

## Antisymmetric operator algebras, I

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**Abstract.** By an analogy to antisymmetric function algebras, the concept of antisymmetry of a subspace of a  $C^*$ -algebra is introduced. The main theorem gives a sufficient condition for the antisymmetry of the image (and of the norm closure of the image) of a representation of a function algebra. As an application, some results are obtained concerning antisymmetry of spaces defined by the von Neumann calculus and the  $H^\infty$  calculus for a contraction.

**1. Introduction.** To begin with we introduce basic notations, definitions and we recall some known results needed in the sequel.  $C$  denotes the complex plane,  $R$  — the real line.  $\partial D$  stands for the topological boundary of a set  $D \subset C$ . It is an elementary topological fact that  $\partial D \subset R$  implies  $D = \partial D$  if  $D \subset C$  is compact. Let  $A$  be a complex Banach algebra with unit 1. The set  $A^{-1}$  of all invertible elements of  $A$  is open. If  $a \in A$ , then  $\text{sp}_A(a)$  denotes the spectrum of  $a$ . The spectral radius  $r(a)$  of  $a \in A$  is defined as  $\sup \{|\lambda|: \lambda \in \text{sp}_A(a)\}$  and the equality  $r(a) = \lim \|a^n\|^{1/n}$  holds true. If  $B \subset A$  is a Banach subalgebra of  $A$  containing 1, then for every  $b \in B$  we have  $\text{sp}_A(b) \subset \text{sp}_B(b)$  and  $\partial \text{sp}_B(b) \subset \partial \text{sp}_A(b)$ . If  $B$  is a maximal commutative subalgebra of  $A$  containing 1, then  $\text{sp}_A(a) = \text{sp}_B(a)$  for  $a \in B$ . If  $A$  is commutative, we denote by  $\text{spec}(A)$  the maximal ideal space of  $A$  and for every  $a \in A$  we have:  $\text{sp}_A(a) = \{a(a): a \in \text{spec}(A)\}$ . A commutative Banach algebra  $A$  is called *semi-simple* if  $r(a) = 0$  implies  $a = 0$  for  $a \in A$ .

The following known lemma is one of main tools used in this paper; we prove it here for the sake of completeness:

**LEMMA A.** *Let  $A$  be a Banach algebra with unit 1. Take  $a \in A$ . Then for every  $\lambda \in \text{sp}_A(a)$ ,  $\varepsilon > 0$ ,  $x \in A$  such that  $ax = xa$  and  $\|x - a\| < \varepsilon$  there is  $\mu \in \text{sp}_A(x)$  satisfying  $|\lambda - \mu| < \varepsilon$ .*

**Proof.** Fix  $\lambda, \varepsilon, x$  as above. Denote by  $B$  a maximal commutative subalgebra of  $A$  containing 1,  $a, x$ . By the assumptions,  $\lambda \in \text{sp}_A(a) = \text{sp}_B(a) = \{a(a): a \in \text{spec}(B)\}$ . Hence there is  $a \in \text{spec}(B)$  such that  $\lambda = a(a)$ . We have now:  $|a(x) - a(a)| \leq \|x - a\| < \varepsilon$  and, since  $\mu \stackrel{\text{df}}{=} a(x)$  belongs to  $\text{sp}_B(x) = \text{sp}_A(x)$ , the proof is finished.

The above results concerning Banach algebras may be found, for instance, in [1].

Let  $A$  be a complex  $C^*$ -algebra with unit 1.  $C_A$  denotes the set of all scalar multiples of 1. If  $S$  is a subset of  $A$ , we define  $S^* = \{a^*: a \in S\}$ . If  $H$  is a complex Hilbert space,  $L(H)$  denotes the algebra of all linear bounded operators in  $H$ ,  $I_H$  (or simply  $I$ ) is the identity operator in  $H$ ,  $C_H$  stands for  $C_{L(H)}$ . If  $T \in L(H)$ , we write  $\text{sp}(T)$  for  $\text{sp}_{L(H)}(T)$ .  $\mathcal{A}(T)$  denotes the algebra of all polynomials in  $T$ ,  $\overline{\mathcal{A}(T)}$  is its norm closure,  $C^*(T)$  is the  $C^*$ -algebra generated by  $T$  and  $I$ . A subset  $\mathcal{S}$  of  $L(H)$  is called *irreducible* if no non-trivial (closed) subspace of  $H$  reduces every  $T \in \mathcal{S}$ . If  $X$  is a compact Hausdorff space, we write  $C(X)$  for the algebra of all complex continuous functions on  $X$  with the uniform norm. If  $X \subset C^n$ ,  $P(X)$  denotes the algebra of restrictions of all polynomials to  $X$  and  $\overline{P(X)}$  denotes its uniform closure.

We denote by  $\hat{X}$  the polynomially convex hull of  $X \subset C$ , which is known to be the union of  $X$  and all bounded components of  $C - X$ . Moreover,  $\hat{X}$  may be identified with  $\text{spec}(\overline{P(X)})$  (see [3]).

In this paper we initiate a study of antisymmetric operator algebras. Main problems have the origin in the function algebras theory. Let us recall that a function algebra  $A \subset C(X)$  is called *antisymmetric* [3] if every real function in  $A$  is constant. Let us introduce the basic definition:

**DEFINITION.** A subspace  $S$  of a  $C^*$ -algebra  $A$  with unit 1, such that  $1 \in S$ , is called *antisymmetric*, if  $a = a^*$  implies  $a \in C_A$  for  $a \in S$ .

Let us point out that  $S$  is not assumed to be closed in any sense. As we will see later, problems concerning the antisymmetry of the closure of an antisymmetric algebra are rather delicate.

We will deal mostly with antisymmetric algebras. A natural question appearing now is the following: How to characterize operators  $T \in L(H)$  for which algebras  $\mathcal{A}(T)$  and  $\overline{\mathcal{A}(T)}$  are antisymmetric? More generally, given an antisymmetric subspace  $S$  of a  $C^*$ -algebra  $A$  and a linear mapping  $\varphi: S \rightarrow L(H)$  preserving unit, what must we assume about  $\varphi$  to get the antisymmetry of  $\varphi(S)$  or  $\overline{\varphi(S)}$ ? We will present partial answers for these questions. In Section 1 we prove some immediate properties of antisymmetric subspaces. We show in Example 1 that a Banach algebra isomorphism need not preserve antisymmetry, while a  $*$ -isomorphism of  $C^*$ -algebras does. In Section 2 we investigate the antisymmetry of  $\mathcal{A}(T)$ , where  $T \in L(H)$  is a normal operator. We show also two examples (Examples 2 and 3) of antisymmetric algebras, whose closures are not antisymmetric. In Section 3 we prove the main theorem, which gives a sufficient condition for the antisymmetry of the image of a representation of an algebra  $A \subset C(X)$ . Section 4 deals with some consequences of this theorem in

connection with the von Neumann and  $H^\infty$  functional calculi for a contraction  $T$ . The conditions proved there depend on the algebra  $\overline{P(\text{sp}(T))}$ .

First we recall some classical examples of antisymmetric algebras. The most obvious one is the algebra  $C_A$ , where  $A$  is a  $C^*$ -algebra with unit. If  $\Gamma = \{z \in C: |z| = 1\}$  is the unit circle, then the algebra  $\overline{P(\Gamma)} \subset C(\Gamma)$  is antisymmetric. Any linear space consisting of analytic functions in a domain (i.e. a connected open set) in  $C^n$  is antisymmetric, by the Cauchy–Riemann equations.

In the following proposition we collect some immediate properties of antisymmetric subspaces and algebras.

**PROPOSITION 1.** (i) *A subspace  $S$  of a  $C^*$ -algebra with unit 1 containing 1 is antisymmetric if and only if  $S \cap S^* = C_A$ .*

(ii) *An antisymmetric subspace (resp. subalgebra) of a  $C^*$ -algebra with unit is contained in a maximal antisymmetric subspace (resp. subalgebra).*

(iii) *The strong closure  $\bar{A}^s$  of a commutative irreducible subalgebra  $A$  of  $L(H)$ , which contains  $I$ , is antisymmetric.*

(iv) *A strongly closed algebra  $A \subset L(H)$  with  $I$  is antisymmetric if and only if  $A$  contains no projections except 0 and  $I$ .*

(v) *Let  $A$  be a  $C^*$ -algebra with unit 1 and let  $S \subset A$  be an antisymmetric subspace of  $A$ . If a linear mapping  $\varphi: S \rightarrow L(H)$ ,  $\varphi(1) = I$ , has a linear extension  $\varphi_1: A \rightarrow L(H)$  to  $A$  such that  $\varphi_1$  preserves involution and has trivial kernel, then  $\varphi(S)$  is antisymmetric.*

**Proof.** (i) follows from the decomposition of any element  $a \in A$  into the sum  $a = a_1 + ia_2$ , where  $a_1, a_2 \in A$  are self-adjoint. To prove (ii), it suffices to apply the Kuratowski–Zorn lemma. Now we prove (iii). The irreducibility of  $A$  implies that only projections in the commutant  $A'$  of  $A$  are 0 and  $I$ . Since  $A$  is commutative,  $A \subset A'$  and  $\bar{A}^s \subset A'$ . If  $T = T^* \in A$ , then all spectral projections of  $T$  belong to  $\mathcal{A}(T)^s \subset \bar{A}^s \subset A'$ ; hence  $T \in C_H$ . To prove (iv), we argue similarly. If  $T = T^* \in A$ , then all its spectral projections belong to  $A$ , and so, if  $A$  contains no projections except 0 and  $I$ , we get  $T \in C_H$ . The converse is trivial. To prove (v), suppose  $\varphi(a) = \varphi(a)^*$  for some  $a \in S$ . Then  $\varphi_1(a - a^*) = 0$  and hence  $a = a^*$ . The antisymmetry of  $S$  yields the desired result. The proposition is proved.

Observe that (v) of Proposition 1 implies, in particular, that a  $*$ -isomorphism of  $C^*$  algebras preserves the antisymmetry of their subalgebras. The following example shows that an isomorphism of Banach algebras only need not preserve antisymmetry.

**EXAMPLE 1.** Let  $A$  be a Banach algebra with unit 1. If  $e \in A$ ,  $e \neq 0$ ,  $e \neq 1$ , is an idempotent, then 1,  $e$  are linearly independent. Indeed, if not, then  $e = z1$  with some  $z \in C$ ,  $z \neq 0$ . Since  $e$  is an idempotent,  $z^2 = z$ , we get  $z = 1$ ; thus  $e = 1$  — a contradiction. The algebra  $A_e$  generated by

1 and  $e$  consists of elements of the form  $z1 + we$  ( $z, w \in C$ ) and is two-dimensional, hence Banach. Let  $f \in A$ ,  $f \neq 0$ ,  $f \neq 1$ , be another idempotent. The Banach algebras  $A_e$ ,  $A_f$  are isomorphic via the mapping  $z1 + we \rightarrow z1 + wf$ . Now consider in the Hilbert space  $H = C^2$  two operators

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Both  $T, P$  are idempotents. By the above remarks, commutative Banach algebras  $\mathcal{A}(T)$ ,  $\mathcal{A}(P)$  are isomorphic. But  $\mathcal{A}(T)$  is irreducible, hence antisymmetric, by Proposition 1, (iii); on the other hand,  $\mathcal{A}(P)$  contains  $P = P^*$ , which does not belong to  $C_H$ .

**2. Normal operators.** The simplest situation in the characterization of the antisymmetry of  $\mathcal{A}(T)$  occurs, when we assume that  $T \in L(H)$  is normal. Denote by  $X$  the spectrum  $\text{sp}(T)$  of  $T$ . In this case we have the Gelfand–Naimark  $*$ -isomorphism  $\varphi: C^*(T) \rightarrow C(X)$ , which preserves identities, involution and  $\varphi(T) = z$  — the identity function on  $X$ . Hence  $\varphi$  maps  $\mathcal{A}(T)$  onto  $P(X)$  and  $\overline{\mathcal{A}(T)}$  onto  $\overline{P(X)}$ . By Proposition 1, (v), the antisymmetry of  $\mathcal{A}(T)$  ( $\overline{\mathcal{A}(T)}$  respectively) is completely characterized by the antisymmetry of  $P(X)$  ( $\overline{P(X)}$  resp.).

The following remark gives a simple necessary condition for the antisymmetry of  $P(X)$ :

**Remark 1.** Suppose that  $X \subset C$  is a compact set containing at least two different points. If  $P(X)$  is antisymmetric, then  $X$  is infinite.

**Proof.** If  $X = \{z_1, \dots, z_s\}$  is finite and  $z_1 \neq z_2$ , then we can find a polynomial  $p$  such that  $p(z_1) = 0$ ,  $p(z_i) = 1$ ,  $i = 2, \dots, s$ . This  $p$  is a real, non-constant element of  $P(X)$ .

As a corollary we get:

**COROLLARY 1.** If  $T \in L(H)$ ,  $T \notin C_H$ , is a normal algebraic operator, then  $\mathcal{A}(T)$  is not antisymmetric.

**Proof.** Suppose the converse. Then  $P(\text{sp}(T))$  is antisymmetric. The spectral theorem yields that  $\text{sp}(T)$  contains more than one point. Hence, by Remark 1,  $\text{sp}(T)$  is infinite. But this is impossible, because  $T$  is algebraic.

Looking at the algebra  $\mathcal{A}(T)$  of Example 1, we see that this corollary fails without the assumption that  $T$  is normal.

**PROPOSITION 2.** Suppose that  $T \in L(H)$  is an arbitrary operator with non one-point spectrum. If  $P(\text{sp}(T))$  is antisymmetric, so is  $\mathcal{A}(T)$ .

**Proof.** Let  $p(T) = p(T)^*$  with some polynomial  $p$ . Then  $\text{sp}(p(T)) \subset R$  and, by the spectral mapping theorem,  $p$  is real on  $\text{sp}(T)$ . Therefore  $p = c$  on  $\text{sp}(T)$  with some  $c \in R$ . Applying Remark 1, we get that  $\text{sp}(T)$  is infinite, hence  $p = c$  everywhere in  $C$  and  $p(T) = cI$ .

The next proposition shows that  $\mathcal{A}(T)$  can be antisymmetric even in case when  $\text{sp}(T)$  contains only one point.

**PROPOSITION 3.** *Suppose that  $H$  is finite dimensional. If  $T \in L(H)$  and  $\text{sp}(T) = \{c\}$ , then  $\mathcal{A}(T)$  is antisymmetric.*

**Proof.** Since the algebras  $\mathcal{A}(T)$  and  $\mathcal{A}(T + cI)$  are equal, we may assume  $c = 0$ . Then there is an orthonormal basis in  $H$  in which  $T$  is represented by a matrix  $(a_{ij})$  with  $a_{ij} = 0$  for  $i \leq j$ . Now the proposition follows from standard rules of matrix multiplication.

The algebra  $\mathcal{A}(T)$  of Example 1 is an example that this sufficient condition for the antisymmetry of  $\mathcal{A}(T)$  is not necessary. Indeed,  $\text{sp}(T) = \{0, 1\}$  in Example 1.

Now we want to prove two sufficient conditions for the antisymmetry of  $P(X)$ . Before doing it let us state the following known lemma:

**LEMMA 1.** *Let  $G$  be a compact subset of  $C$  with non-empty interior. If  $f \in C(G)$  is non-constant and analytic in  $\text{int}G$ , then  $\partial f(G) \subset f(\partial G)$ .*

**Proof.** If  $x \in \partial f(G)$ , then  $x \in f(G)$ , since  $f(G)$  is compact. Hence  $x = f(z)$  with some  $z \in G$ .  $z$  cannot belong to  $\text{int}G$ , because  $f$  is non-constant and analytic in  $\text{int}G$ , hence  $f$  is an open mapping. Thus  $z \in \partial G$  and the proof is complete.

**THEOREM 1.** *Let  $X \subset C$  be compact and let  $m$  denote the plane Lebesgue measure. Each of the following conditions is sufficient for the antisymmetry of  $P(X)$ :*

- (a)  $C - X$  is not connected,
- (b)  $m(X) > 0$ .

**Proof.** Suppose first that (a) is satisfied. Let  $G$  be the closure of an arbitrary bounded component of  $C - X$ . Obviously,  $\partial G \subset X$ . Suppose that  $p$  is a polynomial, non-constant on  $G$ , satisfying  $p(X) \subset R$ . Hence  $p(\partial G) \subset R$  and, by Lemma 1,  $\partial p(G) \subset R$ . Now,  $p(G)$  is a compact subset of  $C$  with  $\partial p(G) \subset R$ . By the introductory remarks  $p(G) = \partial p(G)$  — a contradiction, because  $p$  is an open mapping. To prove the sufficiency of (b), take a polynomial  $p$  non-constant on  $X$ .  $j(p)$  denotes its Jacobian, as a function of two real variables.  $j(p) \neq 0$   $m$ -almost everywhere. Hence

$$m(p(X)) = \int_X |j(p)| dm > 0.$$

Therefore  $p$  cannot be real and theorem is proved.

Now Proposition 2 and Theorem 1 imply that if  $T \in L(H)$  has the spectrum  $X = \text{sp}(T)$  satisfying (a) or (b) of Theorem 1, then  $\mathcal{A}(T)$  is antisymmetric. Proposition 3 shows that the condition sufficient for the antisymmetry of  $\mathcal{A}(T)$  is, however, not necessary.

Now we would like to present two examples of antisymmetric algebras with non-antisymmetric norm closures.

EXAMPLE 2. Take two disjoint closed discs  $X_1, X_2 \subset C$  and define  $X = X_1 \cup X_2$ .  $X$  has connected complement and  $P(X)$  is antisymmetric (by Theorem 1, (b), for instance). The characteristic function of  $X_1$  is real and non-constant on  $X$  and, by Mergelyan's theorem, this function belongs to  $\overline{P(X)}$ ; hence  $\overline{P(X)}$  is not antisymmetric. To get an operator-type example it is sufficient to take  $T \in L(H)$  as a normal operator with the spectrum  $X$ .

In this example  $X$  was not connected. One can omit this assumption, as the following example shows:

EXAMPLE 3. Let  $X_1 \subset C$  be the closed unit disc and let  $X_2$  be the closed interval  $[1, 2] \subset R$ . Put  $X = X_1 \cup X_2$ .  $X$  and  $C - X$  are connected.  $P(X)$  is antisymmetric (Theorem 1, (b)). Take  $x, y \in X_2$  such that  $x < y$ . By Urysohn's lemma we find a continuous, real function  $g$  on  $X_2$ , such that  $g = 0$  on  $[1, x]$ ,  $g = 1$  on  $[y, 2]$ . Putting  $f = g$  on  $X_2$ ,  $f = 0$  on  $X_1$ , we get a continuous function  $f$  on  $X$  analytic in  $\text{int } X$ . Mergelyan's theorem [3] yields  $f \in \overline{P(X)}$ ; hence  $\overline{P(X)}$  is not antisymmetric.

The author does not know a complete characterization of those compact  $X \subset C$  for which  $\overline{P(X)}$  is antisymmetric. A partial negative result is given in Lavrientieff's theorem [3]:  $\overline{P(X)} = C(X)$  if and only if  $\text{int } X = \emptyset$  and  $C - X$  is connected. Clearly, if  $\overline{P(X)} = C(X)$ , then  $\overline{P(X)}$  cannot be antisymmetric, except the trivial case, when  $X$  contains only one point. But the following necessary condition can be proved:

PROPOSITION 4. If  $\overline{P(X)}$  is antisymmetric, then  $\hat{X}$  is connected and  $\hat{X} = \text{int } \hat{X}$ .

Proof. It is easy to see that if  $\hat{X}$  is not connected or  $\hat{X} \neq \overline{\text{int } \hat{X}}$ , then there are two disjoint, closed sets  $Y_1, Y_2 \subset \hat{X}$ , such that  $\text{int } \hat{X} \subset Y_1 \cup Y_2$ . By Urysohn's lemma, we find a function  $f$  continuous on  $\hat{X}$  with values in the closed interval  $[0, 1]$  such that  $f|_{Y_1} = 0$ ,  $f|_{Y_2} = 1$ . If  $G$  is a component of  $\text{int } \hat{X}$ , then  $f$  must be constant on  $G$ , because  $\text{int } \hat{X} \subset Y_1 \cup Y_2$ . This proves that  $f$  is analytic in  $\text{int } \hat{X}$ . Since  $C - \hat{X}$  is connected,  $f \in P(\hat{X})$ , by Mergelyan's theorem [3]. Obviously,  $f|_X \in \overline{P(X)}$  and  $f|_X$  is non-constant, because  $f$  is the image of  $f|_X$  via the Gelfand isomorphism. Hence  $\overline{P(X)}$  is not antisymmetric and the proposition is proved.

It is obvious that we assume in the last proposition  $\text{int } \hat{X} \neq \emptyset$ . In fact,  $\text{int } \hat{X} = \emptyset$  if and only if  $\text{int } X = \emptyset$  and  $C - X$  is connected. Hence, if  $\text{int } \hat{X} = \emptyset$ , then  $\overline{P(X)} = C(X)$ , by Lavrientieff's theorem. The necessary condition proved in Proposition 4 for the antisymmetry of  $\overline{P(X)}$  is, however, not sufficient. Consider an example:

EXAMPLE 4. Define the sequence of equilateral, closed triangles

$D_n \subset C$  as follows: Assume that exactly one apex  $r_n$  of  $D_n$  lies on the real axis and the side opposite to it is parallel to the real axis.  $D_0, D_1, D_2$  have lengths of the sides equal to 1 and  $r_0 = 0, r_1 = 1, r_2 = 2$ .  $D_3, D_4$  have sides equal to  $\frac{1}{2}$  and  $r_3 = \frac{1}{2}, r_4 = \frac{3}{2}$ . The next triangles  $D_5, D_6, D_7, D_8$  have sides equal to  $\frac{1}{4}$  and  $r_5 = \frac{1}{4}, r_6 = \frac{3}{4}, r_7 = \frac{5}{4}, r_8 = \frac{7}{4}$ . Continuing this procedure we get the sequence  $D_n \subset C$  of disjoint, closed triangles, whose apexes form a dense subset of the closed interval  $[0, 2]$ . Put  $X = [0, 2] \cup \bigcup D_n$ .  $X$  is connected and  $\hat{X} = X = \overline{\text{int } X}$ . We define the function  $f$  on  $X$  to be constant and equal to  $r_n$  on  $D_n$  and the identity function on  $[0, 2]$ .  $f$  is continuous, real, non-constant on  $X$ , analytic in  $\text{int } X$  and, by Mergelyan's theorem,  $f \in \overline{P(X)}$ . Hence  $\overline{P(X)}$  is not antisymmetric.

I express my thanks to Professor J. Siciak for showing me this example and for several valuable suggestions.

**3. Representations of algebras.** In this section we solve the problem stated in Section 1 for commutative  $C^*$ -algebras and their subalgebras. Consider first an arbitrary  $C^*$ -algebra  $A$  with unit 1 and its norm-closed, not necessarily symmetric, subalgebra  $B$  containing 1.

The following remark is known, but we give here a short proof for the sake of completeness.

**Remark 2.** If  $b \in B$  is self-adjoint, then  $\text{sp}_A(b) = \text{sp}_B(b)$ .

**Proof.** The inclusions  $\text{sp}_A(b) \subset \text{sp}_B(b)$  and  $\partial \text{sp}_B(b) \subset \partial \text{sp}_A(b)$  hold by remarks at the beginning of this paper. Since  $b = b^*$ ,  $\partial \text{sp}_B(b) \subset \partial \text{sp}_A(b) = \text{sp}_A(b) \subset R$ ; therefore  $\text{sp}_B(b)$  is a compact subset of  $C$  with the boundary contained in  $R$ . Hence  $\text{sp}_B(b) = \partial \text{sp}_B(b) \subset \text{sp}_A(b)$ .

Throughout the rest of this section  $X$  is a fixed compact Hausdorff space,  $A \subset C(X)$  is an arbitrary subalgebra containing constants (not necessarily closed),  $\varphi: A \rightarrow L(H)$  is an algebra homomorphism preserving units and  $\mathcal{A}$  is the norm closure of  $\varphi(A)$  in  $L(H)$ . The main theorem is the following:

**THEOREM 2.** (i) If for all  $f \in A$

$$(*) \quad f(X) \subset \text{sp}_{\mathcal{A}}(\varphi(f))$$

and  $A$  is antisymmetric, then  $\varphi(A)$  is antisymmetric.

(ii) Suppose that there is a norm closed subalgebra  $\mathcal{B}$  of  $L(H)$  such that  $\mathcal{A} \subset \mathcal{B}$  and for all  $f \in A$

$$(**) \quad f(X) = \text{sp}_{\mathcal{B}}(\varphi(f)).$$

If the  $C(X)$ -closure  $\bar{A}$  of  $A$  is antisymmetric, so is  $\mathcal{A}$ .

**Proof.** First observe that if  $f \in A$  satisfies  $(*)$ , then

$$(1) \quad \|f\| \leq r(\varphi(f)) \leq \|\varphi(f)\|.$$

Now we will establish some limit relations following from assumptions (i) and (ii). Suppose first that (i) is satisfied and take  $S \in \mathcal{A}$ . There is a sequence  $f_n \in A$  such that  $\varphi(f_n) \rightarrow S$ . (1) implies that  $f_n$  converges uniformly on  $X$  to a function  $f_S \in A$ . Fix  $x \in X$ . Then  $f_n(x) - \varphi(f_n) \rightarrow f_S(x) - S$ . If  $f_S(x) - S$  is invertible in  $\mathcal{A}$ , so are  $f_n(x) - \varphi(f_n)$  except a finite number of  $n$ 's, because the set of invertible elements in  $\mathcal{A}$  is open. Thus

$$(2') \quad f_S(X) \subset \text{sp}_{\mathcal{A}}(S).$$

The same arguments work, when we replace  $\mathcal{A}$  by a norm closed subalgebra  $\mathcal{B}$  of  $L(H)$  containing  $\mathcal{A}$  and satisfying  $f(X) \subset \text{sp}_{\mathcal{B}}(\varphi(f))$  for all  $f \in A$ . Hence, for such an algebra we have

$$(2) \quad f_S(X) \subset \text{sp}_{\mathcal{B}}(S),$$

where  $f_S$  is the function constructed above. If (ii) holds, we have for  $\mathcal{B}$  as in (ii):

$$(3) \quad f_S(X) = \text{sp}_{\mathcal{B}}(S).$$

For the proof take  $y \in \text{sp}_{\mathcal{B}}(S)$ . By Lemma A in Section 1 we can find a sequence  $y_n \in \text{sp}_{\mathcal{B}}(\varphi(f_n))$  converging to  $y$ . By (\*\*) we choose  $x_n \in X$  satisfying  $f_n(x_n) = y_n$ . Now we have

$$|f_S(x_n) - y| \leq |f_S(x_n) - f_n(x_n)| + |y_n - y| \leq \|f_n - f_S\| + |y_n - y|$$

and, by the uniform convergence of  $f_n$  to  $f_S$ , we get  $f_S(x_n) \rightarrow y$ . But  $f_S(X)$  is compact; hence  $y \in f_S(X)$  and (3) is proved.

To prove our theorem in case (i) suppose  $\varphi(f) = \varphi(f)^*$  for some  $f \in A$ . By (\*) and Remark 2,  $f(X) \subset \text{sp}_{\mathcal{A}}(\varphi(f)) = \text{sp}(\varphi(f)) \subset R$ ; hence  $f$  is real on  $X$  and, by the antisymmetry of  $A$ ,  $f$  must be constant. Thus  $\varphi(f) \in C_H$ . To finish the proof, take a self-adjoint element  $S \in \mathcal{A}$ . Using again Remark 2 we get  $\text{sp}_{\mathcal{B}}(S) = \text{sp}(S) \subset R$  and, by (2),  $f_S \in \bar{A}$  is real on  $X$ .  $\bar{A}$  is antisymmetric, hence  $f_S$  is constant and, by (3),  $S$  is a self-adjoint operator with one-point spectrum. By the spectral theorem  $S \in C_H$  and our theorem is completely proved.

Some remarks concerning this theorem are now in order. We preserve the above notations.

**Remark 3.** The function  $f_S$  constructed for  $S \in \mathcal{A}$  does not depend neither on the sequence  $f_n$ , nor on the norm closed subalgebra  $\mathcal{B} \subset L(H)$  satisfying  $\mathcal{A} \subset \mathcal{B}$  and  $f(X) \subset \text{sp}_{\mathcal{B}}(\varphi(f))$  for all  $f \in A$ . This is a consequence of (1).

**Remark 4.** Suppose that  $A$  is closed in  $C(X)$  and that (i) holds.  $\varphi$  is continuous if and only if  $\mathcal{A}$  is semi-simple.

Indeed, if  $\mathcal{A}$  is semi-simple, then the continuity of  $\varphi$  follows from a known theorem ([1], Theorem 8, p. 83). Conversely, if  $\varphi$  is continuous and  $S \in \mathcal{A}$  satisfies  $r(S) = 0$ , then, in particular,  $\text{sp}_{\mathcal{A}}(S) = \{0\}$ . By (2'),

$f_S = 0$ , and taking  $f_n \in A$  as in the proof of Theorem 2 we get  $f_n \rightarrow 0$ , whence  $\varphi(f_n) \rightarrow 0 = S$ .

Remark 5. If  $A$  is closed in  $C(X)$  and  $\varphi$  is continuous, then  $\varphi$  is an isomorphism of Banach algebras  $A$  and  $\mathcal{A}$ , by (1). Example 1 shows that even in this case our theorem need not be true without any additional assumption.

Remark 6. Theorem 2 remains true in case (i), when we assume only that  $A$  is a linear subspace of  $C(X)$  with 1,  $\varphi$  is linear and preserves units and (\*) is satisfied for  $f \in A$ . In this case  $\mathcal{A}$  will denote the smallest Banach subalgebra of  $L(H)$  containing  $\varphi(A)$ .

Now we will show some concrete and immediate applications of Theorem 2. Observe that conditions (\*) and (\*\*) have character of spectral mapping theorems. Taking an operator  $T \in L(H)$  and putting  $X = \text{sp}(T)$  we obtain from (i) of Theorem 2 immediately Proposition 2, in virtue of the spectral mapping theorem. But applying (ii) of this theorem, we get

**COROLLARY 2.** *If  $T$  has spectrum  $\text{sp}(T)$  such that  $\overline{P(\text{sp}(T))}$  is antisymmetric, then  $\overline{\mathcal{A}(T)}$  is antisymmetric.*

**Proof.** We take  $\varphi: P(\text{sp}(T)) \rightarrow L(H)$  as  $\varphi(p) = p(T)$ . Then  $\overline{\varphi(\overline{P(\text{sp}(T))})} = \overline{\mathcal{A}(T)}$  and (\*\*) of Theorem 2 holds true for every  $p \in P(\text{sp}(T))$ .

Note that the map  $p \rightarrow p(T)$  is not assumed to be continuous.

**EXAMPLE 5.** If  $T \in L(H)$  is a contraction, i.e.  $\|T\| \leq 1$ , such that  $\text{sp}(T)$  contains the whole unit circle  $I$ , then  $\overline{\mathcal{A}(T)}$  is antisymmetric.

This follows from the antisymmetry of  $\overline{P(I)}$  and Corollary 2. Let us remark that the minimal unitary dilation  $U$  of a non-unitary contraction  $T$  has spectrum  $\text{sp}(U)$  covering the whole unit circle (see [6]). Thus  $\overline{\mathcal{A}(U)}$  is antisymmetric.

Now we derive from Theorem 2 a corollary concerning  $n$ -tuples of commuting operators.

**COROLLARY 3.** *Let  $T_1, \dots, T_n$  be commuting operators in  $L(H)$ . Consider their arbitrary joint spectrum  $X = \text{sp}(T_1, \dots, T_n)$  for which the spectral mapping theorem holds true, i.e., such that for every polynomial  $p$  in  $n$  variables  $p(X) = \text{sp}(p(T_1, \dots, T_n))$ . If  $\overline{P(X)} \subset C(X)$  is an antisymmetric algebra, then the norm closure of the algebra of all polynomials in  $T_1, \dots, T_n$  is antisymmetric.*

The proof follows immediately by using Theorem 2, putting  $A = P(X)$  and  $\varphi(p) = p(T_1, \dots, T_n)$  for  $p \in P(X)$ .

Joint spectra of operators for which the spectral mapping theorem is true are considered in [4].

**4. Applications to functional calculi.** In this section we will use Theorem 2 to establish some conditions sufficient for the antisymmetry

of images of two functional calculi, namely the von Neumann functional calculus and the  $H^\infty$ -functional calculus due to Sz.-Nagy and Foiaş. We will denote by  $D$  the open unit disc in  $C$  and by  $\Gamma$  its boundary.  $A$  stands in this section for the algebra of all complex functions analytic in  $D$ , continuous in  $\bar{D}$ .  $H^\infty$  denotes the algebra of all bounded complex functions analytic in  $D$ . If  $T \in L(H)$  is a contraction, then one can uniquely decompose it into the orthogonal sum  $T = T_1 \oplus T_2$ , where  $T_1$  is unitary and  $T_2$  has no non-zero subspace reducing it to a unitary operator [5]. A contraction  $T$  is called *completely non-unitary* (c.n.u.) if  $T_1 = 0$  in the above decomposition. For every contraction  $T \in L(H)$  the von Neumann inequality is true:  $\|p(T)\| \leq \|p\|_{\bar{D}} = \|p\|_r$  for every polynomial  $p$ . If  $u \in A$ , then  $u$  is a uniform limit on  $\bar{D}$  of polynomials  $p_n$  and  $u(T) = \lim p_n(T)$ , by the definition. By the von Neumann inequality  $u(T)$  is well-defined and  $u(T)$  does not depend on the sequence  $p_n$  converging to  $u$ . The mapping  $\varphi: A \rightarrow L(H)$  defined by  $\varphi(u) = u(T)$  for  $u \in A$  is an algebra homomorphism preserving units,  $\varphi(z) = T$  (where  $z$  is the identity function on  $\bar{D}$ ). Passing to the limit in the von Neumann inequality we get  $\|\varphi(u)\| \leq \|u\|_{\bar{D}} = \|u\|_r$  for all  $u \in A$ . Moreover, the following spectral mapping theorem is true:  $u(\text{sp}(T)) = \text{sp}(u(T))$  for all  $u \in A$  [5]. We will call  $\varphi$  the *von Neumann functional calculus*. Let now  $T$  be a c.n.u. contraction. Then it is possible to define an operator  $u(T) \in L(H)$  for all  $u \in H^\infty$ . We refer to [5] for the precise construction. The mapping  $\psi: H^\infty \rightarrow L(H)$  given by the formula  $\psi(u) = u(T)$  for  $u \in H^\infty$  has similar properties as  $\varphi$  above and the inequality  $\|\psi(u)\| \leq \|u\|_\infty \stackrel{\text{def}}{=} \sup\{|u(z)|: z \in D\}$  holds true for all  $u \in H^\infty$ . Denote by  $B$  the set of all  $u \in H^\infty$  which are, moreover, continuous in a set  $G_u \subset \Gamma$  such that  $\text{sp}(T) \cap \Gamma \subset G_u$ . It is easy to see that  $B$  is a subalgebra of  $H^\infty$ . It has been proved by Foiaş and Mlak in [2] that  $u(\text{sp}(T)) = \text{sp}(u(T))$  for  $u \in B$ . We call  $\psi$  the  $H^\infty$  functional calculus.

Preserving the above notations we now prove the following

**THEOREM 3.** *Let  $T \in L(H)$  be a contraction (a c.n.u. contraction, respectively), with  $X = \text{sp}(T)$ .*

- (i) *If  $\overline{P(X)} \neq C(X)$ , then  $\varphi(A)$  ( $\psi(B)$ , resp.) is antisymmetric,*
- (ii) *If  $\overline{P(X)}$  is antisymmetric, then  $\overline{\varphi(A)}$  ( $\overline{\psi(B)}$ , resp.) is antisymmetric.*

**Proof.** First consider the von Neumann functional calculus  $\varphi$ . The idea of the proof is contained in the following three steps:

1° Define  $\tilde{A} = \{\tilde{u} = u|_X, u \in A\}$  and prove that the correspondence  $u \rightarrow \tilde{u}$  is one-to-one; define the homomorphism  $\tilde{\varphi}: \tilde{A} \rightarrow L(H)$  putting  $\tilde{\varphi}(\tilde{u}) = \varphi(u)$  (it is, indeed, well-defined).

2° Show that  $\tilde{A}$  is antisymmetric in case (i) and the  $C(X)$ -closure  $\bar{\tilde{A}}$  of  $\tilde{A}$  is antisymmetric in case (ii).

3° Apply Theorem 2 to the homomorphism  $\tilde{\varphi}$ . By 1°, 2° and the spectral mapping theorem, all assumptions of Theorem 2 are satisfied.

We prove 1°. If  $\overline{P(X)}$  is antisymmetric, then  $\overline{P(X)} \neq C(X)$ . Hence we have to prove 1° only in case, when  $\text{int } X \neq \emptyset$  or  $C - X$  is not connected. If  $X = \Gamma$  and  $\tilde{u} = \tilde{v}$ , then  $u = v$  by the Poisson formula. If  $X \neq \Gamma$ , then  $X$  has a cluster point  $x_0$  in  $D$ , i.e., there is a sequence  $x_n$  converging to  $x_0$  and such that  $\{x_n: n \geq 0\} \subset D \cap X$ . If  $\tilde{u} = \tilde{v}$ , then, in particular,  $u(x_n) = v(x_n)$ ,  $n = 0, 1, \dots$ , and since the zeroes of an analytic function are discrete, we have  $u = v$  on  $D$  and  $u = v$  on  $\bar{D}$  by continuity.

Let us now prove 2° in case (i). If  $\text{int } X \neq \emptyset$ , let  $G$  be an arbitrary component of  $\text{int } X$ . Suppose that a function  $\tilde{u} \in \tilde{A}$  is real. By Cauchy-Riemann equations,  $\tilde{u}$  (hence  $u$ ) is constant on  $G \subset D$ ; thus  $u$  is constant on  $\bar{D}$ . If  $X = \Gamma$ , then  $\tilde{A}$  is antisymmetric. Suppose that  $C - X$  is not connected and  $X \neq \Gamma$ . Let now  $G$  be the closure of a bounded component of  $C - X$ . If a function  $\tilde{u} \in \tilde{A}$  is constant on  $\partial G$ , then as in the proof of 1°,  $u$  is constant on  $\bar{D}$ . Assume that  $\tilde{u}$  is real and non-constant on  $\partial G$ . Then we have  $u(\partial G) = \tilde{u}(\partial G) \subset \mathbb{R}$  and, by Lemma 1,  $\partial u(G) \subset \mathbb{R}$ . Arguing as in the proof of Theorem 1, (a), we have  $\text{int } u(G) = \emptyset$ , which is a contradiction with the analyticity of  $u$  in  $\text{int } G$ . We have proved that if  $\overline{P(X)} \neq C(X)$ , then  $\tilde{A}$  is antisymmetric. Assume now that  $\overline{P(X)}$  is antisymmetric. To get the antisymmetry of  $\tilde{A}$ , it is enough to prove  $\tilde{A} \subset \overline{P(X)}$  (this implies  $\tilde{A} \subset \overline{P(X)}$ ). But every function  $u \in A$  is uniformly approximated on  $\bar{D}$  by polynomials. Thus  $\tilde{u} \in \overline{P(X)}$  and our theorem is proved for the functional calculus  $\varphi$ .

The proof for the  $H^\infty$  functional calculus  $\psi$  is almost the same as the proof given above, so we omit details. We just give the proof of 2° in case (ii). We will show that  $\tilde{B} = \{\tilde{u} = u|_X: u \in B\}$  is antisymmetric if  $\overline{P(X)}$  is. It is enough to prove  $\tilde{B} \subset \overline{P(X)}$ . Take  $\tilde{u} \in \tilde{B}$ . The function  $u \in B$  is continuous in  $D \cup G_u$ , where  $X \cap \Gamma \subset G_u \subset \Gamma$ . Hence  $u$  is continuous in  $\hat{X}$  and analytic in  $\text{int } \hat{X}$ ; therefore  $u|_{\hat{X}} \in \overline{P(\hat{X})}$  by Mergelyan's theorem and thus  $\tilde{u} \in \overline{P(X)}$ . Now our theorem is completely proved.

Note, finally, that the assumption  $\overline{P(X)} \neq C(X)$  is not sufficient for the antisymmetry of  $\varphi(A)$ . Indeed, if  $\varphi(A)$  is antisymmetric, so is  $\mathcal{A}(T)$  and now it is enough to diminish the sets  $X$  of Examples 2 and 3 to obtain a counterexample.

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