beta \cup \{8\}) = \beta - \alpha$. Thus $B(4, 8)$ is antipodal.

As stated in Theorem 6.1, $A(4, 2l)$ is a maximal antipodal subset in $P_4(2l)$ and $P_4(2l+1)$ except for the cases of $A(4, 6) \subset P_4(7)$ and $A(4, 8) \subset P_4(8), P_4(9)$. We can directly show that these are not maximal.

$$B(4, 7) \supset \{\alpha \cup \beta \in P_4(7) \mid \alpha, \beta \in \{\{2, 3\}, \{4, 5\}, \{6, 7\}\}\},$$

which is congruent with $A(4, 6)$ in $P_4(7)$. Thus $A(4, 6)$ is not a maximal antipodal subset in $P_4(7)$.

$$B(4, 8) \supset \{\alpha \cup \beta \in P_4(7) \mid \alpha, \beta \in \{\{2, 3\}, \{4, 5\}, \{6, 7\}, \{1, 8\}\}\},$$

which is congruent with $A(4, 8)$ in $P_4(8)$. Thus $A(4, 8)$ is not a maximal antipodal subset in $P_4(8)$. Of course $A(4, 8)$ is not maximal in $P_4(9)$.

In order to state the main theorem of this section we prepare some notation. A subset $A \subset P_k(n)$ is said to be *full* in $P_k(n)$, if A is not congruent with any subset of $P_k(n-1)$. For example $B(4, 7)$ is full in $P_4(7)$, but it is not full in $P_4(8)$. We denote

$$A + m = \{\{\alpha(1) + m, \dots, \alpha(k) + m\} \mid \alpha \in A\}$$

for an integer m .

The following main theorem of this section states that any maximal antipodal subset of $P_4(n)$ is described by $A(4, 2l)$, $B(4, 7)$ and $B(4, 8)$.

Theorem 6.1. *Each of the followings is a maximal antipodal subset of $P_4(n)$:*

n	4, 5	6	7	8, 9	10
	$A(4, 4)$	$A(4, 6)$	$B(4, 7)$	$B(4, 8)$	$A(4, 10), B(4, 8)$

In the case $n > 10$, $A(4, 2\lfloor n/2 \rfloor)$, $B(4, 7) \cup [(a \text{ full MAS in } P_4(n-7)) + 7]$ and $B(4, 8) \cup [(a \text{ MAS in } P_4(n-8)) + 8]$, where MAS is an abbreviation of maximal antipodal subset. Conversely any maximal antipodal subset of $P_4(n)$ is congruent with one of them.

Proof. We prove the theorem in each cases for $n \leq 10$ and by induction on n for $n > 10$. We take $A_1 = \{\{1, 2, 3, 4\}\}$. We have

$$A(A_1) = P_2(\{1, 2, 3, 4\}) \times P_2(\{5, \dots, n\}) \cup P_4(\{5, \dots, n\}).$$

If $n \leq 5$, $A(A_1)$ is empty and $A_1 = A(4, 4)$ is a maximal antipodal subset of $P_4(n)$.

Thus we suppose that $n \geq 6$. The stabilizer $S(A_1)$ is equal to $\text{Sym}(\{1, 2, 3, 4\}) \times \text{Sym}(\{5, \dots, n\})$, which acts transitively on

$$P_2(\{1, 2, 3, 4\}) \times P_2(\{5, \dots, n\}) \quad \text{and} \quad P_4(\{5, \dots, n\}).$$

These are two orbits of $S(A_1)$. Hence we divide the procedure to two cases where we take minimum elements in the two orbits.

(1) We take the minimum element $\{1, 2, 5, 6\}$ of the orbit $P_2(\{1, 2, 3, 4\}) \times P_2(\{5, \dots, n\})$, add it to A_1 and obtain $A_{2,1} = \{\{1, 2, 3, 4\}, \{1, 2, 5, 6\}\}$.

We have

$$\begin{aligned}
A(A_{2,1}) &= \{\alpha \in A(A_1) - \{\{1, 2, 5, 6\}\} \mid \alpha \text{ and } \{1, 2, 5, 6\} \text{ are antipodal}\} \\
&= \{\{3, 4, 5, 6\}\} \cup \{\{1, 2\}, \{3, 4\}\} \times P_2(\{7, \dots, n\}) \\
&\quad \cup P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, \dots, n\}) \\
&\quad \cup P_2(\{5, 6\}) \times P_2(\{7, \dots, n\}) \cup P_4(\{7, \dots, n\}) \\
&= \{\{3, 4, 5, 6\}\} \\
&\quad \cup P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, \dots, n\}) \\
&\quad \cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{7, \dots, n\}) \\
&\quad \cup P_4(\{7, \dots, n\}).
\end{aligned}$$

In $A(A_{2,1})$, $\{3, 4, 5, 6\}$ and other elements are antipodal. We can add it to $A_{2,1}$ and obtain

$$A_{3,1} = \{\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}\} = A(4, 6).$$

We have

$$\begin{aligned}
A(A_{3,1}) &= P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, \dots, n\}) \\
&\quad \cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{7, \dots, n\}) \\
&\quad \cup P_4(\{7, \dots, n\}).
\end{aligned}$$

The stabilizer $S(A_{3,1})$ induces permutations on $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, so it acts transitively on each of

$$\begin{aligned}
&P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, \dots, n\}), \\
&\{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{7, \dots, n\}), \\
&P_4(\{7, \dots, n\}).
\end{aligned}$$

These are three orbits of $S(A_{3,1})$. In the case $n = 6$, we have $A(A_{3,1}) = \emptyset$ and $A_{3,1}$ is a maximal antipodal subset of $P_4(6)$. Thus we suppose that $n \geq 7$. We divide the procedure to the three cases.

(1.1) We take the minimum element $\{1, 3, 5, 7\}$ of the orbit $P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, \dots, n\})$, add it to $A_{3,1}$ and obtain

$$A_{4,1} = \{\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}, \{1, 3, 5, 7\}\}.$$

We have

$$\begin{aligned}
A(A_{4,1}) &= \{\alpha \in A(A_{3,1}) - \{\{1, 3, 5, 7\}\} \mid \alpha \text{ and } \{1, 3, 5, 7\} \text{ are antipodal}\} \\
&= \{\{1, 4, 6, 7\}, \{2, 3, 6, 7\}, \{2, 4, 5, 7\}\} \\
&\quad \cup \{\{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 6\}\} \times P_1(\{8, \dots, n\}) \\
&\quad \cup \{\{1, 2, 7\}, \{3, 4, 7\}, \{5, 6, 7\}\} \times P_1(\{8, \dots, n\}) \\
&\quad \cup P_4(\{8, \dots, n\}).
\end{aligned}$$

Any element of $\{\{1, 4, 6, 7\}, \{2, 3, 6, 7\}, \{2, 4, 5, 7\}\}$ and any element of $A(A_{4,1})$ are antipodal. We can add these three elements to $A_{4,1}$ and obtain

$$A_{7,1} = \{\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}, \{1, 3, 5, 7\}, \\ \{1, 4, 6, 7\}, \{2, 3, 6, 7\}, \{2, 4, 5, 7\}\}.$$

From the description of $A_{7,1}$ we get

$$A_{7,1}^c = \{\alpha^c \mid \alpha \in A_{7,1}\} \\ = \{\{5, 6, 7\}, \{3, 4, 7\}, \{1, 2, 7\}, \{2, 4, 6\}, \\ \{2, 3, 5\}, \{1, 4, 5\}, \{1, 3, 6\}\}.$$

Acting the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ on $A_{7,1}^c$ we obtain $B(3, 7)$.

So $A_{7,1}$ and $B(4, 7) = B^c(3, 7)$ are congruent. We have

$$A(A_{7,1}) = \{\{1, 2, 7\}, \{3, 4, 7\}, \{5, 6, 7\}, \\ \{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 6\}\} \times P_1(\{8, \dots, n\}) \\ \cup P_4(\{8, \dots, n\}) \\ = A_{7,1}^c \times P_1(\{8, \dots, n\}) \cup P_4(\{8, \dots, n\}).$$

Since the group of all projective transformations on F_2P^2 acts transitively on the set of all projective lines in it, the stabilizer $S(A_{7,1})$ acts transitively on $A_{7,1}^c \times P_1(\{8, \dots, n\})$. It also acts transitively on $P_4(\{8, \dots, n\})$. We can also directly see that $S(A_{7,1})$ acts transitively on $A_{7,1}^c \times P_1(\{8, \dots, n\})$. Hence

$$A_{7,1}^c \times P_1(\{8, \dots, n\}), \quad P_4(\{8, \dots, n\})$$

are two orbits of $S(A_{7,1})$. In the case $n = 7$ we have $A(A_{7,1}) = \emptyset$ and $A_{7,1}$ is a maximal antipodal subset of $P_4(7)$. Thus we suppose that $n \geq 8$ in the case (1.1).

(1.1.1) We take the minimum element $\{1, 2, 7, 8\}$ of the orbit $A_{7,1}^c \times P_1(\{8, \dots, n\})$, add it to $A_{7,1}$ and obtain

$$A_{8,1} = \{\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}, \{1, 3, 5, 7\}, \\ \{1, 4, 6, 7\}, \{2, 3, 6, 7\}, \{2, 4, 5, 7\}, \{1, 2, 7, 8\}\}.$$

We have

$$A(A_{8,1}) \\ = \{\{3, 4, 7\}, \{5, 6, 7\}, \{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 6\}\} \times \{\{8\}\} \\ \cup P_4(\{9, \dots, n\}).$$

$\{\{3, 4, 7\}, \{5, 6, 7\}, \{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 6\}\} \times \{\{8\}\}$ is antipodal and any element of it and any element of $A(A_{8,1})$ are antipodal, so we can add these six elements to $A_{8,1}$ and obtain

$$\begin{aligned} A_{14} &= \{\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}, \{1, 3, 5, 7\}, \\ &\quad \{1, 4, 6, 7\}, \{2, 3, 6, 7\}, \{2, 4, 5, 7\}\} \\ &\quad \cup \{\{1, 2, 7\}, \{3, 4, 7\}, \{5, 6, 7\}, \{1, 3, 6\}, \\ &\quad \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 6\}\} \times \{\{8\}\} \\ &= A_{7,1} \cup A_{7,1}^c \times \{\{8\}\}, \end{aligned}$$

which is congruent with $B(4, 8)$. We have

$$A(A_{14}) = P_4(\{9, \dots, n\}),$$

which is equivalent to $A(B(4, 8)) = P_4(\{9, \dots, n\})$. In the cases $n = 8, 9, 10, 11$ $A(A_{14})$ is empty and A_{14} is a maximal antipodal subset of $P_4(n)$. In the case $n > 11$ any maximal antipodal subset of $P_4(n)$ we reach in this case is equal to $A_{14} \cup B$ for a maximal antipodal subset B of $P_4(\{9, \dots, n\})$.

(1.1.2) We take the minimum element $\{8, 9, 10, 11\}$ of the orbit $P_4(\{8, \dots, n\})$, add it to $A_{7,1}$ and obtain

$$\begin{aligned} A_{8,2} &= \{\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}, \{1, 3, 5, 7\}, \\ &\quad \{1, 4, 6, 7\}, \{2, 3, 6, 7\}, \{2, 4, 5, 7\}, \{8, 9, 10, 11\}\}. \end{aligned}$$

We have

$$\begin{aligned} A(A_{8,2}) &= A_{7,1}^c \times P_1(\{12, \dots, n\}) \\ &\quad \cup P_2(\{8, 9, 10, 11\}) \times P_2(\{12, \dots, n\}) \cup P_4(\{12, \dots, n\}). \end{aligned}$$

The stabilizer $S(A_{8,2})$ is equal to

$$S(A_{7,1}^c) \times \text{Sym}(\{8, 9, 10, 11\}) \times \text{Sym}(\{12, \dots, n\}),$$

which acts transitively on each of

$$\begin{aligned} &A_{7,1}^c \times P_1(\{12, \dots, n\}), \\ &P_2(\{8, 9, 10, 11\}) \times P_2(\{12, \dots, n\}), \\ &P_4(\{12, \dots, n\}). \end{aligned}$$

These are three orbits of $S(A_{8,2})$. We divide the procedure to two cases of (1.1.2.1) $A_{7,1}^c \times P_1(\{12, \dots, n\})$ and (1.1.2.2) $P_2(\{8, 9, 10, 11\}) \times P_2(\{12, \dots, n\})$, $P_4(\{12, \dots, n\})$.

(1.1.2.1) In the case $n \geq 12$ we take the minimum element $\{1, 2, 7, 12\}$ of the orbit $A_{7,1}^c \times P_1(\{12, \dots, n\})$. In a way similar to the case (1.1.1) we reach the union of a maximal antipodal subset in $P_4(\{1, \dots, 7\}) \cup$

$\{12\}$) which is congruent with $B(4, 8)$ and a maximal antipodal subset of

$$P_4(\{8, 9, 10, 11\} \cup \{13, \dots, n\}).$$

This is congruent with the union of $B(4, 8)$ and a maximal antipodal subset in $P_4(\{9, \dots, n\})$.

(1.1.2.2) We take the minimum element of the orbit $P_2(\{8, 9, 10, 11\}) \times P_2(\{12, \dots, n\})$, $P_4(\{12, \dots, n\})$ and repeat the procedure. If we take an element of $A_{7,1}^c \times P_1(\{12, \dots, n\})$ in this case, then the result reduces to the case (1.1.2.1). Hence it is sufficient to consider maximal antipodal subsets of $P_4(n)$ which are included in

$$\begin{aligned} & A_{8,2} \cup P_2(\{8, 9, 10, 11\}) \times P_2(\{12, \dots, n\}) \cup P_4(\{12, \dots, n\}) \\ & \subset A_{7,1} \cup P_4(\{8, \dots, n\}). \end{aligned}$$

Any maximal antipodal subset of $P_4(n)$ we reach in this case is equal to $A_{7,1} \cup B$ for a maximal antipodal subset of $P_4(\{8, \dots, n\})$. If B is not full in $P_4(\{8, \dots, n\})$, then there exists $m \in \{8, \dots, n\}$ which is not contained in any element of B and

$$A_{7,1} \cup B \subset A_{7,1} \cup (A_{7,1}^c \times \{\{m\}\}) \cup B.$$

Here $A_{7,1} \cup (A_{7,1}^c \times \{\{m\}\}) \cup B$ is antipodal and this contradicts to the maximal property of $A_{7,1} \cup B$. Thus B is full in $P_4(\{8, \dots, n\})$. Conversely we show that $A_{7,1} \cup B$ is a maximal antipodal subset of $P_4(n)$ if B is a full maximal antipodal subset of $P_4(\{8, \dots, n\})$. If $A_{7,1} \cup B$ is not a maximal antipodal subset of $P_4(n)$, then there exists $\alpha \notin A_{7,1} \cup B$ such that α and any element of $A_{7,1} \cup B$ are antipodal. In particular

$$\alpha \in A(A_{7,1}) = A_{7,1}^c \times P_1(\{8, \dots, n\}) \cup P_4(\{8, \dots, n\}).$$

If α belongs to $P_1(\{8, \dots, n\})$, then this contradicts the maximal property of B . If α belongs to $A_{7,1}^c \times P_1(\{8, \dots, n\})$, then $\#(\alpha \cap \{8, \dots, n\}) = 1$. By the assumption of induction B is a certain union of copies of $B(4, 7)$, $B(4, 8)$ and $A(4, 2l)$. Moreover B is full in $P_4(\{8, \dots, n\})$. So there exists an element β in B such that $\#(\alpha \cap \beta) = 1$, which is a contradiction. Therefore $A_{7,1} \cup B$ is a maximal antipodal subset of $P_4(n)$.

(1.2) We take the minimum element $\{1, 2, 7, 8\}$ of the orbit

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{7, \dots, n\}),$$

add it to $A_{3,1} = A(4, 6)$ and obtain

$$A_{4,2} = A(4, 6) \cup \{\{1, 2, 7, 8\}\}.$$

We have

$$\begin{aligned} A(A_{4,2}) &= P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, 8\}) \\ &\cup \{\{3, 4, 7, 8\}, \{5, 6, 7, 8\}\} \\ &\cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{9, \dots, n\}) \\ &\cup P_2(\{7, 8\}) \times P_2(\{9, \dots, n\}) \cup P_4(\{9, \dots, n\}). \end{aligned}$$

$\{\{3, 4, 7, 8\}, \{5, 6, 7, 8\}\}$ is antipodal and any element of it and any element of $A(A_{4,2})$ are antipodal, so we can add it to $A_{4,2}$ and obtain

$$A_{6,1} = A(4, 6) \cup \{\{1, 2, 7, 8\}, \{3, 4, 7, 8\}, \{5, 6, 7, 8\}\} = A(4, 8).$$

We have

$$\begin{aligned} A(A_{6,1}) &= P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, 8\}) \\ &\cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \times P_2(\{9, \dots, n\}) \\ &\cup P_4(\{9, \dots, n\}). \end{aligned}$$

The stabilizer $A(A_{6,1})$ induces permutations on

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\},$$

so it acts transitively on each of

$$\begin{aligned} &P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, 8\}), \\ &\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \times P_2(\{9, \dots, n\}), \\ &P_4(\{9, \dots, n\}). \end{aligned}$$

These are three orbits of $S(A_{6,1})$. Hence we divide the procedure to three cases.

(1.2.1) We take the minimum element $\{1, 3, 5, 7\}$ of the orbit

$$P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, 8\}),$$

add it to $A_{6,1}$ and obtain

$$A_{7,2} = A(4, 8) \cup \{\{1, 3, 5, 7\}\}.$$

Let

$$\begin{aligned} Ev_8 &= \{\{i_1, i_2, i_3, i_4\} \mid i_j \in \{2j-1, 2j\} (1 \leq j \leq 4), \\ &\quad \text{the number of even numbers is even}\}. \end{aligned}$$

We have

$$A(A_{7,2}) = (Ev_8 - \{1, 3, 5, 7\}) \cup P_4(\{9, \dots, n\}).$$

Any element of Ev_8 and any element of $A(A_{7,2})$ are antipodal, so we can add it to $A_{7,2}$ and obtain

$$A(4, 8) \cup Ev_8 = A_{14}.$$

Thus the procedure in this case reduces to the case (1.1.1).

(1.2.2) We take the minimum element $\{1, 2, 9, 10\}$ of the orbit

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \times P_2(\{9, \dots, n\}),$$

add it to $A_{6,1}$ and obtain

$$A_{7,3} = A(4, 8) \cup \{\{1, 2, 9, 10\}\}.$$

We have

$$\begin{aligned} A(A_{7,3}) &= \{\{3, 4, 9, 10\}, \{5, 6, 9, 10\}, \{7, 8, 9, 10\}\} \\ &\cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \times P_2(\{11, \dots, n\}) \\ &\cup P_2(\{9, 10\}) \times P_2(\{11, \dots, n\}) \\ &\cup P_4(\{11, \dots, n\}). \end{aligned}$$

$\{\{3, 4, 9, 10\}, \{5, 6, 9, 10\}, \{7, 8, 9, 10\}\}$ is antipodal. Any element of it and any element of $A(A_{7,3})$ are antipodal, so we can add it to $A_{7,3}$ and obtain

$$\begin{aligned} A_{10} &= A(4, 8) \cup \{\{1, 2, 9, 10\}, \{3, 4, 9, 10\}, \{5, 6, 9, 10\}, \{7, 8, 9, 10\}\} \\ &= A(4, 10). \end{aligned}$$

We have

$$\begin{aligned} A(A_{10}) &= \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}\} \times P_2(\{11, \dots, n\}) \\ &\cup P_4(\{11, \dots, n\}). \end{aligned}$$

Let M be a maximal antipodal subset of $P_4(n)$ we reach in this case. Since $M - A_{10} \subset A(A_{10})$, there exist

$$\begin{aligned} M_1 &\subset \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}\} \times P_2(\{11, \dots, n\}), \\ M_2 &\subset P_4(\{11, \dots, n\}) \end{aligned}$$

satisfying $M = A_{10} \cup M_1 \cup M_2$. Let

$$\begin{aligned} N_1 &= \{\beta \in P_2(\{11, \dots, n\}) \mid \alpha \cup \beta \in M_1 \text{ for some} \\ &\quad \alpha = \{2j - 1, 2j\} (1 \leq j \leq 5)\}. \end{aligned}$$

This is an antipodal subset of $P_2(\{11, \dots, n\})$. So by an action of $\text{Sym}(\{11, \dots, n\})$ we can suppose that

$$N_1 = \{\{11, 12\}, \{13, 14\}, \dots, \{2m - 1, 2m\}\}.$$

Any element of N_1 and any element of M_2 have even intersection. Hence any element of $A(4, 2m)$ and any element of M_2 are antipodal and $A(4, 2m) \cup M_2$ is antipodal. We have

$$M = A_{10} \cup M_1 \cup M_2 \subset A(4, 2m) \cup M_2.$$

Since M is a maximal antipodal subset of $P_4(n)$, we obtain $M = A(4, 2m) \cup M_2$. By repeating the procedure adding $\{1, 2, 11, 12\}$, $\{1, 2, 13, 14\}$ and so on, we reach $A(4, 2m)$ and have

$$A(A(4, 2m)) = \{\{1, 2\}, \dots, \{2m-1, 2m\}\} \times P_2(\{2m+1, \dots, n\}) \\ \cup P_4(\{2m+1, \dots, n\}).$$

Thus there exist

$$M_3 \subset \{\{1, 2\}, \dots, \{2m-1, 2m\}\} \times P_2(\{2m+1, \dots, n\}), \\ M_4 \subset P_4(\{2m+1, \dots, n\})$$

satisfying $M = A(4, 2m) \cup M_3 \cup M_4$. If M_3 is not empty, then there exists $\{2a-1, 2a, b, c\}$ with $1 \leq a \leq m, 2m+1 \leq b, c$. The element $\{b, c\}$ of $P_2(\{2m+1, \dots, n\})$ and any element of M_4 have even intersection. Hence $\{1, 2, b, c\}$ and any element of M_4 are antipodal and $\{b, c\}$ belongs to N_1 , which is a contradiction. Therefore M_3 is empty and $M = A(4, 2m) \cup M_4$. So M_4 is a maximal antipodal subset of $P_4(\{2m+1, \dots, n\})$. By the assumption of induction M_4 is a certain disjoint union of copies of $B(4, 3)$, $B(4, 8)$ and at most $A(4, 2l) + 2m$. If there exists $A(4, 2l) + 2m$, then

$$A(4, 2m) \cup (A(4, 2l) + 2m) \subset A(4, 2(m+l))$$

and moreover any element of $A(4, 2l) + 2m$ and any element of M_4 . So any element of $A(4, 2(m+l))$ and any element of M_4 are antipodal, too. Hence $A(4, 2(m+l)) \cup M_4$ is antipodal and

$$M = A(4, 2m) \cup (A(4, 2l) + 2m) \cup M_4 \subset A(4, 2(m+l)) \cup M_4,$$

which contradicts to the maximality of M . So M_4 is a certain disjoint union of copies of $B(4, 7)$ and $B(4, 8)$. If M_4 is empty, then $M = A(4, 2m)$. If M_4 is not empty, then M includes $B(4, 7)$ or $B(4, 8)$. Thus the procedure in this case reduces to the case (1.1.2.2) or (1.1.1).

(1.2.3) We take the minimum element $\{9, 10, 11, 12\}$ of the orbit $P_4(\{9, \dots, n\})$, add it to $A_{6,1}$ and obtain

$$A_{7,4} = A(4, 8) \cup \{\{9, 10, 11, 12\}\}.$$

We have

$$A(A_{7,4}) = P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, 8\}) \\ \cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \times P_2(\{9, 10, 11, 12\}) \\ \cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \times P_2(\{13, \dots, n\}) \\ \cup P_2(\{9, 10, 11, 12\}) \times P_2(\{13, \dots, n\}) \\ \cup P_4(\{13, \dots, n\}).$$

The stabilizer $S(A_{7,4})$ induces permutations on $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$ and $\{9, 10, 11, 12\}$, so it transitively acts on each of

$$\begin{aligned} & P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, 8\}), \\ & \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \times P_2(\{9, 10, 11, 12\}), \\ & \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \times P_2(\{13, \dots, n\}), \\ & P_2(\{9, 10, 11, 12\}) \times P_2(\{13, \dots, n\}), \\ & P_4(\{13, \dots, n\}). \end{aligned}$$

These are five orbits of $S(A_{7,4})$. Hence we divide the procedure to five cases.

(1.2.3.1) We take the minimum element $\{1, 3, 5, 7\}$ of the orbit $P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{7, 8\})$. This reduces to the case (1.1).

(1.2.3.2) We take the minimum element $\{1, 2, 9, 10\}$ of the orbit

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \times P_2(\{9, 10, 11, 12\}).$$

This reduces to the case (1.2.2).

(1.2.3.3) We take the minimum element $\{1, 2, 13, 14\}$ of the orbit

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \times P_2(\{13, \dots, n\}).$$

This reduces to the case (1.2.2) by the action of the permutation $(9\ 13)(10\ 14)$.

(1.2.3.4) We consider the orbits of $P_2(\{9, 10, 11, 12\}) \times P_2(\{13, \dots, n\})$ and $P_4(\{13, \dots, n\})$. When we proceed the procedure in this case, if we take one element of the orbits of (1.2.3.1), (1.2.3.2) and (1.2.3.3), this reduces to the cases of (1.1) or (1.2.2). Thus it is sufficient to take elements of $P_2(\{9, 10, 11, 12\}) \times P_2(\{13, \dots, n\})$ and $P_4(\{13, \dots, n\})$. For any maximal antipodal subset M we reach in this case there exists an antipodal subset M_1 of $P_4(\{9, \dots, n\})$ which satisfies $M = A(4, 8) \cup M_1$. However, this does not happen, because $A(4, 8)$ is not a maximal antipodal subset of $P_4(8)$.

(1.3) We take the minimum element $\{7, 8, 9, 10\}$ of the orbit $P_4(\{7, \dots, n\})$, add it to $A_{3,1}$ and obtain

$$A_{4,3} = A(4, 6) \cup \{\{7, 8, 9, 10\}\}.$$

We have

$$\begin{aligned} A(A_{4,3}) &= P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{11, \dots, n\}) \\ &\cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{7, 8, 9, 10\}) \\ &\cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{11, \dots, n\}) \\ &\cup P_2(\{7, 8, 9, 10\}) \times P_2(\{11, \dots, n\}) \\ &\cup P_4(\{11, \dots, n\}). \end{aligned}$$

The stabilizer $S(A_{4,3})$ induces permutations on $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ and $\{7, 8, 9, 10\}$, so it acts transitively on each of

$$\begin{aligned} & P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{11, \dots, n\}), \\ & \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{7, 8, 9, 10\}), \\ & \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{11, \dots, n\}), \\ & P_2(\{7, 8, 9, 10\}) \times P_2(\{11, \dots, n\}), \\ & P_4(\{11, \dots, n\}). \end{aligned}$$

These are five orbits of $S(A_{4,3})$. Hence we divide the procedure to five cases.

(1.3.1) We take the minimum element $\{1, 3, 5, 11\}$ of the orbit $P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{11, \dots, n\})$ and add it to $A_{4,3}$. This case reduces to the case (1.1) by the action of the permutation $(7\ 11)(8\ 12)$.

(1.3.2) We take the minimum element $\{1, 2, 7, 8\}$ of the orbit

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{7, 8, 9, 10\})$$

and add it to $A_{4,3}$. This case reduces to the case (1.2).

(1.3.3) We take the minimum element $\{1, 2, 11, 12\}$ of the orbit

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{11, \dots, n\})$$

and add it to $A_{4,3}$. This case reduces to the case (1.2) by the action of the permutation $(7\ 11)(8\ 12)$.

(1.3.4) We take the minimum element $\{7, 8, 11, 12\}$ of the orbit

$$P_2(\{7, 8, 9, 10\}) \times P_2(\{11, \dots, n\}),$$

add it to $A_{4,3}$ and obtain

$$A_5 = A(4, 6) \cup \{\{7, 8, 9, 10\}, \{7, 8, 11, 12\}\}.$$

We have

$$\begin{aligned} A(A_5) &= P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{13, \dots, n\}) \\ &\cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times \{\{7, 8\}, \{9, 10\}, \{11, 12\}\} \\ &\cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{13, \dots, n\}) \\ &\cup \{\{9, 10, 11, 12\}\} \\ &\cup \{\{7, 8\}, \{9, 10\}, \{11, 12\}\} \times P_2(\{13, \dots, n\}) \\ &\cup P_1(\{7, 8\}) \times P_1(\{9, 10\}) \times P_1(\{11, 12\}) \times P_1(\{13, \dots, n\}) \\ &\cup P_4(\{13, \dots, n\}). \end{aligned}$$

In $A(A_5)$ the element $\{9, 10, 11, 12\}$ and other elements are antipodal, so we can add it to A_5 and obtain

$$A_{6,2} = A(4, 6) \cup (A(4, 6) + 6).$$

We have

$$\begin{aligned}
 A(A_{6,2}) &= P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{13, \dots, n\}) \\
 &\quad \cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times \{\{7, 8\}, \{9, 10\}, \{11, 12\}\} \\
 &\quad \cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\} \times P_2(\{13, \dots, n\}) \\
 &\quad \cup P_1(\{7, 8\}) \times P_1(\{9, 10\}) \times P_1(\{11, 12\}) \times P_1(\{13, \dots, n\}) \\
 &\quad \cup P_4(\{13, \dots, n\}).
 \end{aligned}$$

The stabilizer $S(A_{6,2})$ induces permutations on $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ and $\{\{7, 8\}, \{9, 10\}, \{11, 12\}\}$ and interchanging these two, so it acts transitively on each of

$$\begin{aligned}
 &P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{13, \dots, n\}) \\
 &\quad \cup P_1(\{7, 8\}) \times P_1(\{9, 10\}) \times P_1(\{11, 12\}) \times P_1(\{13, \dots, n\}), \\
 &\quad \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times \{\{7, 8\}, \{9, 10\}, \{11, 12\}\}, \\
 &\quad \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\} \times P_2(\{13, \dots, n\}) \\
 &\quad P_4(\{13, \dots, n\}).
 \end{aligned}$$

These are four orbits of $S(A_{6,2})$. Hence we divide the procedure to four cases.

(1.3.4.1) We take the minimum element $\{1, 3, 5, 13\}$ of the orbit

$$\begin{aligned}
 &P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times P_1(\{5, 6\}) \times P_1(\{13, \dots, n\}) \\
 &\quad \cup P_1(\{7, 8\}) \times P_1(\{9, 10\}) \times P_1(\{11, 12\}) \times P_1(\{13, \dots, n\})
 \end{aligned}$$

and add it to $A_{6,2}$. This reduces to the case (1.1).

(1.3.4.2) We take the minimum element $\{1, 2, 7, 8\}$ of the orbit

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times \{\{7, 8\}, \{9, 10\}, \{11, 12\}\}$$

and add it to $A_{6,2}$. This reduces to the case (1.3.2).

(1.3.4.3) We take the minimum element $\{1, 2, 13, 14\}$ of the orbit

$$\begin{aligned}
 &\{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{13, \dots, n\}) \\
 &\quad \cup \{\{7, 8\}, \{9, 10\}, \{11, 12\}\} \times P_2(\{13, \dots, n\})
 \end{aligned}$$

and add it to $A_{6,2}$. This reduces to the case (1.2).

(1.3.4.4) We consider the orbit $P_4(\{13, \dots, n\})$. When we proceed the procedure in this case, if we take one element of the orbits of (1.3.4.1), (1.3.4.2) and (1.3.4.3), this reduces to the cases of (1.1), (1.2) or (1.3.2). Thus it is sufficient to take elements of $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{13, \dots, n\})$ and $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \times P_2(\{13, \dots, n\})$. For any maximal antipodal subset M we reach in this case there exists an antipodal subset M_1 of $P_4(\{13, \dots, n\})$ which satisfies $M = A_{6,2} \cup M_1$. However, this does

not happen, because $A_{6,2} = A(4, 6) \cup (A(4, 6) + 6)$ is not a maximal antipodal subset of $P_4(12)$.

(1.3.5) We consider the orbit $P_4(\{11, \dots, n\})$. When we proceed the procedure in this case, if we take one element of the orbits of (1.3.1), (1.3.2), (1.3.3) and (1.3.4), this reduces to the cases already mentioned above. Thus it is sufficient to take elements of $P_4(\{13, \dots, n\})$. For any maximal antipodal subset M we reach in this case there exists an antipodal subset M_1 of $P_4(\{11, \dots, n\})$ which satisfies $M = A_{4,3} \cup M_1$. However, this does not happen, because $A_{4,3}$ is not a maximal antipodal subset of $P_4(10)$.

(2) We take the minimum element $\{5, 6, 7, 8\}$ of the orbit $P_4(\{5, \dots, n\})$, add it to A_1 and obtain

$$A_{2,2} = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}.$$

We have

$$\begin{aligned} A(A_{2,2}) &= P_2(\{1, 2, 3, 4\}) \times P_2(\{5, 6, 7, 8\}) \\ &\quad \cup P_2(\{1, 2, 3, 4\}) \times P_2(\{9, \dots, n\}) \\ &\quad \cup P_2(\{5, 6, 7, 8\}) \times P_2(\{9, \dots, n\}) \\ &\quad \cup P_4(\{9, \dots, n\}). \end{aligned}$$

The stabilizer $S(A_{2,2})$ induces permutations on $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$ and interchanging these two, so it acts transitively on each of

$$\begin{aligned} &P_2(\{1, 2, 3, 4\}) \times P_2(\{5, 6, 7, 8\}), \\ &P_2(\{1, 2, 3, 4\}) \times P_2(\{9, \dots, n\}) \cup P_2(\{5, 6, 7, 8\}) \times P_2(\{9, \dots, n\}), \\ &P_4(\{9, \dots, n\}). \end{aligned}$$

These are three orbits of $S(A_{2,2})$. Hence we divide the procedure to three cases.

(2.1) We take the minimum element $\{1, 2, 5, 6\}$ of the orbit $P_2(\{1, 2, 3, 4\}) \times P_2(\{5, 6, 7, 8\})$ and add it to $A_{2,2}$. This reduces to the case (1).

(2.2) We take the minimum element $\{1, 2, 9, 10\}$ of the orbit

$$P_2(\{1, 2, 3, 4\}) \times P_2(\{9, \dots, n\}) \cup P_2(\{5, 6, 7, 8\}) \times P_2(\{9, \dots, n\})$$

and add it to $A_{2,2}$. This reduces to the case (1).

(2.3) We take the minimum element $\{9, 10, 11, 12\}$ of the orbit $P_4(\{9, \dots, n\})$ and add it to $A_{2,2}$. When we proceed the procedure in this case, if we take one element of the orbits of (2.1) and (2.2), this reduces to the cases already mentioned above. Thus it is sufficient to take elements of $P_4(\{9, \dots, n\})$. For any maximal antipodal subset M we reach in this case there exists an antipodal subset M_1 of $P_4(\{9, \dots, n\})$ which

satisfies $M = A_{2,2} \cup M_1$. However, this does not happen, because $A_{2,2}$ is not a maximal antipodal subset of $P_4(8)$. \square

Example 6.1. We show the maximal antipodal subsets of $P_4(11)$.

$$A(4, 10), \quad B(4, 8), \quad B(4, 7) \cup (A(4, 4) + 7).$$

Corollary 6.1. Any disjoint union of p subsets congruent with $A(4, 2l)$ where $l \geq 2$ and $l \neq 4$ hold, q subsets congruent with $B(4, 7)$ and r subsets congruent with $B(4, 8)$ is a maximal antipodal subset of $P_4(n)$ in the following cases:

- (1) $p = 0, 1, q > 0, r \geq 0, n = 2lp + 7q + 8r,$
- (2) $p = 1, q = 0, r \geq 0, n = 2lp + 7q + 8r, 2lp + 7q + 8r + 1,$
- (3) $p = 0, q = 0, r > 0, n = 2lp + 7q + 8r, 2lp + 7q + 8r + 1, 2lp + 7q + 8r + 2, 2lp + 7q + 8r + 3.$

Conversely any maximal antipodal subset of $P_4(n)$ is congruent with one of the above subsets.

Proof. By Theorem 6.1 any maximal antipodal subset of $P_4(n)$ is equal to a disjoint union of p subsets congruent with $A(4, 2l)$ where $l \geq 2$ and $l \neq 4$ hold, q subsets congruent with $B(4, 7)$ and r subsets congruent with $B(4, 8)$. We have to find a necessary and sufficient condition that such a disjoint union is a maximal antipodal subset of $P_4(n)$. Since $A(4, 2l) \cup (A(4, 2m) + 2l) \subset A(4, 2(l + m))$, we have $p = 0, 1$. We consider two cases where $q > 0$ and $q = 0$. Because of the relation $B(4, 7) \subset B(4, 8)$, we see that $B(4, 7)$ is not a maximal antipodal subset of $P_4(8)$. In the case where $q > 0$ we have $n = 2lp + 7q + 8r$. We suppose $q = 0$. If l is not four, $A(4, 2l)$ is a maximal antipodal subset of $P_4(2l)$ and $P_4(2l + 1)$ but is not maximal in $P_4(2l + 2)$. $B(4, 8)$ is a maximal antipodal subset of $P_4(8), P_4(9), P_4(10)$ and $P_4(11)$ but is not maximal in $P_4(12)$. Therefore in the case where $p = 1$ we have $n = 2lp + 7q + 8r, 2lp + 7q + 8r + 1$ and in the case where $p = 0$ we have $n = 2lp + 7q + 8r, 2lp + 7q + 8r + 1, 2lp + 7q + 8r + 2, 2lp + 7q + 8r + 3$. \square

Corollary 6.2. The 2-number $\#_2\tilde{G}_4(\mathbb{R}^n)$ is as follows:

n	5	6	7	8, ..., 11	more than 11
$\#_2\tilde{G}_4(\mathbb{R}^n)$	2	6	14	28	$[n/2]([n/2] - 1)$

Proof. We have

$$\#A(4, 2l) = \binom{l}{2} = \frac{1}{2}l(l - 1), \quad \#B(4, 7) = 7, \quad \#B(4, 8) = 14$$

and obtain the corollary by Corollary 3.1. \square

Remark 6.1. Some 4-forms correspond to maximal antipodal subsets of $P_4(n)$ in a way similar to Remark 4.2. The half of the second power product of the Kähler form on $\mathbb{C}^l = \mathbb{R}^{2l}$ corresponds to $A(4, 2l)$. The fundamental 4-form on \mathbb{H}^2 defined by Kraines [3] corresponds to $B(4, 8)$. This form is invariant under the action of $Sp(2)Sp(1)$.

7. MAXIMAL ANTIPODAL SUBSETS OF $P_k(n)$ FOR HIGHER k

We show some results on maximal antipodal subsets of $P_k(n)$ for higher k , by generalizing some arguments in previous sections.

For $k = 2k'$ we define

$$A(k, 2l) = \{\alpha_1 \cup \cdots \cup \alpha_{k'} \in P_k(2l) \mid \alpha_i \in \{\{1, 2\}, \dots, \{2l-1, 2l\}\}\}.$$

We can see easily that $A(k, 2l)$ is an antipodal subset of $P_k(2l)$. This is a generalization of $A(4, 2l)$ defined in the previous section.

Proposition 7.1. *Let $k = 2k'$. If $l \geq 3k' - 1$, then $A(k, 2l)$ is a maximal antipodal subset of $P_k(2l)$ and $P_k(2l+1)$.*

Proof. We suppose that $A(k, 2l)$ is not a maximal antipodal subset of $P_k(2l)$. We can take $\beta \in P_k(2l) - A(k, 2l)$ which is antipodal with every elements of $A(k, 2l)$. Since $\beta \notin A(k, 2l)$, there exists $1 \leq i \leq l$ satisfying $\#(\beta \cap \{2i-1, 2i\}) = 1$. Let

$$B = \{\{1, 2\}, \dots, \{2l-1, 2l\}\}.$$

Since $\#\beta = k$, there exist $\alpha_1, \dots, \alpha_k \in B$ satisfying $\beta \subset \alpha_1 \cup \cdots \cup \alpha_k$. $B - \{\alpha_1, \dots, \alpha_k\}$ has $l - k$ elements and $l - k \geq 3k' - 1 - 2k' = k' - 1$. Hence we can take

$$\beta_1, \dots, \beta_{k'-1} \in B - \{\alpha_1, \dots, \alpha_k\}.$$

$\{2i-1, 2i\} \cup \beta_1 \cup \cdots \cup \beta_{k'-1} \in A(k, 2l)$ and the cardinality of

$$\beta \cap (\{2i-1, 2i\} \cup \beta_1 \cup \cdots \cup \beta_{k'-1}) = \beta \cap \{2i-1, 2i\}$$

is one, thus β and $\{2i-1, 2i\} \cup \beta_1 \cup \cdots \cup \beta_{k'-1}$ are not antipodal, which is a contradiction. Therefore $A(k, 2l)$ is a maximal antipodal subset of $P_k(2l)$.

Next we suppose that $A(k, 2l)$ is not a maximal antipodal subset of $P_k(2l+1)$. We can take $\beta \in P_k(2l+1) - A(k, 2l)$ which is antipodal with every elements of $A(k, 2l)$. Since $A(k, 2l)$ is a maximal antipodal subset of $P_k(2l)$, β contains $2l+1$. The cardinality of $\#\beta$ is even, thus there is $1 \leq i \leq l$ satisfying $\#(\beta \cap \{2i-1, 2i\}) = 1$. Hence we can see that this is a contradiction in a way similar to the previous case. Therefore $A(k, 2l)$ is a maximal antipodal subset of $P_k(2l+1)$. \square

Remark 7.1. The $1/k!$ times the k' -th power product of the Kähler form on $\mathbb{C}^l = \mathbb{R}^{2l}$ corresponds to $A(2k', 2l)$.

In order to construct a maximal antipodal subset including $A(4m, 8m)$ in $P_{4m}(8m)$, we prepare the following lemma.

Lemma 7.1. For $\alpha = \{\alpha_1, \dots, \alpha_{2m}\} \in P_1(\{1, 2\}) \times \dots \times P_1(\{4m - 1, 4m\}) \subset P_{2m}(4m)$ we define

$$\alpha^e = \{i \mid \alpha_i \text{ is even}\}, \quad \alpha^o = \{i \mid \alpha_i \text{ is odd}\}.$$

For any $\alpha, \beta \in P_1(\{1, 2\}) \times \dots \times P_1(\{4m - 1, 4m\})$ we have

$$\#(\alpha \cap \beta) = 2\#(\alpha^e \cap \beta^e) + \#\beta^o - \#\alpha^e.$$

Proof. We note that

$$\alpha^e \cup \alpha^o = \beta^e \cup \beta^o = \{1, \dots, 2m\}$$

are disjoint unions. We have

$$\alpha \cap \beta = \{\alpha_i \mid i \in \alpha^e \cap \beta^e\} \cup \{\alpha_i \mid i \in \alpha^o \cap \beta^o\},$$

which is a disjoint union. Thus we obtain

$$\begin{aligned} \#(\alpha \cap \beta) &= \#(\alpha^e \cap \beta^e) + \#(\alpha^o \cap \beta^o) \\ &= \#(\alpha^e \cap \beta^e) + (\#\beta^o - \#(\alpha^e \cap \beta^o)) \\ &= 2\#(\alpha^e \cap \beta^e) + \#\beta^o - \#\alpha^e. \end{aligned}$$

□

We define Ev_{4m} for a natural number m by

$$Ev_{4m} = \{\{\alpha_1, \dots, \alpha_{2m}\} \mid \alpha_i \in \{2i - 1, 2i\} (1 \leq i \leq 2m), \\ \text{the number of even numbers is even}\}.$$

This is a subset of $P_1(\{1, 2\}) \times P_1(\{3, 4\}) \times \dots \times P_1(\{4m - 1, 4m\}) \subset P_{2m}(4m)$ and a generalization of Ev_8 defined in the proof of Theorem 6.1.

Proposition 7.2. $A(4m, 8m) \cup Ev_{8m}$ is a maximal antipodal subset of $P_{4m}(8m)$ for $m \geq 1$.

Proof. We first show that $A(4m, 8m) \cup Ev_{8m}$ is an antipodal subset of $P_{4m}(8m)$. For any α in $A(4m, 8m)$ and β in Ev_{8m} , $\#(\alpha \cap \beta) = 2m$ by their definitions, thus they are antipodal. We take any elements α and β in Ev_{8m} . Lemma 7.1 implies

$$\#(\alpha \cap \beta) = 2\#(\alpha^e \cap \beta^e) + \#\beta^o - \#\alpha^e,$$

which is even. Hence $\#(\alpha \cap \beta)$ is even and α and β are antipodal. Therefore $A(4m, 8m) \cup Ev_{8m}$ is antipodal.

Next we show that $A(4m, 8m) \cup Ev_{8m}$ is a maximal antipodal subset of $P_{4m}(8m)$. For this purpose we show that for any α in $P_{4m}(8m) - A(4m, 8m) \cup Ev_{8m}$ there exists an element in $A(4m, 8m) \cup Ev_{8m}$ which is not antipodal to α . There are two possibilities of α as follows:

- (1) $\#(\alpha \cap \{2i - 1, 2i\}) = 2$ for some i and $\#(\alpha \cap \{2j - 1, 2j\}) = 1$ for some j ,
- (2) $\alpha \in P_1(\{1, 2\}) \times \cdots \times P_1(\{8m - 1, 8m\}) - Ev_{8m}$.

In the case (1), we put

$$\begin{aligned} \{i \mid \#(\alpha \cap \{2i - 1, 2i\}) = 2\} &= \{i_1, \dots, i_a\}, \\ \{j \mid \#(\alpha \cap \{2j - 1, 2j\}) = 1\} &= \{j_1, \dots, j_b\}. \end{aligned}$$

These imply $4m = \#\alpha = 2a + b$, hence b is even. Thus $2m = a + b/2$. In the case where $b/2$ is odd, we put

$$\begin{aligned} \beta &= \{2i_1 - 1, 2i_1\} \cup \cdots \cup \{2i_a - 1, 2i_a\} \\ &\cup \{2j_1 - 1, 2j_1\} \cup \cdots \cup \{2j_{b/2} - 1, 2j_{b/2}\} \in A(4m, 8m) \end{aligned}$$

and

$$\#(\alpha \cap \beta) = 2a + \frac{b}{2}$$

is odd, hence α and β are not antipodal. In the case where $b/2$ is even, we take i_* satisfying

$$\alpha \cap \{2i_* - 1, 2i_*\} = \emptyset.$$

We put

$$\begin{aligned} \gamma &= \{2i_1 - 1, 2i_1\} \cup \cdots \cup \{2i_a - 1, 2i_a\} \\ &\cup \{2j_1 - 1, 2j_1\} \cup \cdots \cup \{2j_{b/2-1} - 1, 2j_{b/2-1}\} \cup \{2i_* - 1, 2i_*\} \in A(4m, 8m) \end{aligned}$$

and

$$\#(\alpha \cap \gamma) = 2a + \frac{b}{2} - 1$$

is odd, hence α and γ are not antipodal.

In the case (2), $\#\alpha^e$ is odd. For any $\delta \in Ev_{8m}$ we obtain

$$\#(\alpha \cap \delta) = 2\#(\alpha^e \cap \delta^e) + \#\delta^e - \#\alpha^e$$

by Lemma 7.1. Hence $\#(\alpha \cap \delta)$ is odd, and α, δ are not antipodal.

Anyway there exists an element in $A(4m, 8m) \cup Ev_{8m}$ which is not antipodal to α . Therefore $A(4m, 8m) \cup Ev_{8m}$ is a maximal antipodal subset in $P_{4m}(8m)$. \square

Remark 7.2. We have already proved the statement of Proposition 7.2 in the case $m = 1$ in the proof of Theorem 6.1, where we showed $A(4, 8) \cup Ev_8 = A_{14}$ is congruent with $B(4, 8)$ in $P_4(8)$. Proposition 7.1 shows that $A(k, 2l)$ is a maximal antipodal subset of $P_k(2l)$ and

$P_k(2l + 1)$ for $l \geq 3k' - 1$. On the other hand, $A(4m, 8m)$ is not a maximal antipodal subset of $P_{4m}(8m)$ by Proposition 7.2. Moreover the case of $m = 1$ shows that the condition $l \geq 3k' - 1$ of Proposition 7.1 is sharp.

Lemma 7.2. *If $A \subset P_k(n)$ is an antipodal subset, then $A \times \{\{n+1\}\} = \{\sigma \cup \{n+1\} \mid \sigma \in A\}$ is an antipodal subset of $P_{k+1}(n+1)$.*

Proof. For $\alpha, \beta \in A$ we have

$$(\beta \cup \{n+1\}) - (\alpha \cup \{n+1\}) = \beta - \alpha.$$

Thus $A \times \{\{n+1\}\}$ is antipodal, if A is antipodal. \square

For a subset $A \subset P_k(2k+1)$, we have $A^c \subset P_{k+1}(2k+1)$ and $A \times \{\{2k+2\}\} \subset P_{k+1}(2k+2)$. Using these we define the *twisted double* $TD(A)$ of A by

$$TD(A) = A^c \cup A \times \{\{2k+2\}\}.$$

This is a generalization of the construction of $B(4, 8)$ from $B(3, 7)$ in the previous section. In the case where $k = 3$, we have $TD(B(3, 7)) = B(4, 8)$.

Proposition 7.3. *We assume that k is odd. If $A \subset P_k(2k+1)$ is an antipodal subset, then $TD(A)$ is an antipodal subset of $P_{k+1}(2k+2)$. Moreover, if A is a maximal antipodal subset of $P_k(2k+1)$, then $TD(A)$ is a maximal antipodal subset of $P_{k+1}(2k+2)$.*

Proof. If A is antipodal, A^c is an antipodal subset of $P_{k+1}(2k+1)$ by Lemma 4.1, and $A \times \{\{2k+2\}\}$ is an antipodal subset of $P_{k+1}(2k+2)$ by Lemma 7.2. In order to prove that $TD(A)$ is antipodal, it is sufficient to show that α^c and $\beta \cup \{2k+2\}$ are antipodal for $\alpha, \beta \in A$. We have

$$(*) \quad \alpha^c \cap (\beta \cup \{2k+2\}) = \beta - \alpha,$$

where α^c is the complement of α in $\{1, \dots, 2k+1\}$. The cardinality of $(*)$ is even and $k+1$ is even, thus α^c and $\beta \cup \{2k+2\}$ are antipodal in $P_{k+1}(2k+2)$.

We suppose that A is a maximal antipodal subset of $P_k(2k+1)$. We take an element $\gamma \in P_{k+1}(2k+2)$ which is antipodal with every elements of $TD(A)$. In the case where γ does not contain $2k+2$, γ belongs to $P_{k+1}(2k+1)$. A^c is a maximal antipodal subset of $P_{k+1}(2k+1)$ and γ is antipodal with every elements of A^c , hence $\gamma \in A^c \subset TD(A)$. In the case where γ contains $2k+2$, there is δ in $P_k(2k+1)$ satisfying $\gamma = \delta \cup \{2k+2\}$. For any $\beta \in A$ γ and $\beta \cup \{2k+2\}$ are antipodal, thus the cardinality of

$$(\beta \cup \{2k+2\}) - (\delta \cup \{2k+2\}) = \beta - \delta$$

is even. Hence β and δ are antipodal. Because of the maximality of A in $P_k(2k+1)$ we get $\delta \in A$ and $\gamma \in A \times \{\{2k+2\}\} \subset TD(A)$. In any case we have $\gamma \in TD(A)$. Therefore $TD(A)$ is a maximal antipodal subset of $P_{k+1}(2k+2)$. \square

ACKNOWLEDGMENTS

The author would like to thank Makiko Tanaka for useful discussion on the subject of this paper during preparing the manuscript. He is also indebted to the referee, whose comments improved the manuscript.

The author was partly supported by the Grant-in-Aid for Science Research (C) 2012 (No. 24540064), Japan Society for the Promotion of Science.

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